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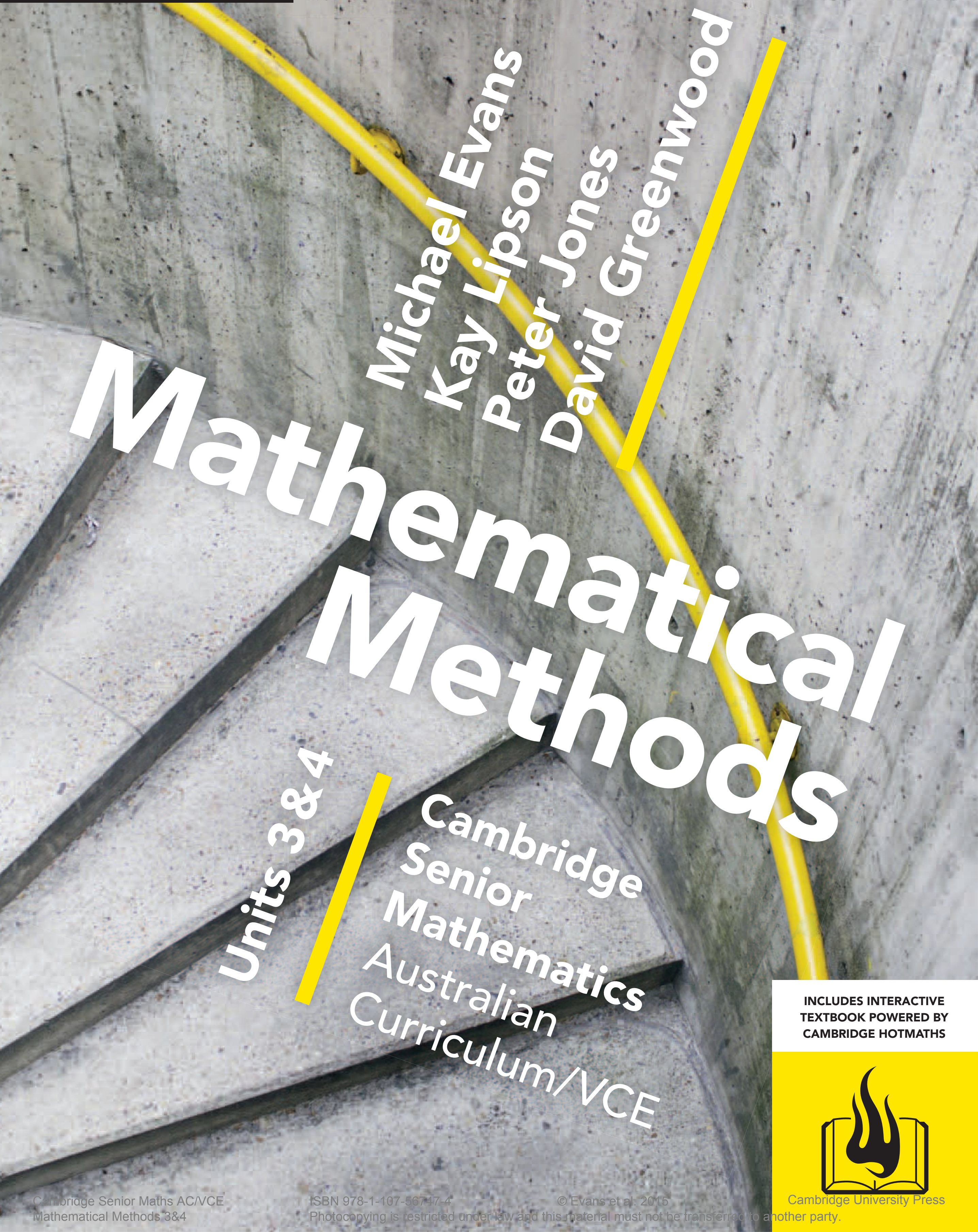

Michael Evans
Kay Lipson
Peter Jones
David Greenwood

Mathematical Methods

Units 3 & 4

Cambridge Senior Mathematics
Australian Curriculum/VCE

INCLUDES INTERACTIVE TEXTBOOK POWERED BY CAMBRIDGE HOTMATHS

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Included in the Interactive Textbook and PDF textbook only

Appendix B: Guide to the TI-Nspire CAS Calculator (OS4) in VCE Mathematics

Appendix C: Guide to the Casio ClassPad II CAS Calculator in VCE Mathematics

Introduction

Cambridge Mathematical Methods Australian Curriculum/VCE Units 3&4 provides a complete teaching and learning resource for the VCE Study Design to be implemented in 2016. It has been written with understanding as its chief aim and with ample practice offered through the worked examples and exercises. All the work has been trialled in the classroom, and the approaches offered are based on classroom experience and the responses of teachers to earlier versions of this book.

Mathematical Methods Units 3 and 4 provide an introductory study of functions, algebra, calculus, probability and statistics and their applications in a variety of practical and theoretical contexts. The changes in this course reflect the requirements of the Australian Curriculum subject Mathematical Methods.

The book has been carefully prepared to reflect the prescribed course. New material such as the sampling distribution for proportions and the corresponding confidence intervals has been included and there has been substantial reorganisation of differentiation based on feedback from teachers.

The book contains four revision chapters. These provide technology free, multiple-choice questions and extended-response questions.

The TI-Nspire calculator examples and instructions have been completed by Russell Brown and those for the Casio ClassPad have been completed by Maria Schaffner.

The integration of the features of the textbook and the new digital components of the package, powered by Cambridge HOTmaths, are illustrated on pages x to xi.

About Cambridge HOTmaths

Cambridge HOTmaths is a comprehensive, award-winning mathematics learning system—an interactive online maths learning, teaching and assessment resource for students and teachers, for individuals or whole classes, for school and at home. Its digital engine or platform is used to host and power the interactive textbook and the Online Teaching Suite. All this is included in the price of the textbook.

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For more detail, see the guide in the online Interactive Textbook

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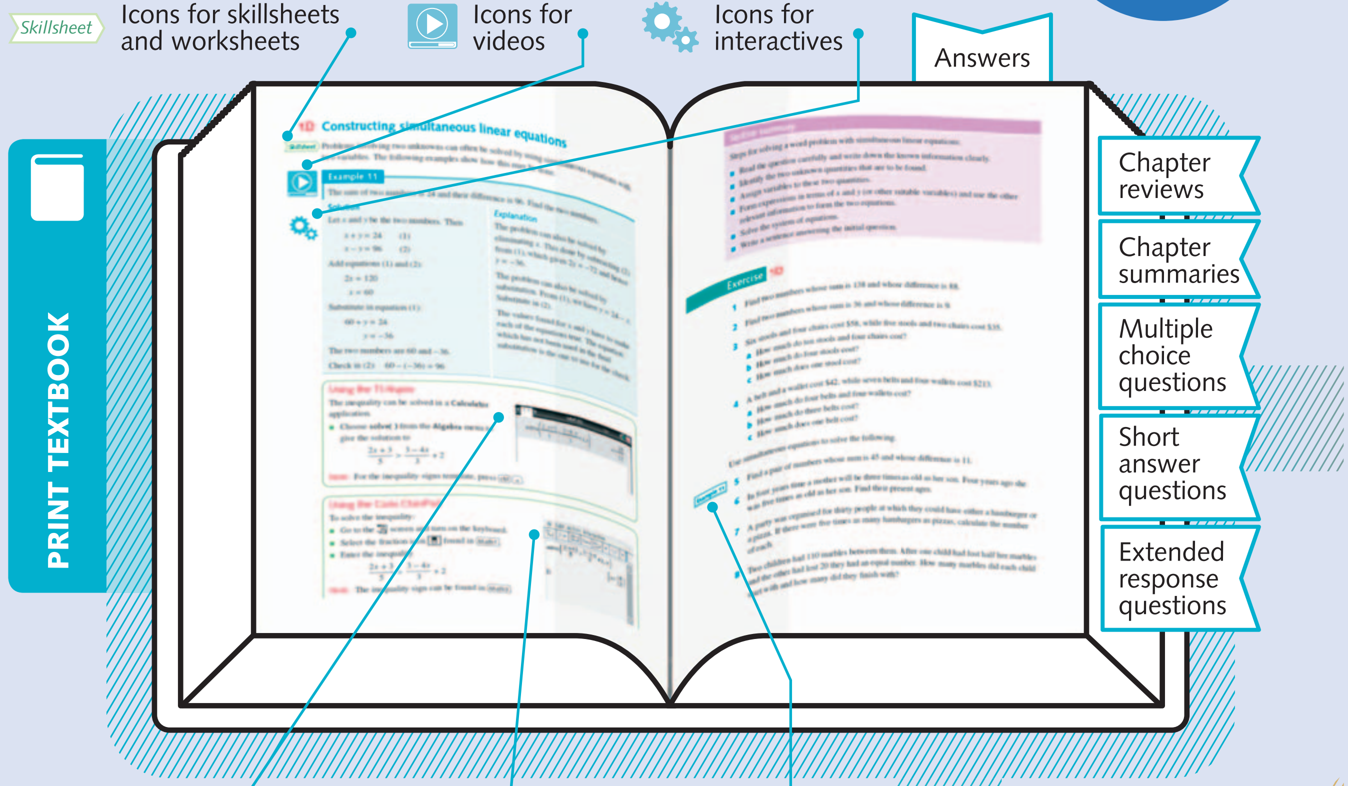
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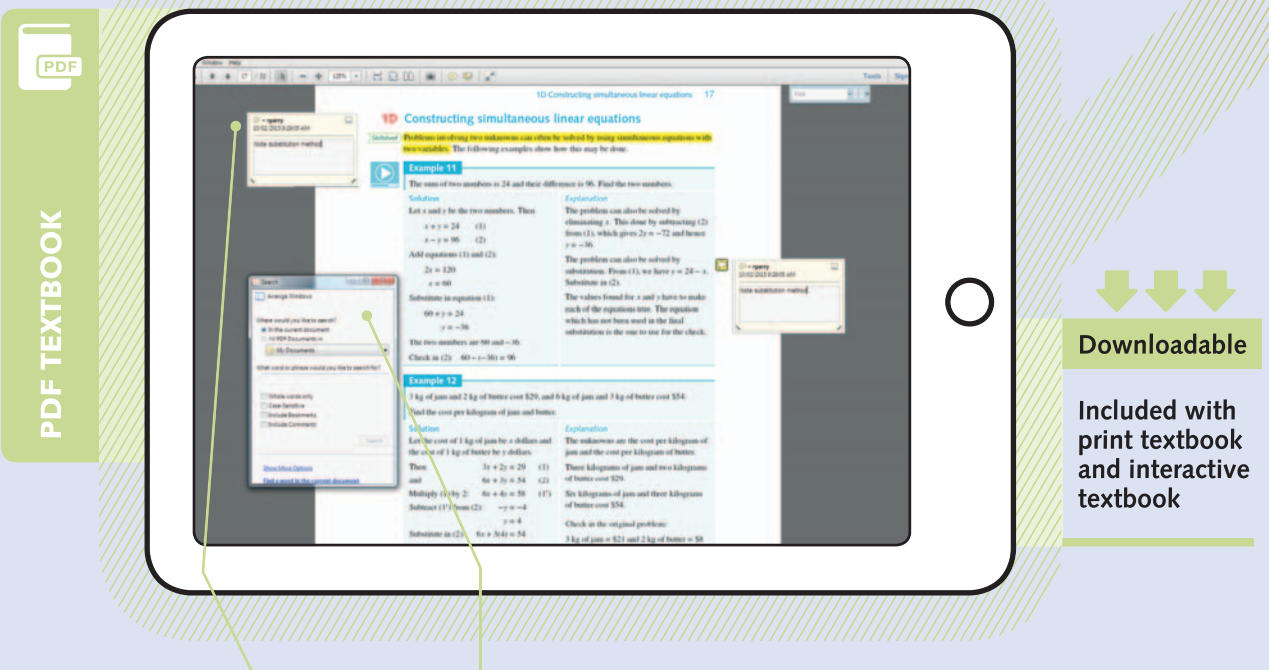

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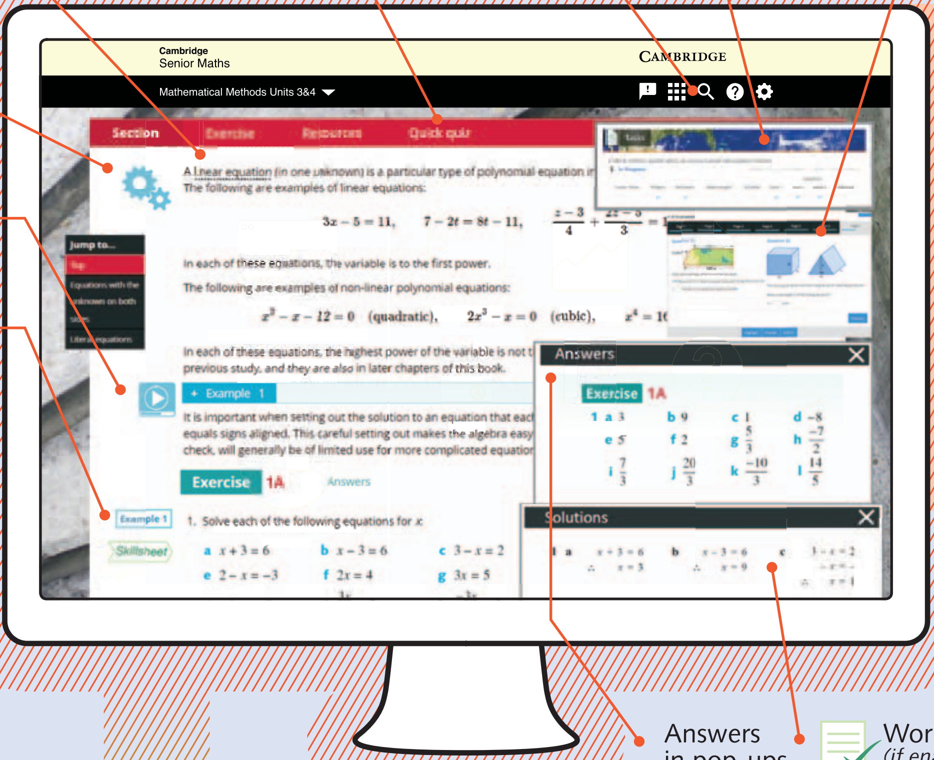
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Solutions

A linear equation (in one unknown) is a particular type of polynomial equation in

The following are examples of linear equations:

$3x - 5 = 11, \quad 7 - 2t = 8t - 11, \quad \frac{z-3}{4} + \frac{2z-5}{3} = 1$

In each of these equations, the variable is to the first power.

The following are examples of non-linear polynomial equations:

$x^2 - x - 12 = 0$ (quadratic), $\quad 2x^3 - x = 0$ (cubic), $\quad x^4 = 1$

In each of these equations, the highest power of the variable is not t

previous study, and they are also in later chapters of this book.

It is important when setting out the solution to an equation that eac

equals signs aligned. This careful setting out makes the algebra easy

check, will generally be of limited use for more complicated equatio

1. Solve each of the following equations for x:

a $x + 3 = 6$ b $x - 3 = 6$ c $3 - x = 2$

e $2 - x = -3$ f $2x = 4$ g $3x = 5$

Exercise 1A

1 a 3 b 9 c 1 d −8

e 5 f 2 g $\frac{5}{3}$ h $\frac{-7}{2}$

i $\frac{7}{3}$ j $\frac{20}{3}$ k $\frac{-10}{3}$ l $\frac{14}{5}$

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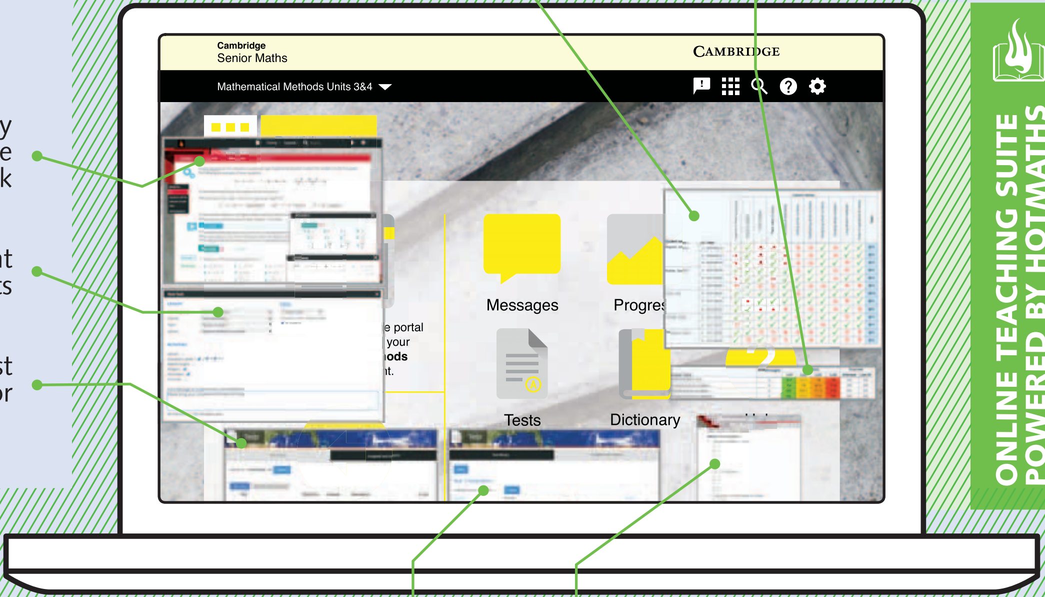
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1

Functions and relations

Objectives

- To revise **set notation**, including the notation for **sets of numbers**.
- To understand the concepts of **relation** and **function**.
- To find the **domain** and **range** of a given relation.
- To find the **implied (maximal) domain** of a function.
- To work with **restrictions of a function**, **piecewise-defined functions**, **odd functions** and **even functions**.
- To decide whether or not a given function is **one-to-one**.
- To find the **inverse** of a one-to-one function.
- To understand **sums** and **products** of functions.
- To use **addition of ordinates** to help sketch the graph of a sum of two functions.
- To define and use **composite functions**.
- To understand the concepts of **strictly increasing** and **strictly decreasing**.
- To work with **power functions** and their graphs.
- To apply a knowledge of functions to **solving problems**.

In this chapter we introduce the notation that will be used throughout the rest of the book. You will have met much of it before and this will serve as revision. The language introduced in this chapter helps to express important mathematical ideas precisely. Initially they may seem unnecessarily abstract, but later in the book you will find them used more and more in practical situations.

In Chapters 2 to 8 we will study different families of functions. In Chapter 2 we will revise linear functions, in Chapter 4, polynomial functions in general and in Chapters 5 and 6, exponential, logarithmic and circular functions.

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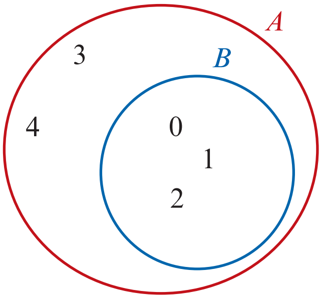
1A Set notation and sets of numbers

▶ Set notation

Set notation is used widely in mathematics and in this book where appropriate. This section summarises all of the set notation you will need.

- A **set** is a collection of objects. The objects that are in the set are known as **elements** or members of the set.
- If x is an element of a set A, we write $x \in A$. This can also be read as 'x is a member of the set A' or 'x belongs to A' or 'x is in A'.
- If x is **not an element** of A, we write $x \notin A$.
- A set B is called a **subset** of a set A if every element of B is also an element of A. We write $B \subseteq A$. This expression can also be read as 'B is contained in A' or 'A contains B'.



For example, let $B = \{0, 1, 2\}$ and $A = \{0, 1, 2, 3, 4\}$. Then

$$3 \in A, \quad 3 \notin B \quad \text{and} \quad B \subseteq A$$

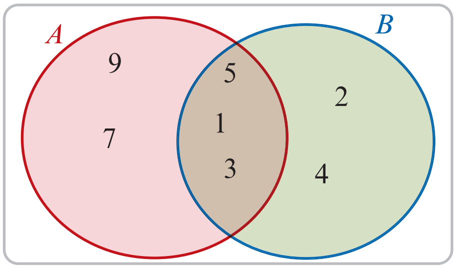
as illustrated in the Venn diagram opposite.

- The set of elements common to two sets A and B is called the **intersection** of A and B, and is denoted by $A \cap B$. Thus $x \in A \cap B$ if and only if $x \in A$ and $x \in B$.
- If the sets A and B have no elements in common, we say A and B are **disjoint**, and write $A \cap B = \emptyset$. The set $\emptyset$ is called the **empty set**.
- The set of elements that are in A or in B (or in both) is called the **union** of sets A and B, and is denoted by $A \cup B$.

For example, let $A = \{1, 3, 5, 7, 9\}$ and $B = \{1, 2, 3, 4, 5\}$. The intersection and union are illustrated by the Venn diagram shown opposite:

$$A \cap B = \{1, 3, 5\}$$

$$A \cup B = \{1, 2, 3, 4, 5, 7, 9\}$$



Example 1

For $A = \{1, 2, 3, 7\}$ and $B = \{3, 4, 5, 6, 7\}$, find:

a $A \cap B$

b $A \cup B$

Solution	*Explanation*
a $A \cap B = \{3, 7\}$	The elements 3 and 7 are common to sets A and B.
b $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$	The set $A \cup B$ contains all elements that belong to A or B (or both).

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The **set difference** of two sets A and B is given by

$$A \setminus B = \{ x : x \in A, \ x \notin B \}$$

The set $A \setminus B$ contains the elements of A that are not elements of B.

Example 2

For $A = \{1, 2, 3, 7\}$ and $B = \{3, 4, 5, 6, 7\}$, find:

a $A \setminus B$ **b** $B \setminus A$

Solution	*Explanation*
a $A \setminus B = \{1, 2, 3, 7\} \setminus \{3, 4, 5, 6, 7\}$ $= \{1, 2\}$	The elements 1 and 2 are in A but not in B.
b $B \setminus A = \{3, 4, 5, 6, 7\} \setminus \{1, 2, 3, 7\}$ $= \{4, 5, 6\}$	The elements 4, 5 and 6 are in B but not in A.

Sets of numbers

We begin by recalling that the elements of $\{1, 2, 3, 4, \dots\}$ are called the **natural numbers**, and the elements of $\{\dots, -2, -1, 0, 1, 2, \dots\}$ are called **integers**.

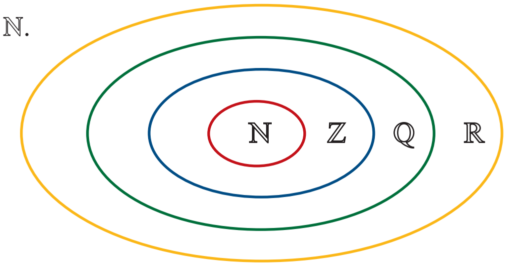
The numbers of the form $\frac{p}{q}$, with p and q integers, $q \neq 0$, are called **rational numbers**.

The real numbers which are not rational are called **irrational** (e.g. π and $\sqrt{2}$).

The rationals may be characterised as being those real numbers that can be written as a terminating or recurring decimal.

- The set of real numbers will be denoted by $\mathbb{R}$.
- The set of rational numbers will be denoted by $\mathbb{Q}$.
- The set of integers will be denoted by $\mathbb{Z}$.
- The set of natural numbers will be denoted by $\mathbb{N}$.

It is clear that $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$, and this may be represented by the diagram on the right.



Describing a set

It is not always possible to list the elements of a set. There is an alternative way of describing sets that is especially useful for infinite sets.

The set of all x such that ___ is denoted by $\{ x : \text{\_\_\_} \}$. Thus, for example:

- $\{ x : 0 < x < 1 \}$ is the set of all real numbers strictly between 0 and 1
- $\{ x : x \geq 3 \}$ is the set of all real numbers greater than or equal to 3
- $\{ x : x > 0, \ x \in \mathbb{Q} \}$ is the set of all positive rational numbers
- $\{ 2n : n = 0, 1, 2, \dots \}$ is the set of all non-negative even numbers
- $\{ 2n + 1 : n = 0, 1, 2, \dots \}$ is the set of all non-negative odd numbers.

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Interval notation

Among the most important subsets of $\mathbb{R}$ are the **intervals**. The following is an exhaustive list of the various types of intervals and the standard notation for them. We suppose that a and b are real numbers with $a < b$.

$$\begin{array}{rcl}
(a,b) = \{\, x : a < x < b \,\} & \qquad & [a,b] = \{\, x : a \le x \le b \,\} \\
(a,b] = \{\, x : a < x \le b \,\} & \qquad & [a,b) = \{\, x : a \le x < b \,\} \\
(a,\infty) = \{\, x : a < x \,\} & \qquad & [a,\infty) = \{\, x : a \le x \,\} \\
(-\infty,b) = \{\, x : x < b \,\} & \qquad & (-\infty,b] = \{\, x : x \le b \,\}
\end{array}$$

Intervals may be represented by diagrams as shown in Example 3.

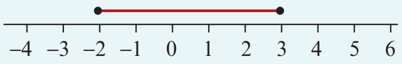
Example 3

Illustrate each of the following intervals of real numbers:

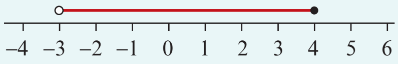
a $[-2,3]$ **b** $(-3,4]$ **c** $(-\infty,5]$ **d** $(-2,4)$ **e** $(-3,\infty)$

Solution

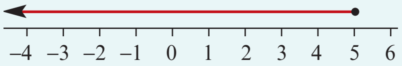
a $[-2,3]$



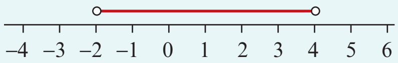
b $(-3,4]$



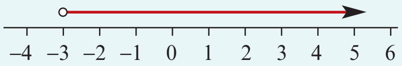
c $(-\infty,5]$



d $(-2,4)$



e $(-3,\infty)$



Notes:

- The ‘closed’ circle (●) indicates that the number is included.
- The ‘open’ circle (○) indicates that the number is not included.

The following are subsets of the real numbers for which we have special notation:

- Positive real numbers: $\mathbb{R}^+ = \{\, x : x > 0 \,\}$
- Negative real numbers: $\mathbb{R}^- = \{\, x : x < 0 \,\}$
- Real numbers excluding zero: $\mathbb{R} \setminus \{0\}$

Section summary

- If x is an element of a set A, we write $x \in A$.
- If x is not an element of a set A, we write $x \notin A$.
- If every element of B is an element of A, we say B is a **subset** of A and write $B \subseteq A$.
- The set $A \cap B$ is the **intersection** of A and B, where $x \in A \cap B$ if and only if $x \in A$ and $x \in B$.
- The set $A \cup B$ is the **union** of A and B, where $x \in A \cup B$ if and only if $x \in A$ or $x \in B$.

- The set $A \setminus B$ is the **set difference** of A and B, where $A \setminus B = \{ x : x \in A,\ x \notin B \}$.
- If the sets A and B have no elements in common, we say A and B are **disjoint** and write $A \cap B = \varnothing$. The set $\varnothing$ is called the **empty set**.
- Sets of numbers:
  - Real numbers: $\mathbb{R}$
  - Rational numbers: $\mathbb{Q}$
  - Integers: $\mathbb{Z}$
  - Natural numbers: $\mathbb{N}$
- For real numbers a and b with $a < b$, we can consider the following **intervals**:

$$(a, b) = \{ x : a < x < b \} \qquad [a, b] = \{ x : a \le x \le b \}$$
$$(a, b] = \{ x : a < x \le b \} \qquad [a, b) = \{ x : a \le x < b \}$$
$$(a, \infty) = \{ x : a < x \} \qquad [a, \infty) = \{ x : a \le x \}$$
$$(-\infty, b) = \{ x : x < b \} \qquad (-\infty, b] = \{ x : x \le b \}$$

Exercise 1A

Example 1

1 For $A = \{3, 8, 11, 18, 22, 23, 24\}$, $B = \{8, 11, 25, 30, 32\}$ and $C = \{1, 8, 11, 25, 30\}$, find:

a $A \cap B$ **b** $A \cap B \cap C$ **c** $A \cup C$
d $A \cup B$ **e** $A \cup B \cup C$ **f** $(A \cap B) \cup C$

Example 2

2 For $A = \{3, 8, 11, 18, 22, 23, 24\}$, $B = \{8, 11, 25, 30, 32\}$ and $C = \{1, 8, 11, 25, 30\}$, find:

a $A \setminus B$ **b** $B \setminus A$ **c** $A \setminus C$ **d** $C \setminus A$

Example 3

3 Illustrate each of the following intervals on a number line:

a $[-2, 3)$ **b** $(-\infty, 4]$ **c** $[-3, -1]$ **d** $(-3, \infty)$ **e** $(-4, 3)$ **f** $(-1, 4]$

4 For $X = \{2, 3, 5, 7, 9, 11\}$, $Y = \{7, 9, 15, 19, 23\}$ and $Z = \{2, 7, 9, 15, 19\}$, find:

a $X \cap Y$ **b** $X \cap Y \cap Z$ **c** $X \cup Y$ **d** $X \setminus Y$
e $Z \setminus Y$ **f** $X \cap Z$ **g** $[-2, 8] \cap X$ **h** $(-3, 8] \cap Y$
i $(2, \infty) \cap Y$ **j** $(3, \infty) \cup Y$

5 For $X = \{a, b, c, d, e\}$ and $Y = \{a, e, i, o, u\}$, find:

a $X \cap Y$ **b** $X \cup Y$ **c** $X \setminus Y$ **d** $Y \setminus X$

6 For $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $B = \{2, 4, 6, 8, 10\}$ and $C = \{1, 3, 6, 9\}$, find:

a $B \cap C$ **b** $B \setminus C$ **c** $A \setminus B$ **d** $(A \setminus B) \cup (A \setminus C)$
e $A \setminus (B \cap C)$ **f** $(A \setminus B) \cap (A \setminus C)$ **g** $A \setminus (B \cup C)$ **h** $A \cap B \cap C$

7 Use the appropriate interval notation (i.e. $[a, b]$, (a, b), etc.) to describe each of the following sets:

a $\{ x : -3 \le x < 1 \}$ **b** $\{ x : -4 < x \le 5 \}$ **c** $\{ y : -\sqrt{2} < y < 0 \}$
d $\left\{ x : -\dfrac{1}{\sqrt{2}} < x < \sqrt{3} \right\}$ **e** $\{ x : x < -3 \}$ **f** $\mathbb{R}^+$
g $\mathbb{R}^-$ **h** $\{ x : x \ge -2 \}$

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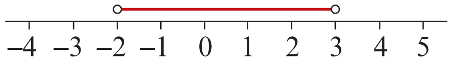
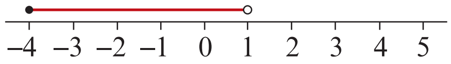
8 Describe each of the following subsets of the real number line using the interval notation $[a, b)$, (a, b), etc.:

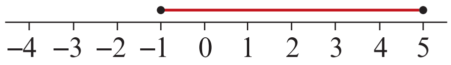
a



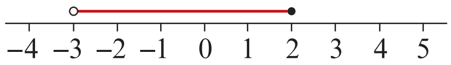
b



c



d



9 Illustrate each of the following intervals on a number line:

a $(-3, 2]$ **b** $(-4, 3)$ **c** $(-\infty, 3)$ **d** $[-4, -1]$ **e** $[-4, \infty)$ **f** $[-2, 5)$

10 For each of the following, use one number line on which to represent the sets:

a $[-3, 6]$, $[2, 4]$, $[-3, 6] \cap [2, 4]$ **b** $[-3, 6]$, $\mathbb{R} \setminus [-3, 6]$

c $[-2, \infty)$, $(-\infty, 6]$, $[-2, \infty) \cap (-\infty, 6]$ **d** $(-8, -2)$, $\mathbb{R}^- \setminus (-8, -2)$

1B Identifying and describing relations and functions

Relations, domain and range

An **ordered pair**, denoted (x, y), is a pair of elements x and y in which x is considered to be the first coordinate and y the second coordinate.

A **relation** is a set of ordered pairs. The following are examples of relations:

a $S = \{(1, 1), (1, 2), (3, 4), (5, 6)\}$

b $T = \{(-3, 5), (4, 12), (5, 12), (7, -6)\}$

Every relation determines two sets:

- The set of all the first coordinates of the ordered pairs is called the **domain**.
- The set of all the second coordinates of the ordered pairs is called the **range**.

For the above examples:

a domain of $S = \{1, 3, 5\}$, range of $S = \{1, 2, 4, 6\}$

b domain of $T = \{-3, 4, 5, 7\}$, range of $T = \{5, 12, -6\}$

Some relations may be defined by a **rule** relating the elements in the domain to their corresponding elements in the range. In order to define the relation fully, we need to specify both the rule and the domain. For example, the set

$$\{\, (x, y) : y = x + 1,\ x \in \{1, 2, 3, 4\} \,\}$$

is the relation

$$\{(1, 2), (2, 3), (3, 4), (4, 5)\}$$

The **domain** is the set $X = \{1, 2, 3, 4\}$ and the **range** is the set $Y = \{2, 3, 4, 5\}$.

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When the domain of a relation is not explicitly stated, it is understood to consist of all real numbers for which the defining rule has meaning. For example:

- $S = \{ (x, y) : y = x^2 \}$ is assumed to have domain $\mathbb{R}$
- $T = \{ (x, y) : y = \sqrt{x} \}$ is assumed to have domain $[0, \infty)$.

Example 4

Sketch the graph of each of the following relations and state the domain and range of each:

a $\{ (x, y) : y = x^2 \}$

b $\{ (x, y) : y \leq x + 1 \}$

c $\{(-2, -1), (-1, -1), (-1, 1), (0, 1), (1, -1)\}$

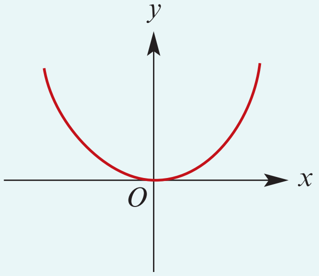
d $\{ (x, y) : x^2 + y^2 = 1 \}$

e $\{ (x, y) : 2x + 3y = 6,\ x \geq 0 \}$

f $\{ (x, y) : y = 2x - 1,\ x \in [-1, 2] \}$

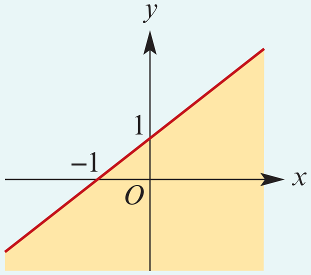
Solution

a



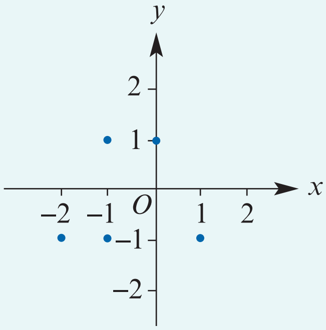
Domain = $\mathbb{R}$; Range = $\mathbb{R}^+ \cup \{0\}$

b



Domain = $\mathbb{R}$; Range = $\mathbb{R}$

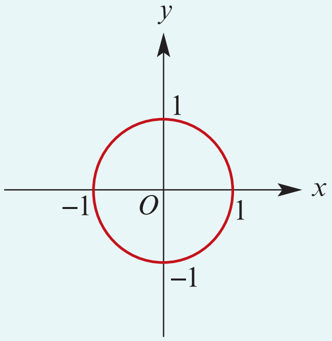
c



Domain = $\{-2, -1, 0, 1\}$

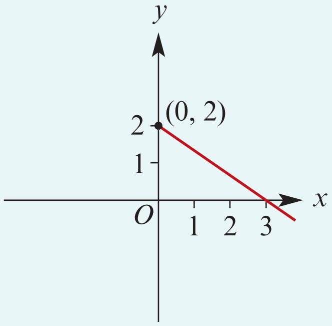
Range = $\{-1, 1\}$

d



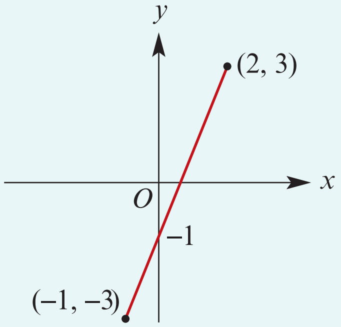
Domain = $[-1, 1]$; Range = $[-1, 1]$

e



Domain = $[0, \infty)$; Range = $(-\infty, 2]$

f



Domain = $[-1, 2]$; Range = $[-3, 3]$

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Sometimes set notation is not used in the specification of a relation.

For the previous example:

a is written as $y = x^2$
b is written as $y \le x + 1$
e is written as $2x + 3y = 6$, $x \ge 0$

Functions

A **function** is a relation such that for each x-value there is only one corresponding y-value. This means that, if (a, b) and (a, c) are ordered pairs of a function, then $b = c$. In other words, a function cannot contain two different ordered pairs with the same first coordinate.

Example 5

Which of the following sets of ordered pairs defines a function?

a $S = \{(-3, -4), (-1, -1), (-6, 7), (1, 5)\}$

b $T = \{(-4, 1), (-4, -1), (-6, 7), (-6, 8)\}$

Solution

a S is a function because for each x-value there is only one y-value.

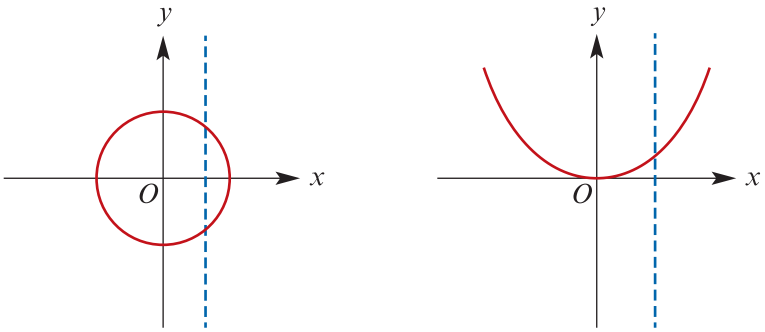
b T is not a function, because there is an x-value with two different y-values: the two ordered pairs $(-4, 1)$ and $(-4, -1)$ in T have the same first coordinate.

One way to identify whether a relation is a function is to draw a graph of the relation and then apply the following test.

Vertical-line test

If a vertical line can be drawn anywhere on the graph and it only ever intersects the graph a maximum of once, then the relation is a **function**.

For example:



$x^2 + y^2 = 1$ is not a function $\qquad$ $y = x^2$ is a function

Functions are usually denoted by lowercase letters such as f, g, h.

The definition of a function tells us that, for each x in the domain of f, there is a unique element y in the range such that $(x, y) \in f$. The element y is called 'the **image** of x under f' or 'the **value** of f at x', and the element x is called 'a **pre-image** of y'.

For $(x, y) \in f$, the element y is determined by x, and so we also use the notation $f(x)$, read as 'f of x', in place of y.

This gives an alternative way of writing functions:

- For the function $\{ (x, y) : y = x^2 \}$, write $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$.
- For the function $\{ (x, y) : y = 2x - 1,\ x \in [0, 4] \}$, write $f : [0, 4] \to \mathbb{R}$, $f(x) = 2x - 1$.
- For the function $\left\{ (x, y) : y = \dfrac{1}{x} \right\}$, write $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = \dfrac{1}{x}$.

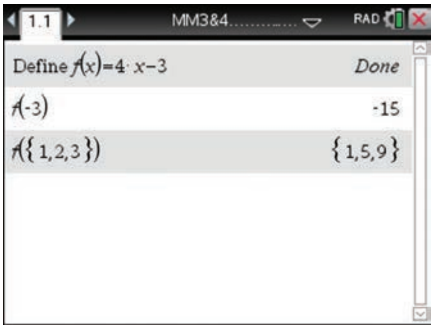
If the domain is $\mathbb{R}$, we often just write the rule: for example, $f(x) = x^2$.

Note that in using the notation $f : X \to Y$, the set X is the domain, but Y is not necessarily the range. It is a set that contains the range and is called the **codomain**. With this notation for functions, the domain of f is written as **dom f** and range of f as **ran f**.

Using the TI-Nspire

Function notation can be used with a CAS calculator.

- Use menu > **Actions** > **Define** to define the function $f(x) = 4x - 3$.
- To find the value of $f(-3)$, type $f(-3)$ followed by enter.
- To evaluate $f(1)$, $f(2)$ and $f(3)$, type $f(\{1, 2, 3\})$ followed by enter.



Define f(x)=4·x−3 Done

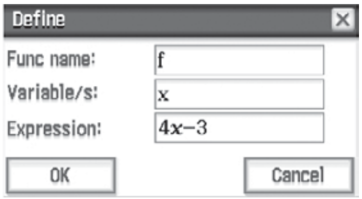
f(-3) -15

f({1,2,3}) {1,5,9}

Using the Casio ClassPad

Function notation can be used with a CAS calculator.

- In Main $\sqrt{\alpha}$, select **Interactive** > **Define**.
- Enter the expression $4x - 3$ as shown and tap OK.

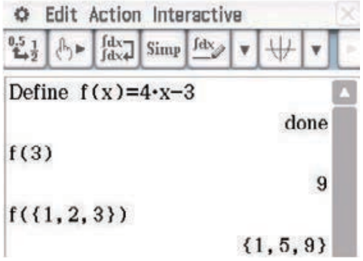


Func name: f

Variable/s: x

Expression: 4x−3

- Enter $f(3)$ in the entry line and tap EXE.
- Enter $f(\{1, 2, 3\})$ to obtain the values of $f(1)$, $f(2)$ and $f(3)$.



Define f(x)=4·x−3

done

f(3)

9

f({1,2,3})

{1,5,9}

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Example 6

For $f(x) = 2x^2 + x$, find:

a $f(3)$ **b** $f(-2)$ **c** $f(x-1)$

Solution

a $f(3) = 2(3)^2 + 3$
$= 21$

b $f(-2) = 2(-2)^2 - 2$
$= 6$

c
$$\begin{aligned} f(x-1) &= 2(x-1)^2 + (x-1) \\ &= 2(x^2 - 2x + 1) + x - 1 \\ &= 2x^2 - 3x + 1 \end{aligned}$$

Example 7

For $g(x) = 3x^2 + 1$:

a Find $g(-2)$ and $g(4)$.

b Express each the following in terms of x:

i $g(-2x)$ **ii** $g(x-2)$ **iii** $g(x+2)$ **iv** $g(x^2)$

Solution

a $g(-2) = 3(-2)^2 + 1 = 13$ and $g(4) = 3(4)^2 + 1 = 49$

b **i**
$$\begin{aligned} g(-2x) &= 3(-2x)^2 + 1 \\ &= 3 \times 4x^2 + 1 \\ &= 12x^2 + 1 \end{aligned}$$

ii
$$\begin{aligned} g(x-2) &= 3(x-2)^2 + 1 \\ &= 3(x^2 - 4x + 4) + 1 \\ &= 3x^2 - 12x + 13 \end{aligned}$$

iii
$$\begin{aligned} g(x+2) &= 3(x+2)^2 + 1 \\ &= 3(x^2 + 4x + 4) + 1 \\ &= 3x^2 + 12x + 13 \end{aligned}$$

iv
$$\begin{aligned} g(x^2) &= 3(x^2)^2 + 1 \\ &= 3x^4 + 1 \end{aligned}$$

Example 8

Consider the function defined by $f(x) = 2x - 4$ for all $x \in \mathbb{R}$.

a Find the value of $f(2)$, $f(-1)$ and $f(t)$.

b For what values of t is $f(t) = t$?

c For what values of x is $f(x) \geq x$?

d Find the pre-image of 6.

Solution

a
$$\begin{aligned} f(2) &= 2(2) - 4 = 0 \\ f(-1) &= 2(-1) - 4 = -6 \\ f(t) &= 2t - 4 \end{aligned}$$

b
$$\begin{aligned} f(t) &= t \\ 2t - 4 &= t \\ t - 4 &= 0 \\ \therefore \quad t &= 4 \end{aligned}$$

c
$$\begin{aligned} f(x) &\geq x \\ 2x - 4 &\geq x \\ x - 4 &\geq 0 \\ \therefore \quad x &\geq 4 \end{aligned}$$

d
$$\begin{aligned} f(x) &= 6 \\ 2x - 4 &= 6 \\ x &= 5 \end{aligned}$$

Thus 5 is the pre-image of 6.

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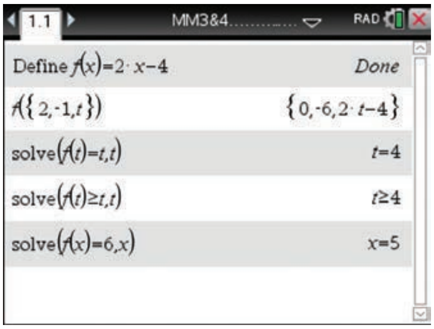
Using the TI-Nspire

- Use menu > **Actions** > **Define** to define the function and menu > **Algebra** > **Solve** to solve as shown.
- The symbol ≥ can be found using ctrl = or using ctrl menu > **Symbols**.



Define $f(x)=2\cdot x-4$ Done

$f(\{2,-1,t\})$ $\{0,-6,2\cdot t-4\}$

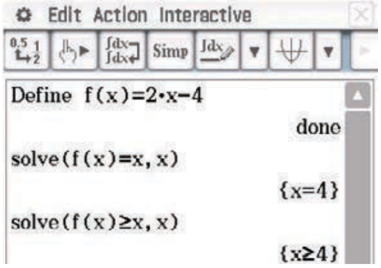
solve$(f(t)=t,t)$ $t=4$

solve$(f(t)\geq t,t)$ $t\geq 4$

solve$(f(x)=6,x)$ $x=5$

Using the Casio ClassPad

- In $\overset{\text{Main}}{\sqrt{\alpha}}$, define the function $f(x) = 2x - 4$ using **Interactive** > **Define**.
- Now enter and highlight the equation $f(x) = x$.
- Select **Interactive** > **Equation/Inequality** > **solve**. Ensure the variable is set as x and tap OK.
- To enter the inequality, find the symbol ≥ in the Math3 keyboard.



Define f(x)=2•x−4

done

solve(f(x)=x, x)

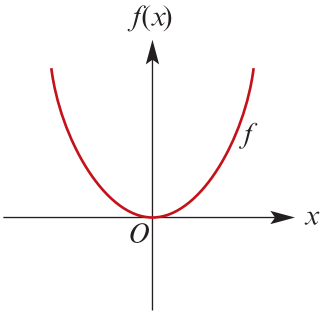
{x=4}

solve(f(x)≥x, x)

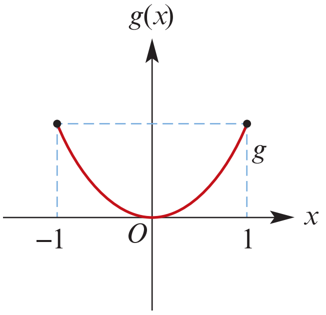
{x≥4}

Restriction of a function

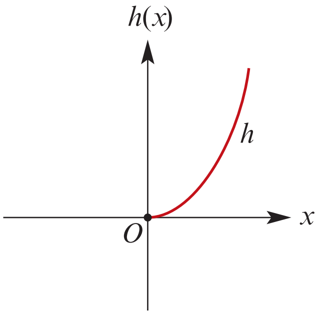
Consider the following functions:



$f(x) = x^2, \ x \in \mathbb{R}$



$g(x) = x^2, \ -1 \leq x \leq 1$



$h(x) = x^2, \ x \in \mathbb{R}^+ \cup \{0\}$

The different letters, f, g and h, used to name the functions emphasise the fact that there are three different functions, even though they each have the same rule. They are different because they are defined for different domains.

We call g and h **restrictions** of f, since their domains are subsets of the domain of f.

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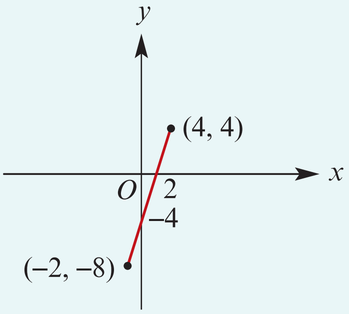
Example 9

For each of the following, sketch the graph and state the range:

a $f\colon [-2,4] \to \mathbb{R},\ f(x) = 2x - 4$ **b** $g\colon (-1,2] \to \mathbb{R},\ g(x) = x^2$

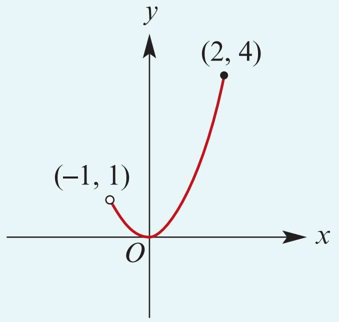
Solution

a



Range $= [-8, 4]$

b



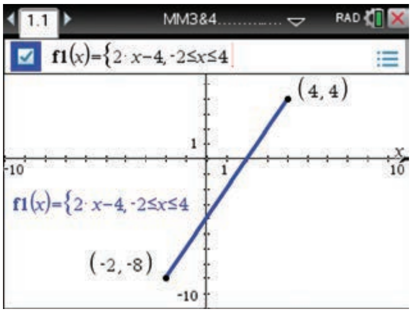
Range $= [0, 4]$

Using the TI-Nspire

Domain restrictions can be entered with the function if required.

For example: $f1(x) = 2x - 4 \mid -2 \le x \le 4$

Domain restrictions are entered using the 'with' symbol $\mid$, which is accessed using ctrl = or by using the **Symbols** palette ctrl 📖 and scrolling to the required symbol. The inequality symbols are also accessed from this palette.


1.1
MM3&4
RAD
(4, 4)
1
-10
1
10
x
(-2, -8)
-10

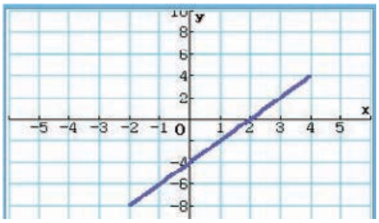
☑ $f1(x)=\{2\cdot x-4,\ -2\le x\le 4$

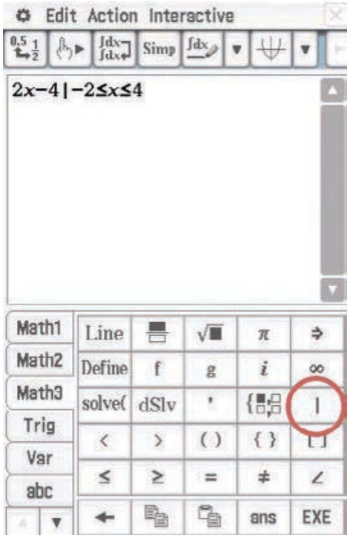
Using the Casio ClassPad

Domain restrictions can be entered with the function if required.

For example: $2x - 4 \mid -2 \le x \le 4$

Domain restrictions are entered using the 'with' symbol $\mid$, which is accessed from the Math3 palette in the soft keyboard. The inequality symbols are also accessed from Math3.




Edit Action Interactive
Simp
Math1
Math2
Math3
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≠
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ans
EXE

$2x-4\mid -2\le x\le 4$

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Section summary

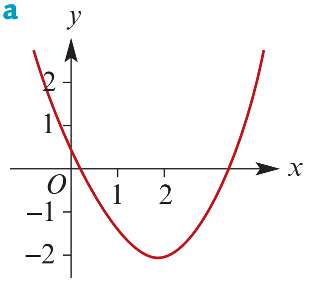
- An **ordered pair**, denoted (x, y), is a pair of elements x and y in which x is considered to be the first coordinate and y the second coordinate.
- A **relation** is a set of ordered pairs.
  - The set of all the first coordinates of the ordered pairs is called the **domain**.
  - The set of all the second coordinates of the ordered pairs is called the **range**.
- Some relations may be defined by a rule relating the elements in the domain to their corresponding elements in the range: for example, $\{(x, y) : y = x + 1,\ x \in \mathbb{R}^+ \cup \{0\}\}$.
- A **function** is a relation such that for each x-value there is only one corresponding y-value.
- **Vertical-line test**: If a vertical line can be drawn anywhere on the graph and it only ever intersects the graph a maximum of once, then the relation is a function.
- For an ordered pair (x, y) of a function f, we say that y is the **image** of x under f or that y is the value of f at x, and we say that x is a **pre-image** of y. Since the y-value is determined by the x-value, we use the notation $f(x)$, read as 'f of x', in place of y.
- Notation for defining functions: For example, we write $f\colon [0, 4] \to \mathbb{R}$, $f(x) = 2x - 1$ to define a function f with domain $[0, 4]$ and rule $f(x) = 2x - 1$.
- A **restriction** of a function has the same rule but a 'smaller' domain.

Exercise 1B

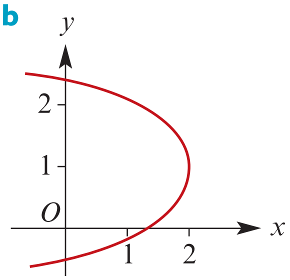
Skillsheet

1 State the domain and range for the relations represented by each of the following graphs:

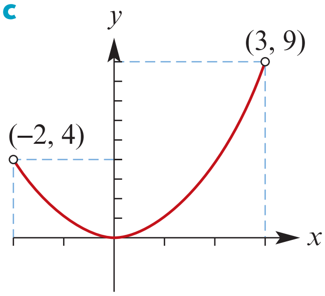
a



b



c



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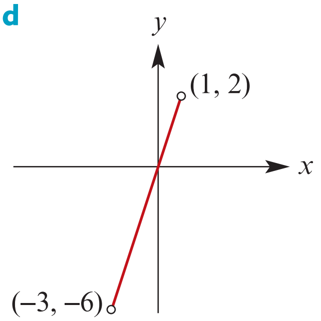
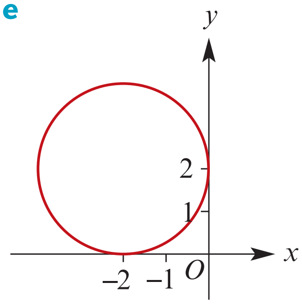
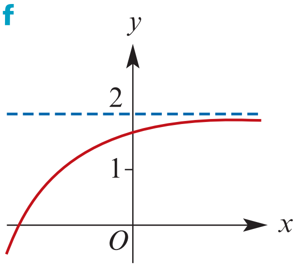


e



f



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Example 4 **2** Sketch a graph of each of the following relations and state its domain and range:

a $\{(x, y) : y = x^2 + 1\}$
b $\{(x, y) : x^2 + y^2 = 9\}$
c $\{(x, y) : 3x + 12y = 24,\ x \geq 0\}$
d $y = \sqrt{2x}$
e $\{(x, y) : y = 5 - x,\ x \in [0, 5]\}$
f $y = x^2 + 2,\ x \in [0, 4]$
g $y = 3x - 2,\ x \in [-1, 2]$
h $y = 4 - x^2$
i $\{(x, y) : y \leq 1 - x\}$

Example 5 **3** Which of the following relations are functions? State the domain and range for each.

a $\{(-1, 1), (-1, 2), (1, 2), (3, 4), (2, 3)\}$
b $\{(-2, 0), (-1, -1), (0, 3), (1, 5), (2, -4)\}$
c $\{(-1, 1), (-1, 2), (-2, -2), (2, 4), (4, 6)\}$
d $\{(-1, 4), (0, 4), (1, 4), (2, 4), (3, 4)\}$

4 Which of the following relations are functions? State the domain and range for each.

a $\{(x, 4) : x \in \mathbb{R}\}$
b $\{(2, y) : y \in \mathbb{Z}\}$
c $y = -2x + 4$
d $y \geq 3x + 2$
e $\{(x, y) : x^2 + y^2 = 16\}$

Example 6 **5** Let $f(x) = 2x^2 + 4x$ and $g(x) = 2x^3 + 2x - 6$.

a Evaluate $f(-1)$, $f(2)$, $f(-3)$ and $f(2a)$.
b Evaluate $g(-1)$, $g(2)$, $g(3)$ and $g(a - 1)$.

Example 7 **6** Consider the function $g(x) = 3x^2 - 2$.

a Find $g(-2)$ and $g(4)$.
b Express the following in terms of x:
  i $g(-2x)$
  ii $g(x - 2)$
  iii $g(x + 2)$
  iv $g(x^2)$

Example 8 **7** Consider the function $f(x) = 2x - 3$. Find:

a the image of 3
b the pre-image of 11
c $\{x : f(x) = 4x\}$
d $\{x : f(x) > x\}$

8 Consider the functions $g(x) = 6x + 7$ and $h(x) = 3x - 2$. Find:

a $\{x : g(x) = h(x)\}$
b $\{x : g(x) > h(x)\}$
c $\{x : h(x) = 0\}$

9 Rewrite each of the following using the $f : X \to Y$ notation:

a $\{(x, y) : y = 2x + 3\}$
b $\{(x, y) : 3y + 4x = 12\}$
c $\{(x, y) : y = 2x - 3,\ x \geq 0\}$
d $y = x^2 - 9,\ x \in \mathbb{R}$
e $y = 5x - 3,\ 0 \leq x \leq 2$

Example 9 **10** Sketch the graph of each of the following and state the range of each:

a $y = x + 1,\ x \in [2, \infty)$
b $y = -x + 1,\ x \in [2, \infty)$
c $y = 2x + 1,\ x \in [-4, \infty)$
d $y = 3x + 2,\ x \in (-\infty, 3)$
e $y = x + 1,\ x \in (-\infty, 3]$
f $y = 3x - 1,\ x \in [-2, 6]$
g $y = -3x - 1,\ x \in [-5, -1]$
h $y = 5x - 1,\ x \in (-2, 4)$

11 For $f(x) = 2x^2 - 6x + 1$ and $g(x) = 3 - 2x$:

a Evaluate $f(2)$, $f(-3)$ and $f(-2)$.

b Evaluate $g(-2)$, $g(1)$ and $g(-3)$.

c Express the following in terms of a:

i $f(a)$ **ii** $f(a + 2)$ **iii** $g(-a)$ **iv** $g(2a)$

v $f(5 - a)$ **vi** $f(2a)$ **vii** $g(a) + f(a)$ **viii** $g(a) - f(a)$

12 For $f(x) = 3x^2 + x - 2$, find:

a $\{ x : f(x) = 0 \}$ **b** $\{ x : f(x) = x \}$ **c** $\{ x : f(x) = -2 \}$

d $\{ x : f(x) > 0 \}$ **e** $\{ x : f(x) > x \}$ **f** $\{ x : f(x) \leq -2 \}$

13 For $f(x) = x^2 + x$, find:

a $f(-2)$

b $f(2)$

c $f(-a)$ in terms of a

d $f(a) + f(-a)$ in terms of a

e $f(a) - f(-a)$ in terms of a

f $f(a^2)$ in terms of a

14 For $g(x) = 3x - 2$, find:

a $\{ x : g(x) = 4 \}$ **b** $\{ x : g(x) > 4 \}$ **c** $\{ x : g(x) = a \}$

d $\{ x : g(-x) = 6 \}$ **e** $\{ x : g(2x) = 4 \}$ **f** $\left\{ x : \dfrac{1}{g(x)} = 6 \right\}$

15 Find the value of k for each of the following if $f(3) = 3$, where:

a $f(x) = kx - 1$ **b** $f(x) = x^2 - k$ **c** $f(x) = x^2 + kx + 1$

d $f(x) = \dfrac{k}{x}$ **e** $f(x) = kx^2$ **f** $f(x) = 1 - kx^2$

16 Find the values of x for which the given functions have the given value:

a $f(x) = 5x - 4,\ f(x) = 2$ **b** $f(x) = \dfrac{1}{x},\ f(x) = 5$ **c** $f(x) = \dfrac{1}{x^2},\ f(x) = 9$

d $f(x) = x + \dfrac{1}{x},\ f(x) = 2$ **e** $f(x) = (x + 1)(x - 2),\ f(x) = 0$

1C Types of functions and implied domains

► One-to-one functions

A function is said to be **one-to-one** if different x-values map to different y-values. That is, a function f is one-to-one if $a \neq b$ implies $f(a) \neq f(b)$, for all $a, b \in \text{dom } f$.

An equivalent way to say this is that a function f is one-to-one if $f(a) = f(b)$ implies $a = b$, for all $a, b \in \text{dom } f$.

The function $f(x) = 2x + 1$ is one-to-one because

$$\begin{aligned} f(a) = f(b) &\Rightarrow 2a + 1 = 2b + 1 \\ &\Rightarrow 2a = 2b \\ &\Rightarrow a = b \end{aligned}$$

The function $f(x) = x^2$ is not one-to-one as, for example, $f(3) = 9 = f(-3)$.

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Example 10

Which of the following functions are one-to-one?

a $f = \{(2, -3), (4, 7), (6, 6), (8, 10)\}$

b $g = \{(1, 4), (2, 5), (3, 4), (4, 7)\}$

Solution

a The function f is one-to-one as the second coordinates of all of the ordered pairs are different.

b The function g is not one-to-one as the second coordinates of the ordered pairs are not all different: $g(1) = 4 = g(3)$.

The vertical-line test can be used to determine whether a relation is a function or not. Similarly, there is a geometric test that determines whether a function is one-to-one or not.

Horizontal-line test

If a horizontal line can be drawn anywhere on the graph of a function and it only ever intersects the graph a maximum of once, then the function is **one-to-one**.

Example 11

Which of the following functions are one-to-one?

a $y = x^2$

b $y = 2x + 1$

c $f(x) = 5$

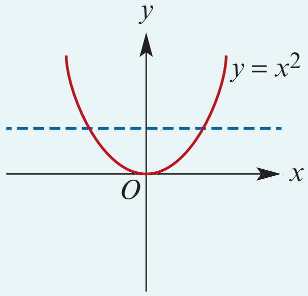
d $y = x^3$

e $y = \sqrt{9 - x^2}$

f $y = \dfrac{1}{x}$

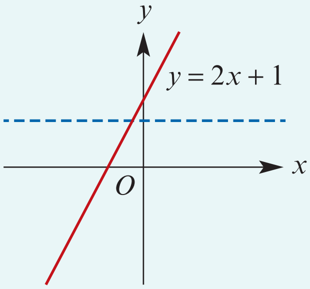
Solution

a



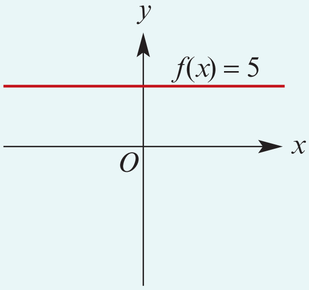
not one-to-one

b



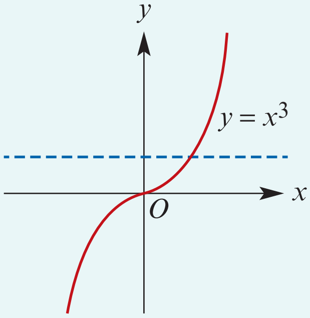
one-to-one

c



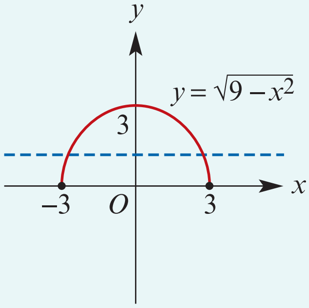
not one-to-one

d



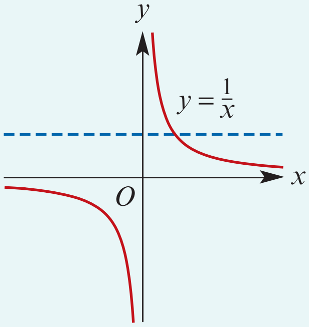
one-to-one

e



not one-to-one

f



one-to-one

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A function that is not one-to-one is **many-to-one**.

Implied domains

If the domain of a function is not specified, then the domain is the largest subset of $\mathbb{R}$ for which the rule is defined; this is called the **implied domain** or the **maximal domain**.

Thus, for the function $f(x) = \sqrt{x}$, the implied domain is $[0, \infty)$. We write:

$$f\colon [0, \infty) \to \mathbb{R},\ f(x) = \sqrt{x}$$

Example 12

Find the implied domain and the corresponding range for the functions with rules:

a $f(x) = 2x - 3$ **b** $f(x) = \dfrac{1}{(x-2)^2}$ **c** $f(x) = \sqrt{x+6}$ **d** $f(x) = \sqrt{4 - x^2}$

Solution

a $f(x) = 2x - 3$ is defined for all x. The implied domain is $\mathbb{R}$. The range is $\mathbb{R}$.

b $f(x) = \dfrac{1}{(x-2)^2}$ is defined for $x \neq 2$. The implied domain is $\mathbb{R} \setminus \{2\}$. The range is $\mathbb{R}^+$.

c $f(x) = \sqrt{x+6}$ is defined for $x + 6 \geq 0$, i.e. for $x \geq -6$.
Thus the implied domain is $[-6, \infty)$. The range is $\mathbb{R}^+ \cup \{0\}$.

d $f(x) = \sqrt{4 - x^2}$ is defined for $4 - x^2 \geq 0$, i.e. for $x^2 \leq 4$.
Thus the implied domain is $[-2, 2]$. The range is $[0, 2]$.

Example 13

Find the implied domain of the functions with the following rules:

a $f(x) = \dfrac{2}{2x-3}$ **b** $g(x) = \sqrt{5 - x}$

c $h(x) = \sqrt{x-5} + \sqrt{8-x}$ **d** $f(x) = \sqrt{x^2 - 7x + 12}$

Solution

a $f(x)$ is defined when $2x - 3 \neq 0$, i.e. when $x \neq \frac{3}{2}$. Thus the implied domain is $\mathbb{R} \setminus \{\frac{3}{2}\}$.

b $g(x)$ is defined when $5 - x \geq 0$, i.e. when $x \leq 5$. Thus the implied domain is $(-\infty, 5]$.

c $h(x)$ is defined when $x - 5 \geq 0$ and $8 - x \geq 0$, i.e. when $x \geq 5$ and $x \leq 8$. Thus the implied domain is $[5, 8]$.

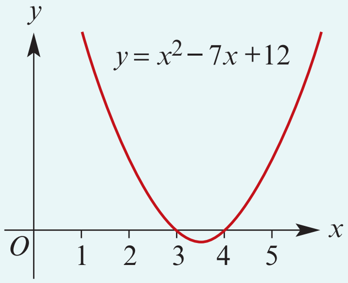
d $f(x)$ is defined when

$$x^2 - 7x + 12 \geq 0$$

which is equivalent to

$$(x - 3)(x - 4) \geq 0$$

Thus $f(x)$ is defined when $x \geq 4$ or $x \leq 3$.
The implied domain is $(-\infty, 3] \cup [4, \infty)$.



Piecewise-defined functions

Functions which have different rules for different subsets of their domain are called **piecewise-defined functions**. They are also known as **hybrid functions**.

Example 14

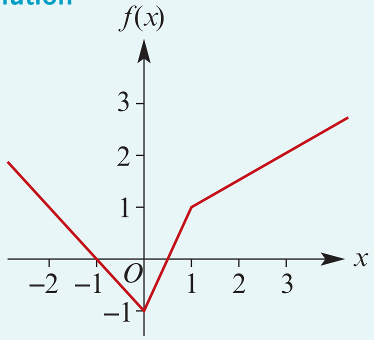
a Sketch the graph of the function f given by:

$$f(x) = \begin{cases} -x-1 & \text{for } x < 0 \\ 2x-1 & \text{for } 0 \le x \le 1 \\ \frac{1}{2}x + \frac{1}{2} & \text{for } x > 1 \end{cases}$$

b State the range of f.

Solution

a



b The range is $[-1, \infty)$.

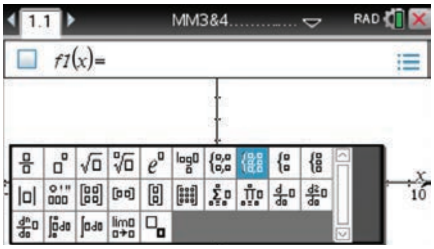
Explanation

- The graph of $y = -x - 1$ is sketched for $x < 0$. Note that when $x = 0$, $y = -1$ for this rule.
- The graph of $y = 2x - 1$ is sketched for $0 \le x \le 1$. Note that when $x = 0$, $y = -1$ and when $x = 1$, $y = 1$ for this rule.
- The graph of $y = \frac{1}{2}x + \frac{1}{2}$ is sketched for $x > 1$. Note that when $x = 1$, $y = 1$ for this rule.

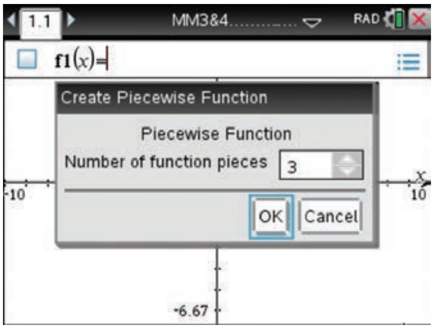
Note: For this function, the sections of the graph 'join up'. This is not always the case.

Using the TI-Nspire

- In a **Graphs** application with the cursor in the entry line, select the piecewise function template as shown. (Access the templates using ⊞ or ctrl menu > **Math Templates**.)
- If the domain of the last function piece is the remaining subset of ℝ, then leave the final condition blank and it will autofill as 'Else' when you press enter.



$f1(x)=$

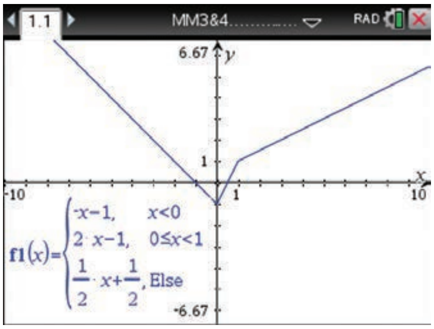


$f1(x)=$

Create Piecewise Function

Piecewise Function

Number of function pieces: 3



$$f1(x)=\begin{cases} -x-1, & x<0 \\ 2\cdot x-1, & 0\le x<1 \\ \frac{1}{2}\cdot x+\frac{1}{2}, & \text{Else} \end{cases}$$

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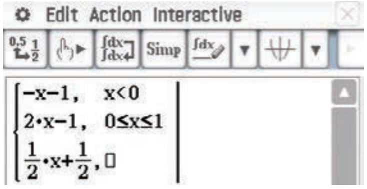
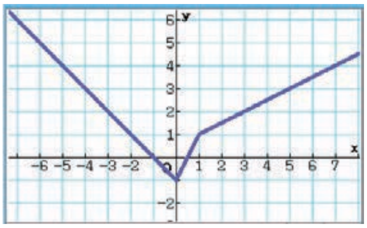
Using the Casio ClassPad

- In $\overset{\text{Main}}{\sqrt{\alpha}}$, open the keyboard and select the Math3 palette.
- Tap the piecewise template $\{\square$ twice.
- Enter the function as shown.

Note: If the domain of the last function piece is the remaining subset of ℝ, then the last domain box can be left empty.



$$\begin{cases} -x-1, & x<0 \\ 2\cdot x-1, & 0\le x\le 1 \\ \frac{1}{2}\cdot x+\frac{1}{2}, & \square \end{cases}$$



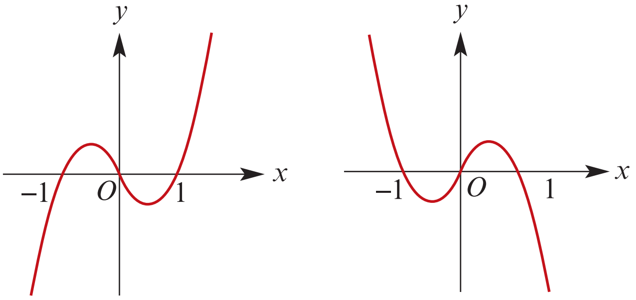
Odd and even functions

Odd functions

An **odd** function has the property that $f(-x) = -f(x)$. The graph of an odd function has rotational symmetry with respect to the origin: the graph remains unchanged after rotation of 180° about the origin.

For example, $f(x) = x^3 - x$ is an odd function, since

$$\begin{aligned} f(-x) &= (-x)^3 - (-x) \\ &= -x^3 + x \\ &= -f(x) \end{aligned}$$



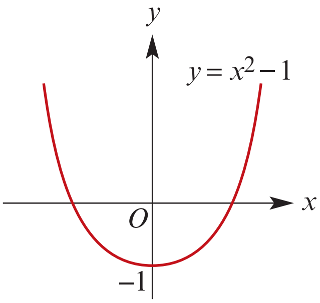
$y = f(x)$ $y = f(-x)$

Even functions

An **even** function has the property that $f(-x) = f(x)$. The graph of an even function is symmetrical about the y-axis.

For example, $f(x) = x^2 - 1$ is an even function, since

$$\begin{aligned} f(-x) &= (-x)^2 - 1 \\ &= x^2 - 1 \\ &= f(x) \end{aligned}$$



The properties of odd and even functions often facilitate the sketching of graphs.

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Example 15

State whether each function is odd or even or neither:

a $f(x) = x^2 + 7$

b $f(x) = x^4 + x^2$

c $f(x) = -2x^3 + 7$

d $f(x) = \dfrac{1}{x}$

e $f(x) = \dfrac{1}{x-3}$

f $f(x) = x^5 + x^3 + x$

Solution

a
$$\begin{aligned} f(-a) &= (-a)^2 + 7 \\ &= a^2 + 7 \\ &= f(a) \end{aligned}$$

The function is even.

b
$$\begin{aligned} f(-a) &= (-a)^4 + (-a)^2 \\ &= a^4 + a^2 \\ &= f(a) \end{aligned}$$

The function is even.

c $f(-1) = -2(-1)^3 + 7 = 9$
but $f(1) = -2 + 7 = 5$
and $-f(1) = -5$

The function is neither even nor odd.

d
$$\begin{aligned} f(-a) &= \frac{1}{-a} \\ &= -\frac{1}{a} \\ &= -f(a) \end{aligned}$$

The function is odd.

e $f(-1) = -\frac{1}{4}$
but $f(1) = -\frac{1}{2}$
and $-f(1) = \frac{1}{2}$

The function is neither even nor odd.

f
$$\begin{aligned} &f(-a) \\ &= (-a)^5 + (-a)^3 + (-a) \\ &= -a^5 - a^3 - a \\ &= -f(a) \end{aligned}$$

The function is odd.

Section summary

- A function f is **one-to-one** if different x-values map to different y-values. Equivalently, a function f is one-to-one if $f(a) = f(b)$ implies $a = b$, for all $a, b \in \operatorname{dom} f$.
- **Horizontal-line test**: If a horizontal line can be drawn anywhere on the graph of a function and it only ever intersects the graph a maximum of once, then the function is one-to-one.
- When the domain of a function is not explicitly stated, it is assumed to consist of all real numbers for which the rule has meaning; this is called the **implied domain** or the **maximal domain** of the function.
- Functions which have different rules for different subsets of their domain are called **piecewise-defined functions**.
- A function f is **odd** if $f(-x) = -f(x)$ for all x in the domain of f.
- A function f is **even** if $f(-x) = f(x)$ for all x in the domain of f.

Exercise 1C

Skillsheet

Example 10

1 State which of the following functions are one-to-one:

a $\{(2,3), (3,4), (5,4), (4,6)\}$

b $\{(1,2), (2,3), (3,4), (4,6)\}$

c $\{(7,-3), (11,5), (6,4), (17,-6), (12,-4)\}$

d $\{(-1,-2), (-2,-2), (-3,4), (-6,7)\}$

Example 11

2 State which of the following functions are one-to-one:

a $\{(x, y) : y = x^2 + 2\}$

b $\{(x, y) : y = 2x + 4\}$

c $f(x) = 2 - x^2$

d $y = x^2,\ x \geq 1$

e $y = \dfrac{1}{x^2},\ x \neq 0$

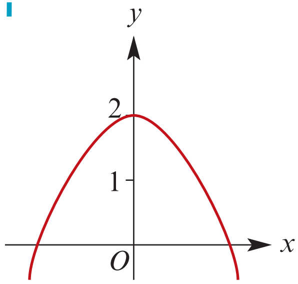
f $y = (x - 1)^3$

3 Each of the following is the graph of a relation.

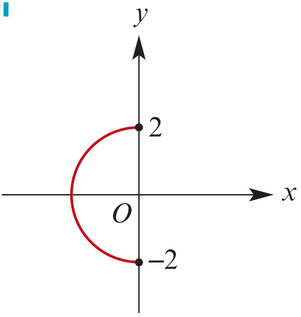
a State which are the graph of a function.

b State which are the graph of a one-to-one function.

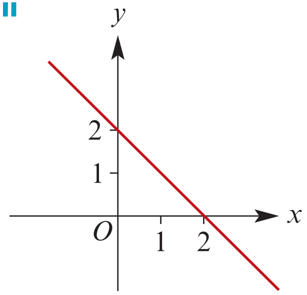
i



ii



iii



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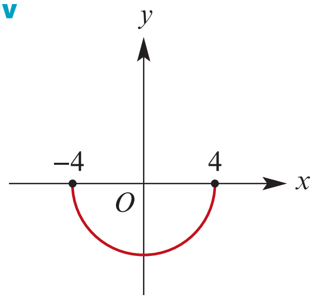
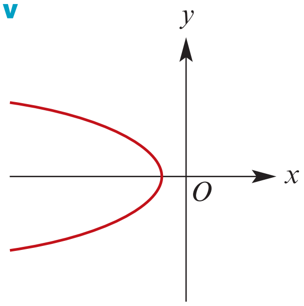


v



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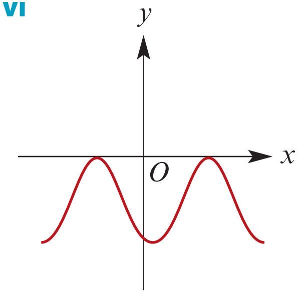

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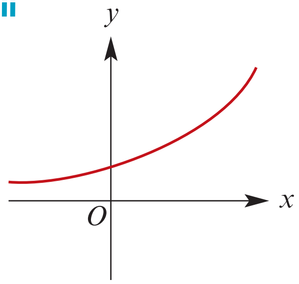

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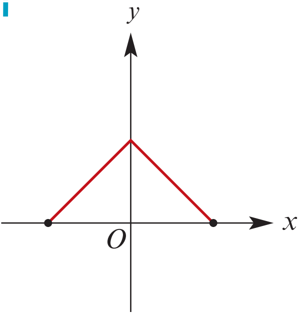
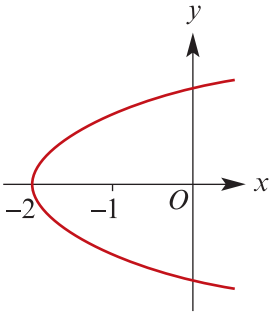


4 The graph of the relation $\{(x, y) : y^2 = x + 2,\ x \geq -2\}$ is shown on the right. From this relation, form two functions and specify the range of each.



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5 **a** Draw the graph of $g\colon \mathbb{R} \to \mathbb{R}$, $g(x) = x^2 + 2$.

b By restricting the domain of g, form two one-to-one functions that have the same rule as g.

Example 12 **6** State the largest possible domain and range for the functions defined by each of the following rules:

a $y = 4 - x$ **b** $y = \sqrt{x}$ **c** $y = x^2 - 2$ **d** $y = \sqrt{16 - x^2}$

e $y = \dfrac{1}{x}$ **f** $y = 4 - 3x^2$ **g** $y = \sqrt{x - 3}$

7 Each of the following is the rule of a function. In each case, write down the implied domain and the range.

a $y = 3x + 2$ **b** $y = x^2 - 2$

c $f(x) = \sqrt{9 - x^2}$ **d** $g(x) = \dfrac{1}{x - 1}$

Example 13 **8** Find the implied domain for each of the following rules:

a $f(x) = \dfrac{1}{x - 3}$ **b** $f(x) = \sqrt{x^2 - 3}$

c $g(x) = \sqrt{x^2 + 3}$ **d** $h(x) = \sqrt{x - 4} + \sqrt{11 - x}$

e $f(x) = \dfrac{x^2 - 1}{x + 1}$ **f** $h(x) = \sqrt{x^2 - x - 2}$

g $f(x) = \dfrac{1}{(x + 1)(x - 2)}$ **h** $h(x) = \sqrt{\dfrac{x - 1}{x + 2}}$

i $f(x) = \sqrt{x - 3x^2}$ **j** $h(x) = \sqrt{25 - x^2}$

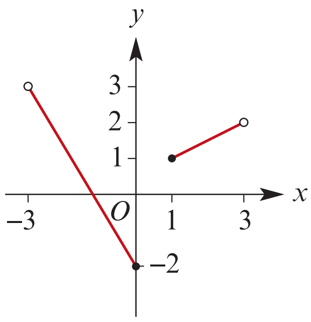
k $f(x) = \sqrt{x - 3} + \sqrt{12 - x}$

Example 14 **9** **a** Sketch the graph of the function

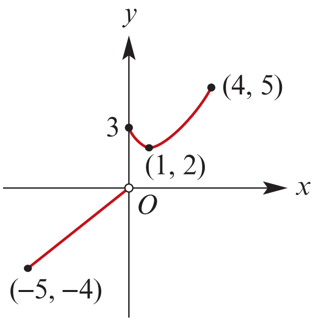
$$f(x) = \begin{cases} -2x - 2, & x < 0 \\ x - 2, & 0 \le x < 2 \\ 3x - 6, & x \ge 2 \end{cases}$$

b What is the range of f?

10 State the domain and range of the function for which the graph is shown.



11 State the domain and range of the function for which the graph is shown.



12 **a** Sketch the graph of the function with rule

$$f(x) = \begin{cases} 2x + 6, & 0 < x \le 2 \\ -x + 5, & -4 \le x \le 0 \\ -4, & x < -4 \end{cases}$$

b State the domain and range of the function.

13 **a** Sketch the graph of the function with rule

$$g(x) = \begin{cases} x^2 + 5, & x > 0 \\ 5 - x, & -3 \le x \le 0 \\ 8, & x < -3 \end{cases}$$

b State the range of the function.

14 Given that

$$f(x) = \begin{cases} \dfrac{1}{x}, & x > 3 \\ 2x, & x \le 3 \end{cases}$$

find:

a $f(-4)$ **b** $f(0)$ **c** $f(4)$

d $f(a + 3)$ in terms of a **e** $f(2a)$ in terms of a **f** $f(a - 3)$ in terms of a

15 Given that

$$f(x) = \begin{cases} \sqrt{x - 1}, & x \ge 1 \\ 4, & x < 1 \end{cases}$$

find:

a $f(0)$ **b** $f(3)$ **c** $f(8)$

d $f(a + 1)$ in terms of a **e** $f(a - 1)$ in terms of a

16 Sketch the graph of the function

$$g(x) = \begin{cases} -x - 2, & x < -1 \\ \dfrac{x - 1}{2}, & -1 \le x < 1 \\ 3x - 3, & x \ge 1 \end{cases}$$

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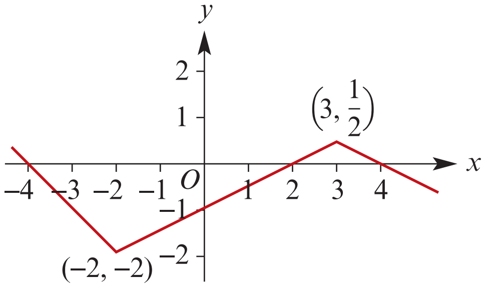
17 Specify the function illustrated by the graph.



Example 15 **18** State whether each of the following functions is odd, even or neither:

a $f(x) = x^4$ **b** $f(x) = x^5$ **c** $f(x) = x^4 - 3x$

d $f(x) = x^4 - 3x^2$ **e** $f(x) = x^5 - 2x^3$ **f** $f(x) = x^4 - 2x^5$

19 State whether each of the following functions is odd, even or neither:

a $f(x) = x^2 - 4$ **b** $f(x) = 2x^4 - x^2$ **c** $f(x) = -4x^3 + 7x$

d $f(x) = \dfrac{1}{2x}$ **e** $f(x) = \dfrac{1}{x+5}$ **f** $f(x) = 3 + 2x^2$

g $f(x) = x^2 - 5x$ **h** $f(x) = 3^x$ **i** $f(x) = x^4 + x^2 + 2$

1D Sums and products of functions

The domain of f is denoted by $\text{dom}\, f$ and the domain of g by $\text{dom}\, g$. Let f and g be functions such that $\text{dom}\, f \cap \text{dom}\, g \neq \varnothing$. The **sum**, $f + g$, and the **product**, fg, as functions on $\text{dom}\, f \cap \text{dom}\, g$ are defined by

$$(f + g)(x) = f(x) + g(x) \qquad \text{and} \qquad (fg)(x) = f(x)\,g(x)$$

The domain of both $f + g$ and fg is the intersection of the domains of f and g, i.e. the values of x for which both f and g are defined.

Example 16

If $f(x) = \sqrt{x-2}$ for all $x \geq 2$ and $g(x) = \sqrt{4-x}$ for all $x \leq 4$, find:

a $f + g$ **b** $(f + g)(3)$ **c** fg **d** $(fg)(3)$

Solution

Note that $\text{dom}\, f \cap \text{dom}\, g = [2, 4]$.

a $(f + g)(x) = f(x) + g(x)$

$= \sqrt{x-2} + \sqrt{4-x}$

$\text{dom}(f + g) = [2, 4]$

b $(f + g)(3) = \sqrt{3-2} + \sqrt{4-3}$

$= 2$

c $(fg)(x) = f(x)\,g(x)$

$= \sqrt{(x-2)(4-x)}$

$\text{dom}(fg) = [2, 4]$

d $(fg)(3) = \sqrt{(3-2)(4-3)}$

$= 1$

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Addition of ordinates

We have seen that, for two functions f and g, a new function $f + g$ can be defined by

$$(f+g)(x) = f(x) + g(x)$$
$$\operatorname{dom}(f+g) = \operatorname{dom} f \cap \operatorname{dom} g$$

We now look at how to graph the new function $f + g$. This is a useful graphing technique and can be combined with other techniques such as finding axis intercepts, stationary points and asymptotes.

Example 17

Sketch the graphs of $f(x) = x + 1$ and $g(x) = 3 - 2x$ and hence the graph of $(f + g)(x)$.

Solution

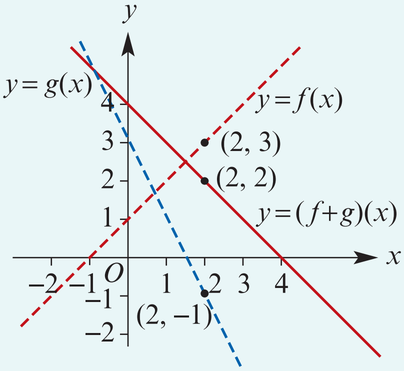
For $f(x) = x + 1$ and $g(x) = 3 - 2x$, we have

$$\begin{aligned}(f+g)(x) &= f(x) + g(x)\\ &= (x+1) + (3-2x)\\ &= 4 - x\end{aligned}$$

For example:

$$\begin{aligned}(f+g)(2) &= f(2) + g(2)\\ &= 3 + (-1) = 2\end{aligned}$$

i.e. the ordinates are added.



Now check that the same principle applies for other points on the graphs. A table of values can be a useful aid to find points that lie on the graph of $y = (f + g)(x)$.

x	$f(x)$	$g(x)$	$(f+g)(x)$
-1	0	5	5
0	1	3	4
$\frac{3}{2}$	$\frac{5}{2}$	0	$\frac{5}{2}$
2	3	-1	2

The table shows that $(-1, 5)$, $(0, 4)$, $(\frac{3}{2}, \frac{5}{2})$ and $(2, 2)$ lie on the graph of $y = (f + g)(x)$.

Example 18

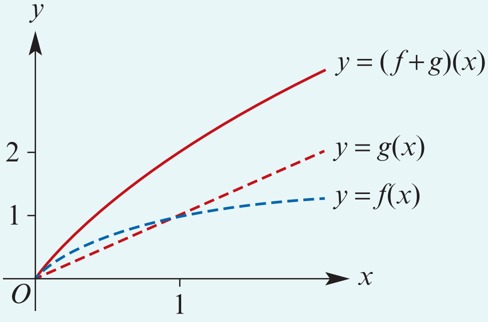
Sketch the graph of $y = (f + g)(x)$, where $f(x) = \sqrt{x}$ and $g(x) = x$.

Solution

The function with rule

$$(f+g)(x) = \sqrt{x} + x$$

is defined by the addition of the two functions f and g.



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Example 19

Sketch the graph of $y = (f - g)(x)$, where $f(x) = x^2$ and $g(x) = \sqrt{x}$.

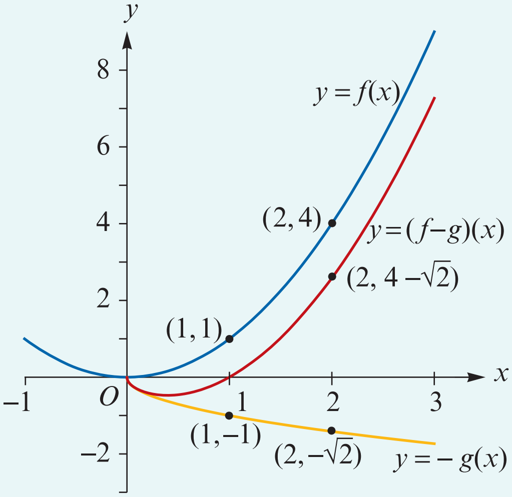
Solution

The function with rule

$$(f - g)(x) = x^2 - \sqrt{x}$$

is defined by the addition of the two functions f and $-g$.

The implied domain of $f - g$ is $[0, \infty)$.



Section summary

- **Sum of functions** $(f + g)(x) = f(x) + g(x)$, where $\text{dom}(f + g) = \text{dom}\, f \cap \text{dom}\, g$
- **Difference of functions** $(f - g)(x) = f(x) - g(x)$, where $\text{dom}(f - g) = \text{dom}\, f \cap \text{dom}\, g$
- **Product of functions** $(f \cdot g)(x) = f(x) \cdot g(x)$, where $\text{dom}(f \cdot g) = \text{dom}\, f \cap \text{dom}\, g$
- **Addition of ordinates** This technique can be used to help sketch the graph of the sum of two functions. Key points to consider when sketching $y = (f + g)(x)$:
  - When $f(x) = 0$, $(f + g)(x) = g(x)$.
  - When $g(x) = 0$, $(f + g)(x) = f(x)$.
  - If $f(x)$ and $g(x)$ are positive, then $(f + g)(x) > g(x)$ and $(f + g)(x) > f(x)$.
  - If $f(x)$ and $g(x)$ are negative, then $(f + g)(x) < g(x)$ and $(f + g)(x) < f(x)$.
  - If $f(x)$ is positive and $g(x)$ is negative, then $g(x) < (f + g)(x) < f(x)$.
  - Look for values of x for which $f(x) + g(x) = 0$.

Exercise 1D

Example 16

1 For each of the following, find $(f + g)(x)$ and $(fg)(x)$ and state the domain for both $f + g$ and fg:

a $f(x) = 3x$ and $g(x) = x + 2$

b $f(x) = 1 - x^2$ for all $x \in [-2, 2]$ and $g(x) = x^2$ for all $x \in \mathbb{R}^+$

c $f(x) = \sqrt{x}$ and $g(x) = \dfrac{1}{\sqrt{x}}$ for $x \in [1, \infty)$

d $f(x) = x^2$, $x \ge 0$ and $g(x) = \sqrt{4 - x}$, $0 \le x \le 4$

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2 Functions f, g, h and k are defined by:

i $f(x) = x^2 + 1,\ x \in \mathbb{R}$ **ii** $g(x) = x,\ x \in \mathbb{R}$

iii $h(x) = \dfrac{1}{x^2},\ x \neq 0$ **iv** $k(x) = \dfrac{1}{x},\ x \neq 0$

a State which of the above functions are odd and which are even.

b Give the rules for the functions $f + h$, fh, $g + k$, gk, $f + g$ and fg, stating which are odd and which are even.

Example 17 **3** Sketch the graphs of $f(x) = x + 2$ and $g(x) = 4 - 3x$ and hence the graph of $(f + g)(x)$.

Example 18 **4** Sketch the graph of $f\colon \mathbb{R}^+ \cup \{0\} \to \mathbb{R},\ f(x) = \sqrt{x} + 2x$ using addition of ordinates.

5 Sketch the graph of $f\colon [-2, \infty) \to \mathbb{R},\ f(x) = \sqrt{x+2} + x$ using addition of ordinates.

6 Sketch the graph of $f\colon \mathbb{R}^+ \cup \{0\} \to \mathbb{R},\ f(x) = -\sqrt{x} + x$ using addition of ordinates.

7 Sketch the graph of $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R},\ f(x) = \dfrac{1}{x} + \dfrac{1}{x^2}$ using addition of ordinates.

8 For each of the following, sketch the graph of $f + g$:

a $f\colon [-2, \infty) \to \mathbb{R},\ f(x) = \sqrt{2 + x}$ and $g\colon \mathbb{R} \to \mathbb{R},\ g(x) = -2x$

b $f\colon (-\infty, 2] \to \mathbb{R},\ f(x) = \sqrt{2 - x}$ and $g\colon [-2, \infty) \to \mathbb{R},\ g(x) = \sqrt{x + 2}$

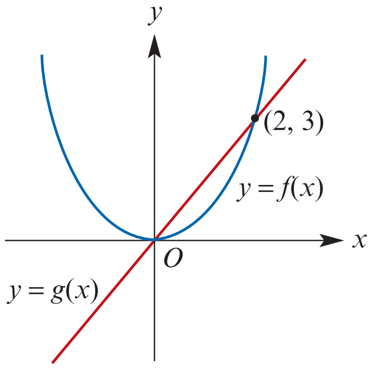
Example 19 **9** Sketch the graph of $y = (f - g)(x)$, where $f(x) = x^3$ and $g(x) = \sqrt{x}$.

10 Sketch the graph of $y = (f - g)(x)$, where $f(x) = 2x^2$ and $g(x) = 3\sqrt{x}$.

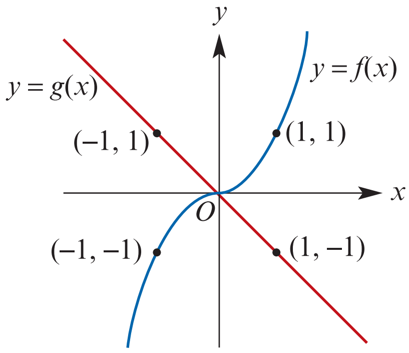
11 Sketch the graph of $f(x) = x^2$ and $g(x) = 3x + 2$ on the one set of axes and hence, using addition of ordinates, sketch the graph of $y = x^2 + 3x + 2$.

12 Copy and add the graph of $y = (f + g)(x)$ using addition of ordinates:

a



b



13 For each of the following, sketch the graph of $f + g$:

a $f\colon \mathbb{R} \to \mathbb{R},\ f(x) = x^2$ and $g\colon \mathbb{R} \to \mathbb{R},\ g(x) = 3$

b $f\colon \mathbb{R} \to \mathbb{R},\ f(x) = x^2 + 2x$ and $g\colon \mathbb{R}^+ \cup \{0\} \to \mathbb{R},\ g(x) = \sqrt{x}$

c $f\colon \mathbb{R} \to \mathbb{R},\ f(x) = -x^2$ and $g\colon \mathbb{R}^+ \cup \{0\} \to \mathbb{R},\ g(x) = \sqrt{x}$

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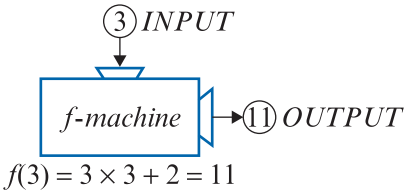
1E Composite functions

A function may be considered to be similar to a machine for which the input (domain) is processed to produce an output (range). For example, the diagram on the right represents an 'f-machine' where $f(x) = 3x + 2$.


3 INPUT
f-machine
11 OUTPUT
f(3) = 3 × 3 + 2 = 11

With many processes, more than one machine operation is required to produce an output.

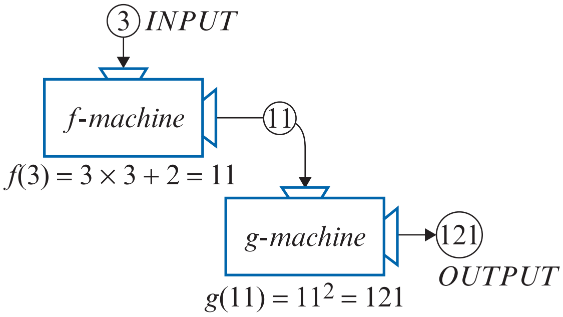
Suppose an output is the result of one function being applied after another.

For example: $f(x) = 3x + 2$

followed by $g(x) = x^2$

This is illustrated on the right.

A new function h is formed. The rule for h is $h(x) = (3x + 2)^2$.


3 INPUT
f-machine
11
f(3) = 3 × 3 + 2 = 11
g-machine
121
OUTPUT
g(11) = 11² = 121

The diagram shows $f(3) = 11$ and then $g(11) = 121$. This may be written:

$$h(3) = g(f(3)) = g(11) = 121$$

The new function h is said to be the **composition** of g with f. This is written $h = g \circ f$ (read 'composition of f followed by g') and the rule for h is given by $h(x) = g(f(x))$.

In the example we have considered:

$$\begin{aligned} h(x) &= g(f(x)) \\ &= g(3x + 2) \\ &= (3x + 2)^2 \end{aligned}$$

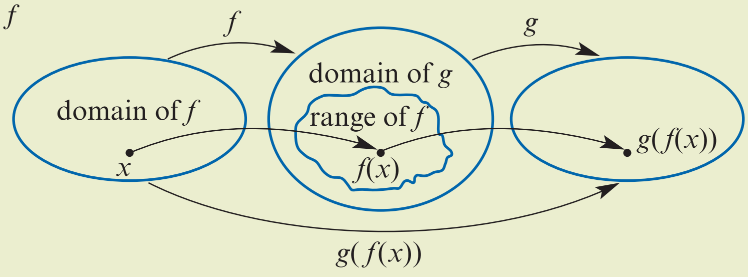
In general, for functions f and g such that

$$\operatorname{ran} f \subseteq \operatorname{dom} g$$

we define the **composite function** of g with f by

$$g \circ f(x) = g(f(x))$$

$$\operatorname{dom}(g \circ f) = \operatorname{dom} f$$


f
g
domain of f
x
domain of g
range of f
f(x)
g(f(x))
g(f(x))

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Example 20

Find both $f \circ g$ and $g \circ f$, stating the domain and range of each, where:

$$f\colon \mathbb{R} \to \mathbb{R},\ f(x) = 2x - 1 \quad \text{and} \quad g\colon \mathbb{R} \to \mathbb{R},\ g(x) = 3x^2$$

Solution

To determine the existence of a composite function, it is useful to form a table of domains and ranges.

We see that $f \circ g$ is defined since $\operatorname{ran} g \subseteq \operatorname{dom} f$, and that $g \circ f$ is defined since $\operatorname{ran} f \subseteq \operatorname{dom} g$.

	Domain	Range
g	$\mathbb{R}$	$\mathbb{R}^+ \cup \{0\}$
f	$\mathbb{R}$	$\mathbb{R}$

$$\begin{aligned} f \circ g(x) &= f(g(x)) \\ &= f(3x^2) \\ &= 2(3x^2) - 1 \\ &= 6x^2 - 1 \end{aligned}$$

$$\begin{aligned} g \circ f(x) &= g(f(x)) \\ &= g(2x - 1) \\ &= 3(2x - 1)^2 \\ &= 12x^2 - 12x + 3 \end{aligned}$$

$$\operatorname{dom}(f \circ g) = \operatorname{dom} g = \mathbb{R}$$

$$\operatorname{ran}(f \circ g) = [-1, \infty)$$

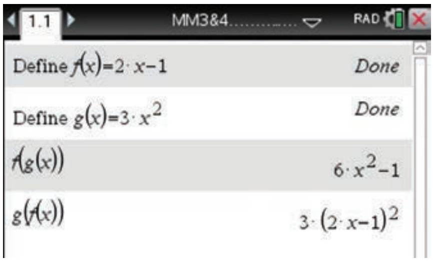
$$\operatorname{dom}(g \circ f) = \operatorname{dom} f = \mathbb{R}$$

$$\operatorname{ran}(g \circ f) = [0, \infty)$$

Note: It can be seen from this example that in general $f \circ g \neq g \circ f$.

Using the TI-Nspire

- Define $f(x) = 2x - 1$ and $g(x) = 3x^2$.
- The rules for $f \circ g$ and $g \circ f$ can now be found using $f(g(x))$ and $g(f(x))$.



Define $f(x)=2\cdot x-1$ Done

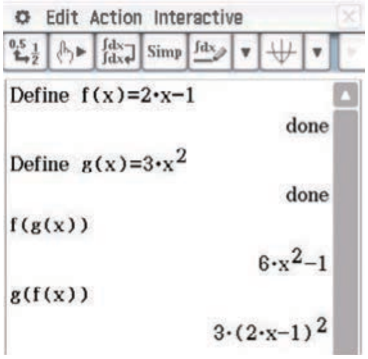
Define $g(x)=3\cdot x^2$ Done

$f(g(x))$ $6\cdot x^2-1$

$g(f(x))$ $3\cdot(2\cdot x-1)^2$

Using the Casio ClassPad

- Define $f(x) = 2x - 1$ and $g(x) = 3x^2$.
- The rules for $f \circ g$ and $g \circ f$ can now be found using $f(g(x))$ and $g(f(x))$.



Define $f(x)=2\cdot x-1$
done

Define $g(x)=3\cdot x^2$
done

$f(g(x))$
$6\cdot x^2-1$

$g(f(x))$
$3\cdot(2\cdot x-1)^2$

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Example 21

For the functions $g(x) = 2x - 1$, $x \in \mathbb{R}$, and $f(x) = \sqrt{x}$, $x \geq 0$:

a State which of $f \circ g$ and $g \circ f$ is defined.

b For the composite function that is defined, state the domain and rule.

Solution

a Range of $f \subseteq$ domain of g

Range of $g \nsubseteq$ domain of f

Thus $g \circ f$ is defined, but $f \circ g$ is not defined.

	Domain	Range
g	$\mathbb{R}$	$\mathbb{R}$
f	$\mathbb{R}^+ \cup \{0\}$	$\mathbb{R}^+ \cup \{0\}$

b
$$\begin{aligned} g \circ f(x) &= g(f(x)) \\ &= g(\sqrt{x}) \\ &= 2\sqrt{x} - 1 \end{aligned}$$

$$\text{dom}(g \circ f) = \text{dom } f = \mathbb{R}^+ \cup \{0\}$$

Example 22

For the functions $f(x) = x^2 - 1$, $x \in \mathbb{R}$, and $g(x) = \sqrt{x}$, $x \geq 0$:

a State why $g \circ f$ is not defined.

b Define a restriction f^* of f such that $g \circ f^*$ is defined, and find $g \circ f^*$.

Solution

a Range of $f \nsubseteq$ domain of g

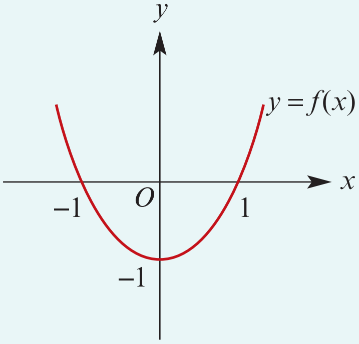
Thus $g \circ f$ is not defined.

	Domain	Range
f	$\mathbb{R}$	$[-1, \infty)$
g	$\mathbb{R}^+ \cup \{0\}$	$\mathbb{R}^+ \cup \{0\}$

b For $g \circ f^*$ to be defined, we need range of $f^* \subseteq$ domain of g, i.e. range of $f^* \subseteq \mathbb{R}^+ \cup \{0\}$.

For the range of f^* to be a subset of $\mathbb{R}^+ \cup \{0\}$, the domain of f must be restricted to a subset of

$$\{\, x : x \leq -1 \,\} \cup \{\, x : x \geq 1 \,\} = \mathbb{R} \setminus (-1, 1)$$



So we define f^* by

$$f^* : \mathbb{R} \setminus (-1, 1) \to \mathbb{R}, \; f^*(x) = x^2 - 1$$

Then
$$\begin{aligned} g \circ f^*(x) &= g(f^*(x)) \\ &= g(x^2 - 1) \\ &= \sqrt{x^2 - 1} \end{aligned}$$

$$\text{dom}(g \circ f^*) = \text{dom } f^* = \mathbb{R} \setminus (-1, 1)$$

The composite function is $g \circ f^* : \mathbb{R} \setminus (-1, 1) \to \mathbb{R}$, $g \circ f^*(x) = \sqrt{x^2 - 1}$

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Section summary

- If range of $f \subseteq$ domain of g, the composition $g \circ f$ is defined and

$$g \circ f(x) = g(f(x)) \quad \text{with } \operatorname{dom}(g \circ f) = \operatorname{dom} f$$

- If range of $g \subseteq$ domain of f, the composition $f \circ g$ is defined and

$$f \circ g(x) = f(g(x)) \quad \text{with } \operatorname{dom}(f \circ g) = \operatorname{dom} g$$

- In general, $f \circ g \neq g \circ f$.

Exercise 1E

Skillsheet

Example 20

1 For each of the following, find $f(g(x))$ and $g(f(x))$:

a $f(x) = 2x - 1,\ g(x) = 2x$ **b** $f(x) = 4x + 1,\ g(x) = 2x + 1$

c $f(x) = 2x - 1,\ g(x) = 2x - 3$ **d** $f(x) = 2x - 1,\ g(x) = x^2$

e $f(x) = 2x^2 + 1,\ g(x) = x - 5$ **f** $f(x) = 2x + 1,\ g(x) = x^2$

2 For the functions $f(x) = 2x - 1$ and $h(x) = 3x + 2$, find:

a $f \circ h(x)$ **b** $h(f(x))$ **c** $f \circ h(2)$ **d** $h \circ f(2)$

e $f(h(3))$ **f** $h(f(-1))$ **g** $f \circ h(0)$

3 For the functions $f(x) = x^2 + 2x$ and $h(x) = 3x + 1$, find:

a $f \circ h(x)$ **b** $h \circ f(x)$ **c** $f \circ h(3)$

d $h \circ f(3)$ **e** $f \circ h(0)$ **f** $h \circ f(0)$

4 For the functions $h\colon \mathbb{R} \setminus \{0\} \to \mathbb{R},\ h(x) = \dfrac{1}{x^2}$ and $g\colon \mathbb{R}^+ \to \mathbb{R},\ g(x) = 3x + 2$, find:

a $h \circ g$ (state rule and domain) **b** $g \circ h$ (state rule and domain)

c $h \circ g(1)$ **d** $g \circ h(1)$

Example 21

5 Consider the functions $f\colon \mathbb{R} \to \mathbb{R},\ f(x) = x^2 - 4$ and $g\colon \mathbb{R}^+ \cup \{0\} \to \mathbb{R},\ g(x) = \sqrt{x}$.

a State the ranges of f and g. **b** Find $f \circ g$, stating its range.

c Explain why $g \circ f$ does not exist.

6 Let f and g be functions given by

$$f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R},\ f(x) = \frac{1}{2}\left(\frac{1}{x} + 1\right) \qquad g\colon \mathbb{R} \setminus \{\tfrac{1}{2}\} \to \mathbb{R},\ g(x) = \frac{1}{2x - 1}$$

Find:

a $f \circ g$ **b** $g \circ f$

and state the range in each case.

7 The functions f and g are defined by $f\colon \mathbb{R} \to \mathbb{R},\ f(x) = x^2 - 2$ and $g\colon [0, \infty) \to \mathbb{R}$, $g(x) = \sqrt{x}$.

a Explain why $g \circ f$ does not exist. **b** Find $f \circ g$ and sketch its graph.

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Example 22 **8** $f\colon (-\infty, 3] \to \mathbb{R}$, $f(x) = 3 - x$ and $g\colon \mathbb{R} \to \mathbb{R}$, $g(x) = x^2 - 1$

a Show that $f \circ g$ is not defined.

b Define a restriction g^* of g such that $f \circ g^*$ is defined and find $f \circ g^*$.

9 $f\colon \mathbb{R}^+ \to \mathbb{R}$, $f(x) = x^{-\frac{1}{2}}$ and $g\colon \mathbb{R} \to \mathbb{R}$, $g(x) = 3 - x$

a Show that $f \circ g$ is not defined.

b By suitably restricting the domain of g, obtain a function g_1 such that $f \circ g_1$ is defined.

10 Let $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ and let $g\colon (-\infty, 3] \to \mathbb{R}$, $g(x) = \sqrt{3 - x}$. State with reasons whether:

a $f \circ g$ exists **b** $g \circ f$ exists.

11 Let $f\colon S \to \mathbb{R}$, $f(x) = \sqrt{4 - x^2}$, where S is the set of all real values of x for which $f(x)$ is defined. Let $g\colon \mathbb{R} \to \mathbb{R}$, where $g(x) = x^2 + 1$.

a Find S.

b Find the range of f and the range of g.

c State whether or not $f \circ g$ and $g \circ f$ are defined and give a reason for each assertion.

12 Let a be a positive number, let $f\colon [2, \infty) \to \mathbb{R}$, $f(x) = a - x$ and let $g\colon (-\infty, 1] \to \mathbb{R}$, $g(x) = x^2 + a$. Find all values of a for which both $f \circ g$ and $g \circ f$ exist.

1F Inverse functions

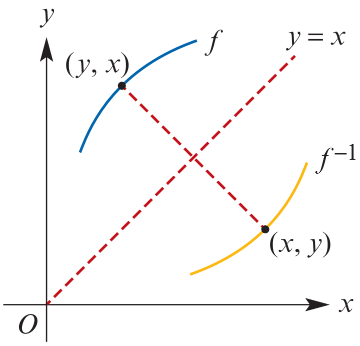
If f is a one-to-one function, then for each number y in the range of f there is exactly one number x in the domain of f such that $f(x) = y$.

Thus if f is a one-to-one function, a new function f^{-1}, called the **inverse** of f, may be defined by:

$$f^{-1}(x) = y \text{ if } f(y) = x, \quad \text{for } x \in \operatorname{ran} f \text{ and } y \in \operatorname{dom} f$$

Note: The function f^{-1} is also a one-to one function, and f is the inverse of f^{-1}.

It is not difficult to see what the relation between f and f^{-1} means geometrically. The point (x, y) is on the graph of f^{-1} if the point (y, x) is on the graph of f. Therefore to get the graph of f^{-1} from the graph of f, the graph of f is to be reflected in the line $y = x$.



From this the following is evident:

$$\operatorname{dom} f^{-1} = \operatorname{ran} f$$
$$\operatorname{ran} f^{-1} = \operatorname{dom} f$$

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A function has an inverse function if and only if it is one-to-one. Using the notation for composition we can write:

$$\begin{aligned} f \circ f^{-1}(x) &= x, \quad \text{for all } x \in \operatorname{dom} f^{-1} \\ f^{-1} \circ f(x) &= x, \quad \text{for all } x \in \operatorname{dom} f \end{aligned}$$

Example 23

Find the inverse function f^{-1} of the function $f(x) = 2x - 3$.

Solution

Method 1

The graph of f has equation $y = 2x - 3$ and the graph of f^{-1} has equation $x = 2y - 3$, that is, x and y are interchanged.

Solve for y:

$$\begin{aligned} x &= 2y - 3 \\ x + 3 &= 2y \\ \therefore \quad y &= \tfrac{1}{2}(x + 3) \end{aligned}$$

Thus $f^{-1}(x) = \frac{1}{2}(x + 3)$ and $\operatorname{dom} f^{-1} = \operatorname{ran} f = \mathbb{R}$.

Method 2

We require f^{-1} such that

$$\begin{aligned} f(f^{-1}(x)) &= x \\ 2f^{-1}(x) - 3 &= x \\ \therefore \quad f^{-1}(x) &= \tfrac{1}{2}(x + 3) \end{aligned}$$

Thus $f^{-1}(x) = \frac{1}{2}(x + 3)$ and $\operatorname{dom} f^{-1} = \operatorname{ran} f = \mathbb{R}$.

Example 24

Find the inverse of each of the following functions, stating the domain and range for each:

a $f\colon [-2, 1] \to \mathbb{R},\ f(x) = 2x + 3$

b $g(x) = \dfrac{1}{5 - x},\ x > 5$

c $h(x) = x^2 - 2,\ x \ge 1$

Solution

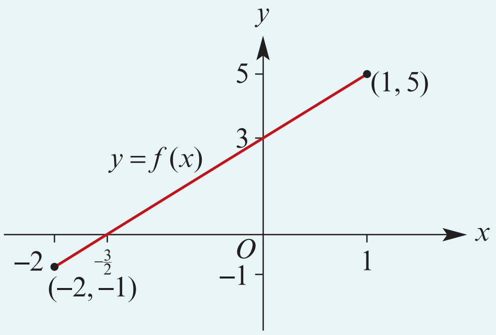
a $f\colon [-2, 1] \to \mathbb{R},\ f(x) = 2x + 3$

$\operatorname{ran} f^{-1} = \operatorname{dom} f = [-2, 1]$
$\operatorname{dom} f^{-1} = \operatorname{ran} f = [-1, 5]$

Let $y = 2x + 3$. Interchange x and y:

$$\begin{aligned} x &= 2y + 3 \\ x - 3 &= 2y \\ y &= \frac{x - 3}{2} \end{aligned}$$

$$\therefore f^{-1}\colon [-1, 5] \to \mathbb{R},\ f^{-1}(x) = \frac{x - 3}{2}$$



b $g(x) = \dfrac{1}{5-x},\ x > 5$

$\operatorname{ran} g^{-1} = \operatorname{dom} g = (5, \infty)$

$\operatorname{dom} g^{-1} = \operatorname{ran} g = (-\infty, 0)$

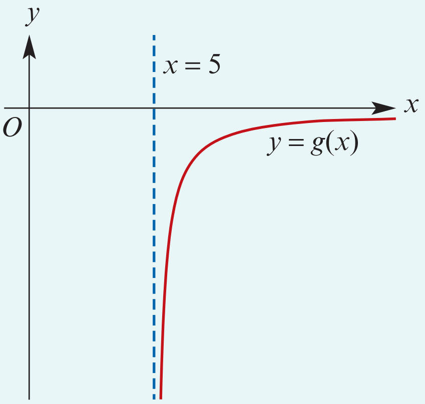
Let $y = \dfrac{1}{5-x}$. Interchange x and y:

$$x = \frac{1}{5-y}$$

$$5 - y = \frac{1}{x}$$

$$y = 5 - \frac{1}{x}$$

$\therefore g^{-1} \colon (-\infty, 0) \to \mathbb{R},\ g^{-1}(x) = 5 - \dfrac{1}{x}$



c $h(x) = x^2 - 2,\ x \geq 1$

$\operatorname{ran} h^{-1} = \operatorname{dom} h = [1, \infty)$

$\operatorname{dom} h^{-1} = \operatorname{ran} h = [-1, \infty)$

Let $y = x^2 - 2$. Interchange x and y:

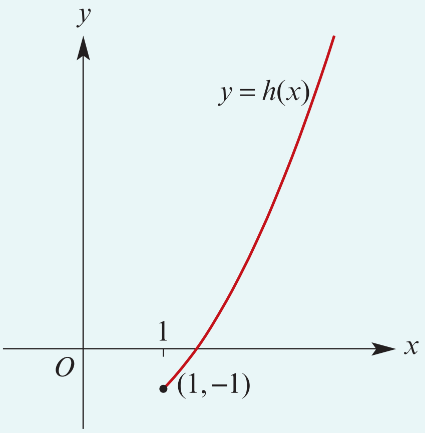
$$x = y^2 - 2$$

$$y^2 = x + 2$$

$$y = \pm\sqrt{x+2}$$

$\therefore h^{-1} \colon [-1, \infty) \to \mathbb{R},\ h^{-1}(x) = \sqrt{x+2}$

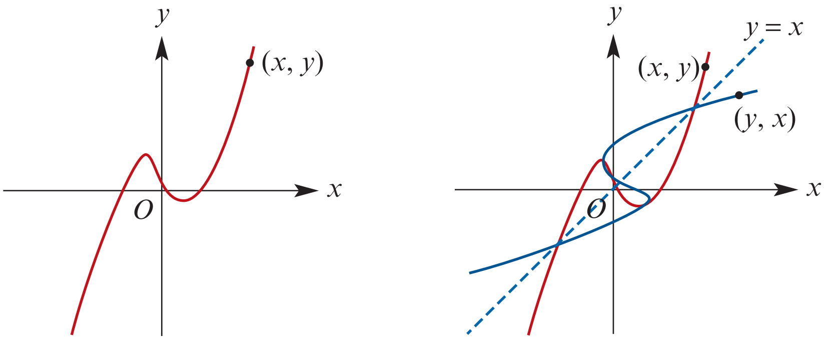
The positive square root is taken because of the known range.



Graphing inverse functions

The transformation which reflects each point in the plane in the line $y = x$ can be described as 'interchanging the x- and y-coordinates of each point in the plane' and can be written as $(x, y) \to (y, x)$. This is read as 'the ordered pair (x, y) is mapped to the ordered pair (y, x)'.

Reflecting the graph of a function in the line $y = x$ produces the graph of its **inverse relation**. Note that the image in the graph below is not a function.



If the function is one-to-one, then the image is the graph of a function. (This is because, if the function satisfies the horizontal-line test, then its reflection will satisfy the vertical-line test.)

Example 25

Find the inverse of the function $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = \frac{1}{x} + 3$ and sketch both functions on one set of axes, showing the points of intersection of the graphs.

Solution

We use method 2.

Let $x \in \operatorname{dom} f^{-1} = \operatorname{ran} f$. Then

$$\begin{aligned} f(f^{-1}(x)) &= x \\ \frac{1}{f^{-1}(x)} + 3 &= x \\ \frac{1}{f^{-1}(x)} &= x - 3 \\ \therefore \quad f^{-1}(x) &= \frac{1}{x-3} \end{aligned}$$

The inverse function is

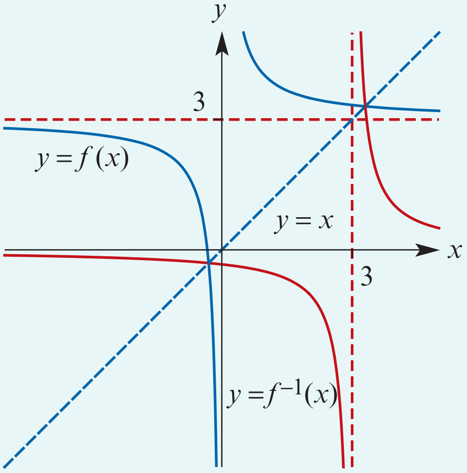
$$f^{-1}: \mathbb{R} \setminus \{3\} \to \mathbb{R}, \; f^{-1}(x) = \frac{1}{x-3}$$

The graphs of f and f^{-1} are shown opposite.

The two graphs intersect when

$$\begin{aligned} f(x) &= f^{-1}(x) \\ \frac{1}{x} + 3 &= \frac{1}{x-3} \\ 3x^2 - 9x - 3 &= 0 \\ x^2 - 3x - 1 &= 0 \end{aligned}$$

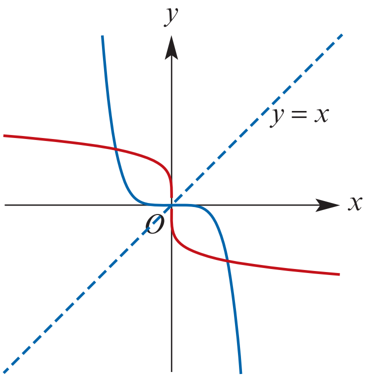
$$\therefore \quad x = \tfrac{1}{2}(3 - \sqrt{13}) \text{ or } x = \tfrac{1}{2}(3 + \sqrt{13})$$



The points of intersection are

$$\left(\tfrac{1}{2}(3 - \sqrt{13}), \tfrac{1}{2}(3 - \sqrt{13})\right) \quad \text{and} \quad \left(\tfrac{1}{2}(3 + \sqrt{13}), \tfrac{1}{2}(3 + \sqrt{13})\right)$$

Note: In this example, the points of intersection of the graphs of $y = f(x)$ and $y = f^{-1}(x)$ can also be found by solving either $f(x) = x$ or $f^{-1}(x) = x$, rather than the more complicated equation $f(x) = f^{-1}(x)$.
However, there can be points of intersection of the graphs of $y = f(x)$ and $y = f^{-1}(x)$ that *do not* lie on the line $y = x$, as shown in the diagram opposite.



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Example 26

Find the inverse of the function with rule $f(x) = 3\sqrt{x+2} + 4$ and sketch both functions on one set of axes.

Solution

Consider $x = 3\sqrt{y+2} + 4$ and solve for y:

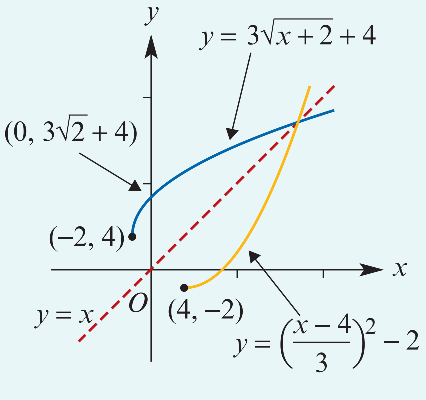
$$\frac{x-4}{3} = \sqrt{y+2}$$

$$y = \left(\frac{x-4}{3}\right)^2 - 2$$

$$\therefore \quad f^{-1}(x) = \left(\frac{x-4}{3}\right)^2 - 2$$

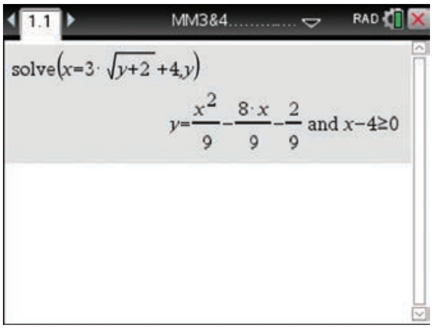
The domain of f^{-1} equals the range of f. Thus

$$f^{-1}\colon [4, \infty) \to \mathbb{R},\ f^{-1}(x) = \left(\frac{x-4}{3}\right)^2 - 2$$



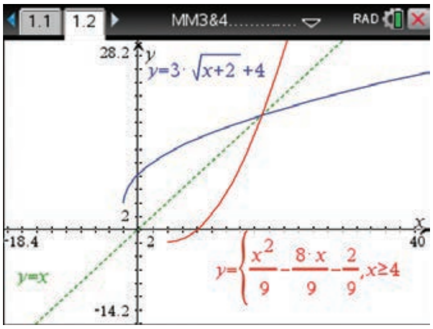
Using the TI-Nspire

- First find the rule for the inverse of $y = 3\sqrt{x+2} + 4$ by solving the equation $x = 3\sqrt{y+2} + 4$ for y.
- Insert a **Graphs** page and enter $f1(x) = 3\sqrt{x+2} + 4$, $f2(x) = \dfrac{x^2}{9} - \dfrac{8x}{9} - \dfrac{2}{9} \mid x \geq 4$ and $f3(x) = x$.



solve$(x=3\cdot\sqrt{y+2}+4,y)$

$y=\frac{x^2}{9}-\frac{8\cdot x}{9}-\frac{2}{9}$ and $x-4\geq 0$

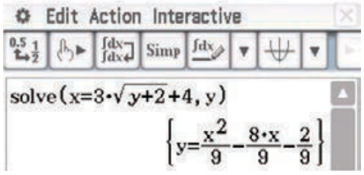


Note: To change the graph label to $y =$, place the cursor on the plot, press ctrl menu > **Attributes**, arrow down to the **Label Style** and select the desired style using the arrow keys. The **Attributes** menu can also be used to change the **Line Style**.

Using the Casio ClassPad

To find the rule for the inverse of $f(x) = 3\sqrt{x+2} + 4$:

- In $\overset{\text{Main}}{\sqrt{\alpha}}$, enter and highlight $x = 3\sqrt{y+2} + 4$.
- Select **Interactive** > **Equation/Inequality** > **solve** and set the variable as y. Then tap OK.

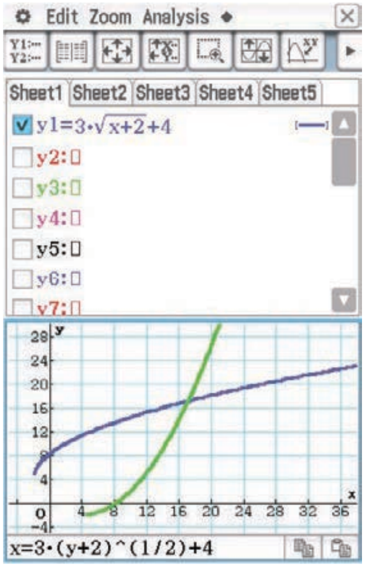


solve$(x=3\cdot\sqrt{y+2}+4,y)$

$\left\{y=\frac{x^2}{9}-\frac{8\cdot x}{9}-\frac{2}{9}\right\}$

To graph the inverse of $f(x) = 3\sqrt{x+2} + 4$:

- In Graph& Table, enter the rule for the function f in $y1$.
- Tick the box and tap ⊞.
- Use ⊞ to adjust the window view.
- To graph the inverse function f^{-1}, select **Analysis** > **Sketch** > **Inverse**.


Edit Zoom Analysis
Sheet1 Sheet2 Sheet3 Sheet4 Sheet5
y2:□
y3:□
y4:□
y5:□
y6:□
y7:□
y
28
24
20
16
12
8
4
0
-4
4
8
12
16
20
24
28
32
36
x

☑ y1=3·√x+2+4

x=3·(y+2)^(1/2)+4

Example 27

Express $\dfrac{x+4}{x+1}$ in the form $\dfrac{a}{x+b} + c$. Hence find the inverse of the function $f(x) = \dfrac{x+4}{x+1}$. Sketch both functions on the one set of axes.

Solution

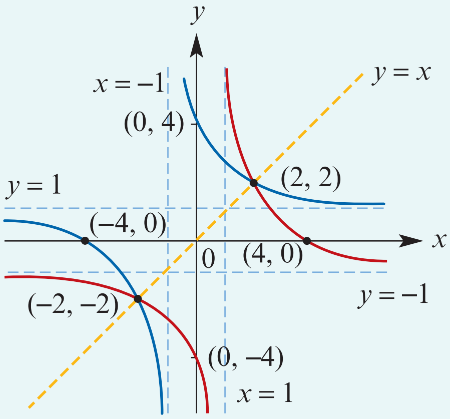
$$\frac{x+4}{x+1} = \frac{3+x+1}{x+1} = \frac{3}{x+1} + \frac{x+1}{x+1} = \frac{3}{x+1} + 1$$

Consider $x = \dfrac{3}{y+1} + 1$ and solve for y:

$$x - 1 = \frac{3}{y+1}$$

$$y + 1 = \frac{3}{x-1}$$

$$\therefore \quad y = \frac{3}{x-1} - 1$$



The range of f is $\mathbb{R} \setminus \{1\}$ and thus the inverse function is

$$f^{-1}\colon \mathbb{R} \setminus \{1\} \to \mathbb{R},\ f^{-1}(x) = \frac{3}{x-1} - 1$$

Note: The graph of f^{-1} is obtained by reflecting the graph of f in the line $y = x$. The two graphs meet where

$$\frac{3}{x+1} + 1 = x, \quad x \neq -1$$

i.e. where $x = \pm 2$. Thus the two graphs meet at the points $(2, 2)$ and $(-2, -2)$.

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Example 28

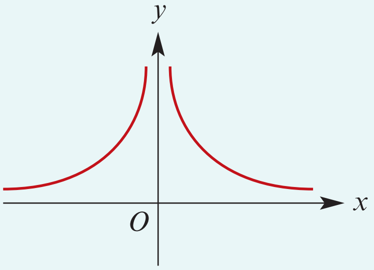
Let f be the function given by $f(x) = \dfrac{1}{x^2}$ for $x \in \mathbb{R} \setminus \{0\}$. Define a suitable restriction g of f such that g^{-1} exists, and find g^{-1}.

Solution

The function f is not one-to-one. Therefore the inverse function f^{-1} is not defined. The following restrictions of f are one-to-one:

$f_1\colon (0, \infty) \to \mathbb{R}, \quad f_1(x) = \dfrac{1}{x^2} \quad$ Range of $f_1 = (0, \infty)$

$f_2\colon (-\infty, 0) \to \mathbb{R}, \quad f_2(x) = \dfrac{1}{x^2} \quad$ Range of $f_2 = (0, \infty)$



Let g be f_1 and determine f_1^{-1}.

Using method 2, we require f_1^{-1} such that

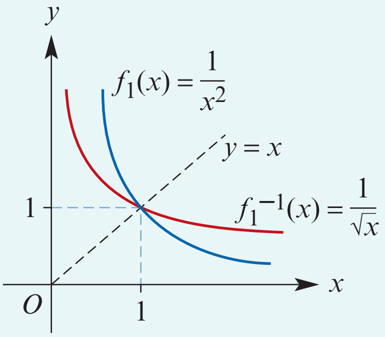
$$f_1(f_1^{-1}(x)) = x$$

$$\frac{1}{(f_1^{-1}(x))^2} = x$$

$$f_1^{-1}(x) = \pm\frac{1}{\sqrt{x}}$$

But ran f_1^{-1} = dom f_1 = $(0, \infty)$ and so

$$f_1^{-1}(x) = \frac{1}{\sqrt{x}}$$



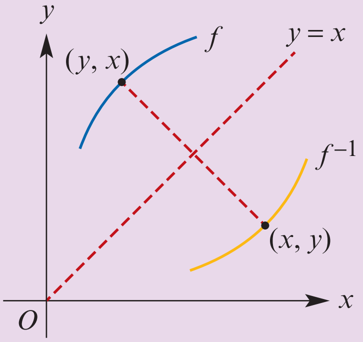
As dom f_1^{-1} = ran f_1 = $(0, \infty)$, the inverse function is $f_1^{-1}\colon (0, \infty) \to \mathbb{R}$, $f_1^{-1}(x) = \dfrac{1}{\sqrt{x}}$

Section summary

- If f is a one-to-one function, then a new function f^{-1}, called the **inverse** of f, may be defined by

$$f^{-1}(x) = y \text{ if } f(y) = x, \quad \text{for } x \in \operatorname{ran} f,\ y \in \operatorname{dom} f$$

- dom f^{-1} = ran f
- ran f^{-1} = dom f
- $f \circ f^{-1}(x) = x$, for all $x \in \operatorname{dom} f^{-1}$
- $f^{-1} \circ f(x) = x$, for all $x \in \operatorname{dom} f$
- The point (x, y) is on the graph of f^{-1} if and only if the point (y, x) is on the graph of f. Thus the graph of f^{-1} is the reflection of the graph of f in the line $y = x$.



Exercise 1F

Skillsheet

Example 23

1 Find the inverse function f^{-1} of the function:

a $f(x) = 2x + 3$

b $f(x) = 4 - 3x$

c $f(x) = 4x + 3$

2 For each of the following, find the rule for the inverse:

a $f: \mathbb{R} \to \mathbb{R}, f(x) = x - 4$

b $f: \mathbb{R} \to \mathbb{R}, f(x) = 2x$

c $f: \mathbb{R} \to \mathbb{R}, f(x) = \dfrac{3x}{4}$

d $f: \mathbb{R} \to \mathbb{R}, f(x) = \dfrac{3x - 2}{4}$

Example 24

3 For each of the following functions, find the inverse and state its domain and range:

a $f: [-2, 6] \to \mathbb{R}, f(x) = 2x - 4$

b $g(x) = \dfrac{1}{9 - x}, \ x > 9$

c $h(x) = x^2 + 2, \ x \geq 0$

d $f: [-3, 6] \to \mathbb{R}, f(x) = 5x - 2$

e $g: (1, \infty) \to \mathbb{R}, g(x) = x^2 - 1$

f $h: \mathbb{R}^+ \to \mathbb{R}, h(x) = \sqrt{x}$

4 Consider the function $g: [-1, \infty) \to \mathbb{R}, g(x) = x^2 + 2x$.

a Find g^{-1}, stating the domain and range.

b Sketch the graph of g^{-1}.

Example 25

5 Find the inverse of the function $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}, f(x) = \dfrac{1}{x} - 3$. Sketch both functions on one set of axes, showing the points of intersection of the graphs.

6 Let $f: [0, 3] \to \mathbb{R}, f(x) = 3 - 2x$. Find $f^{-1}(2)$ and the domain of f^{-1}.

7 For each of the following functions, find the inverse and state its domain and range:

a $f: [-1, 3] \to \mathbb{R}, f(x) = 2x$

b $f: [0, \infty) \to \mathbb{R}, f(x) = 2x^2 - 4$

c $\{(1, 6), (2, 4), (3, 8), (5, 11)\}$

d $h: \mathbb{R}^- \to \mathbb{R}, h(x) = \sqrt{-x}$

e $f: \mathbb{R} \to \mathbb{R}, f(x) = x^3 + 1$

f $g: (-1, 3) \to \mathbb{R}, g(x) = (x + 1)^2$

g $g: [1, \infty) \to \mathbb{R}, g(x) = \sqrt{x - 1}$

h $h: [0, 2] \to \mathbb{R}, h(x) = \sqrt{4 - x^2}$

Example 26

8 For each of the following functions, sketch the graph of the function and on the same set of axes sketch the graph of the inverse function. For each of the functions, state the rule, domain and range of the inverse. It is advisable to draw in the line with equation $y = x$ for each set of axes.

a $y = 2x + 4$

b $f(x) = \dfrac{3 - x}{2}$

c $f: [2, \infty) \to \mathbb{R}, f(x) = (x - 2)^2$

d $f: [1, \infty) \to \mathbb{R}, f(x) = (x - 1)^2$

e $f: (-\infty, 2] \to \mathbb{R}, f(x) = (x - 2)^2$

f $f: \mathbb{R}^+ \to \mathbb{R}, f(x) = \dfrac{1}{x}$

g $f: \mathbb{R}^+ \to \mathbb{R}, f(x) = \dfrac{1}{x^2}$

h $h(x) = \dfrac{1}{2}(x - 4)$

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9 Find the inverse function of each of the following, and sketch the graph of the inverse function:

a $f\colon \mathbb{R}^+ \cup \{0\} \to \mathbb{R},\ f(x) = \sqrt{x} + 2$

b $f\colon \mathbb{R} \setminus \{3\} \to \mathbb{R},\ f(x) = \dfrac{1}{x-3}$

c $f\colon [2, \infty) \to \mathbb{R},\ f(x) = \sqrt{x-2} + 4$

d $f\colon \mathbb{R} \setminus \{3\} \to \mathbb{R},\ f(x) = \dfrac{1}{x-3} + 1$

e $f\colon \mathbb{R} \setminus \{1\} \to \mathbb{R},\ f(x) = \dfrac{5}{x-1} - 1$

f $f\colon (-\infty, 2] \to \mathbb{R},\ f(x) = \sqrt{2-x} + 1$

Example 27 **10** Find the rule for the inverse of each of the following functions:

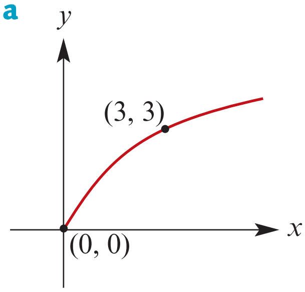
a $f\colon \mathbb{R} \setminus \{1\} \to \mathbb{R},\ f(x) = \dfrac{x+1}{x-1}$

b $f\colon [2, \infty) \to \mathbb{R},\ f(x) = \sqrt{x-2}$

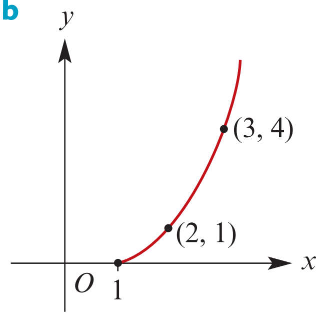
c $f\colon \mathbb{R} \setminus \{\frac{2}{3}\} \to \mathbb{R},\ f(x) = \dfrac{2x+3}{3x-2}$

11 Copy each of the following graphs and on the same set of axes draw the inverse of each of the corresponding functions:

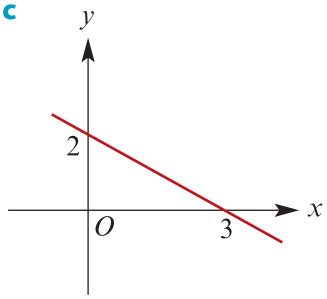
a



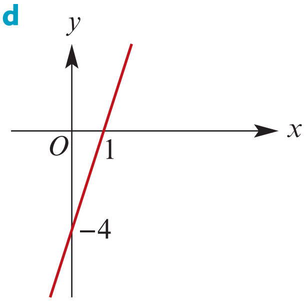
b



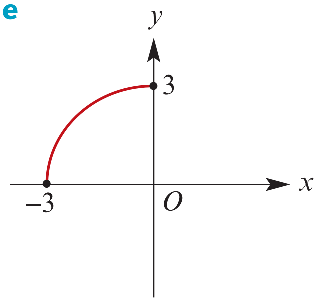
c



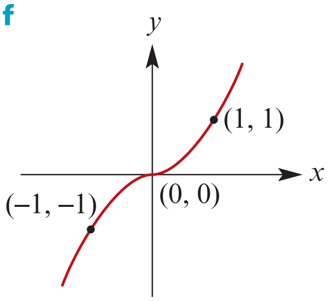
d



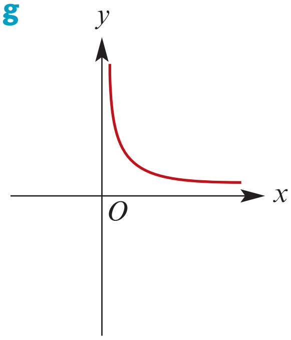
e



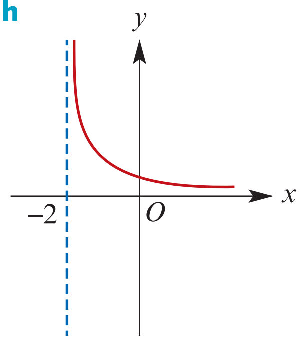
f



g

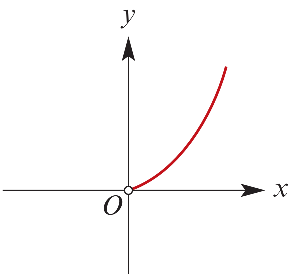


h

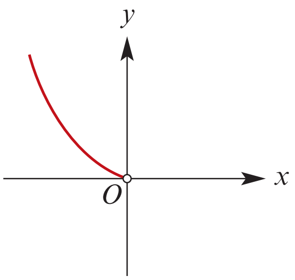


12 Match each of the graphs of **a**, **b**, **c** and **d** with its inverse.

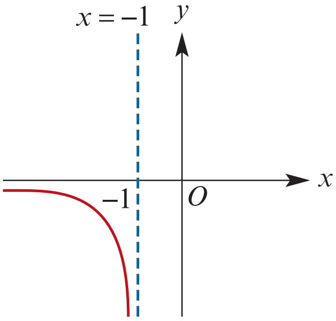
a



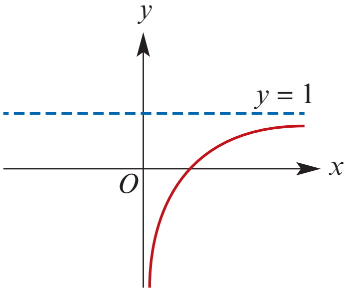
b



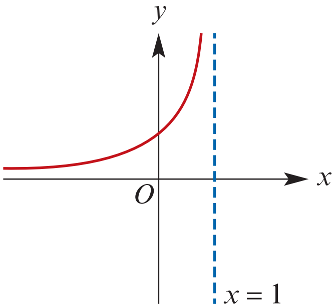
c



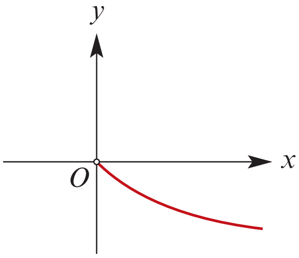
d



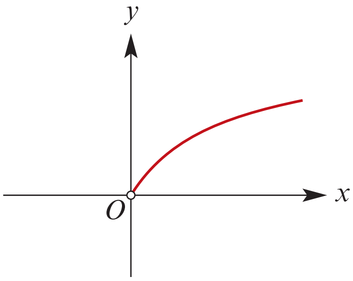
A



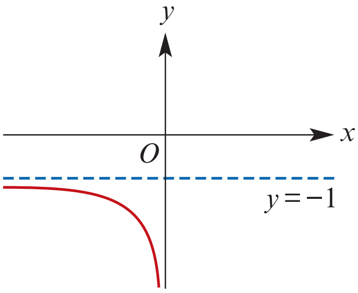
B



C



D



13 **a** Let $f\colon A \to \mathbb{R}$, $f(x) = \sqrt{3 - x}$. If A is the set of all real values of x for which $f(x)$ is defined, find A.

Example 28

b Let $g\colon [b, 2] \to \mathbb{R}$, $g(x) = 1 - x^2$. If b is the smallest real number such that g has an inverse function, find b and $g^{-1}(x)$.

14 Let $g\colon [b, \infty) \to \mathbb{R}$, where $g(x) = x^2 + 4x$. If b is the smallest real number such that g has an inverse function, find b and $g^{-1}(x)$.

15 Let $f\colon (-\infty, a) \to \mathbb{R}$, where $f(x) = x^2 - 6x$. If a is the largest real number such that f has an inverse function, find a and $f^{-1}(x)$.

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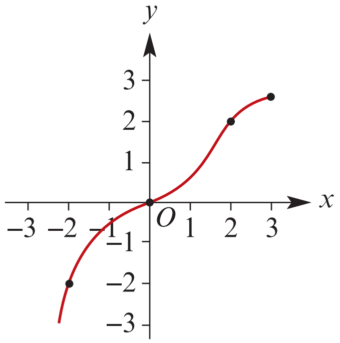
16 For each of the following functions, find the inverse function and state its domain:

a $g(x) = \frac{3}{x}$ **b** $g(x) = \sqrt[3]{x+2} - 4$ **c** $h(x) = 2 - \sqrt{x}$

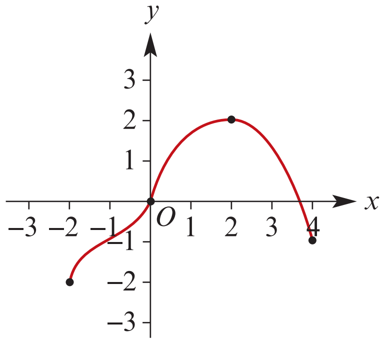
d $f(x) = \frac{3}{x} + 1$ **e** $h(x) = 5 - \frac{2}{(x-6)^3}$ **f** $g(x) = \frac{1}{(x-1)^{\frac{3}{4}}} + 2$

17 For each of the following, copy the graph onto a grid and sketch the graph of the inverse on the same set of axes. In each case, state whether the inverse is or is not a function.

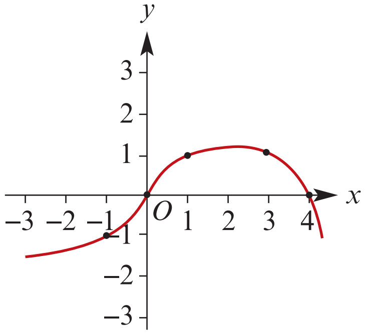
a



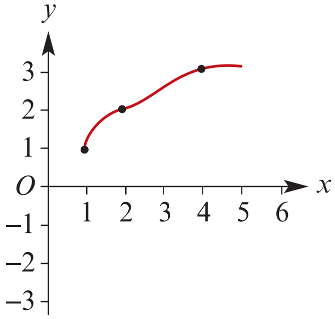
b



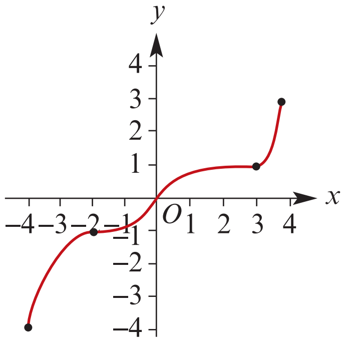
c



d



e



18 Let $f: S \to \mathbb{R}$ be given by $f(x) = \frac{x+3}{2x-1}$, where $S = \mathbb{R} \setminus \{\frac{1}{2}\}$.

a Show that $f \circ f$ is defined.

b Find $f \circ f(x)$ and sketch the graph of $f \circ f$.

c Write down the inverse of f.

1G Power functions

In this section we look at functions of the form $f(x) = x^r$, where r is a rational number. These functions are called **power functions**.

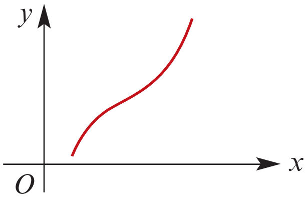
In particular, we look at functions with rules such as

$$f(x) = x^4, \quad f(x) = x^{-4}, \quad f(x) = x^{\frac{1}{4}}, \quad f(x) = x^5, \quad f(x) = x^{-5}, \quad f(x) = x^{\frac{1}{3}}$$

We will not concern ourselves with functions such as $f(x) = x^{\frac{2}{3}}$ at this stage, but return to consider these functions in Chapter 7.

▶ Increasing and decreasing functions

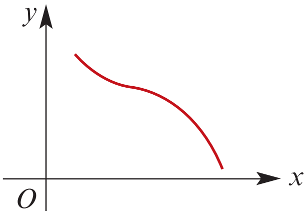
We say a function f is **strictly increasing** on an interval if $x_2 > x_1$ implies $f(x_2) > f(x_1)$.



For example:

- The graph opposite shows a strictly increasing function.
- A straight line with positive gradient is strictly increasing.
- The function $f\colon (0, \infty) \to \mathbb{R}$, $f(x) = x^2$ is strictly increasing.

We say a function f is **strictly decreasing** on an interval if $x_2 > x_1$ implies $f(x_2) < f(x_1)$.



For example:

- The graph opposite shows a strictly decreasing function.
- A straight line with negative gradient is strictly decreasing.
- The function $f\colon (-\infty, 0) \to \mathbb{R}$, $f(x) = x^2$ is strictly decreasing.

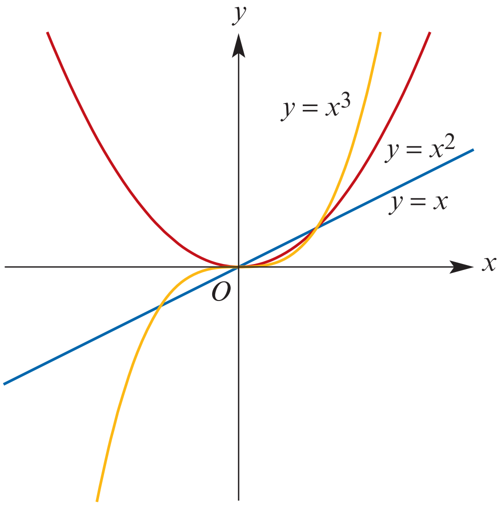
▶ Power functions with positive integer index

We start by considering power functions $f(x) = x^n$ where n is a positive integer.

Taking $n = 1, 2, 3$, we obtain the linear function $f(x) = x$, the quadratic function $f(x) = x^2$ and the cubic function $f(x) = x^3$.

We have studied these functions in Mathematical Methods Units 1 & 2 and have referred to them in the earlier sections of this chapter.

The general shape of the graph of $f(x) = x^n$ depends on whether the index n is odd or even.



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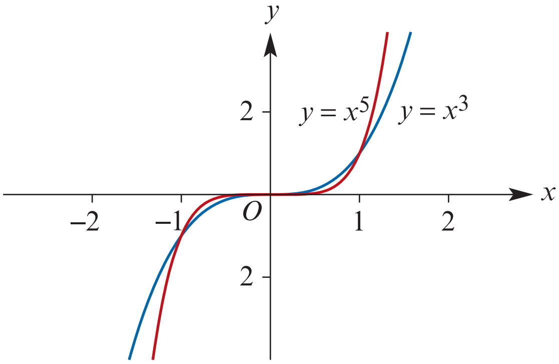
The function $f(x) = x^n$ where n is an odd positive integer

The graph has a similar shape to those shown below. The maximal domain is $\mathbb{R}$ and the range is $\mathbb{R}$.

Some properties of $f(x) = x^n$ where n is an odd positive integer:

- f is an odd function
- f is strictly increasing
- f is one-to-one
- $f(0) = 0$, $f(1) = 1$ and $f(-1) = -1$
- as $x \to \infty$, $f(x) \to \infty$ and as $x \to -\infty$, $f(x) \to -\infty$.

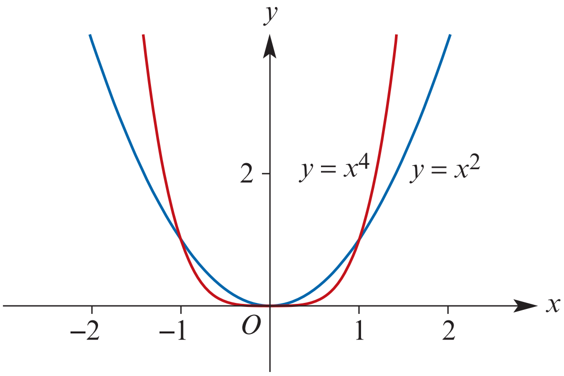


The function $f(x) = x^n$ where n is an even positive integer

The graph has a similar shape to those shown below. The maximal domain is $\mathbb{R}$ and the range is $\mathbb{R}^+ \cup \{0\}$.

Some properties of $f(x) = x^n$ where n is an even positive integer:

- f is an even function
- f strictly increasing for $x > 0$
- f is strictly decreasing for $x < 0$
- $f(0) = 0$, $f(1) = 1$ and $f(-1) = 1$
- as $x \to \pm\infty$, $f(x) \to \infty$.



Power functions with negative integer index

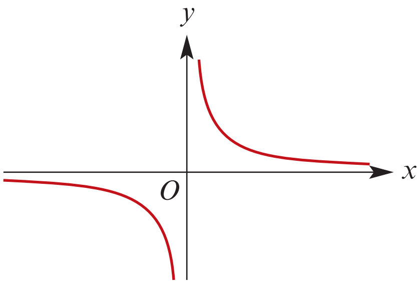
Again, the general shape of the graph depends on whether the index n is odd or even.

The function $f(x) = x^n$ where n is an odd negative integer

Taking $n = -1$, we obtain

$$f(x) = x^{-1} = \frac{1}{x}$$

The graph of this function is shown on the right. The graphs of functions of this type are all similar to this one.



In general, we consider the functions $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}, f(x) = x^{-k}$ for $k = 1, 3, 5, \ldots$

- the maximal domain is $\mathbb{R} \setminus \{0\}$ and the range is $\mathbb{R} \setminus \{0\}$
- f is an odd function
- there is a horizontal asymptote with equation $y = 0$
- there is a vertical asymptote with equation $x = 0$.

Example 29

For the function f with rule $f(x) = \dfrac{1}{x^5}$:

a State the maximal domain and the corresponding range.

b Evaluate each of the following:

i $f(2)$ **ii** $f(-2)$ **iii** $f(\frac{1}{2})$ **iv** $f(-\frac{1}{2})$

c Sketch the graph without using your calculator.

Solution

a The maximal domain is $\mathbb{R} \setminus \{0\}$ and the range is $\mathbb{R} \setminus \{0\}$.

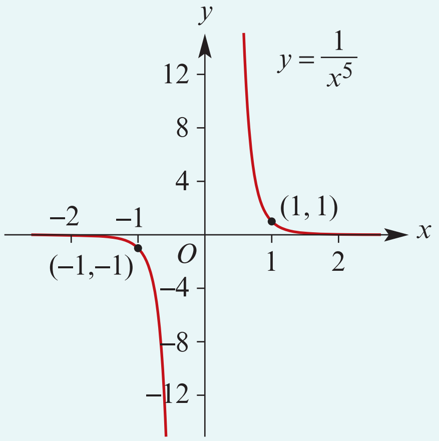
b **i** $f(2) = \dfrac{1}{2^5} = \dfrac{1}{32}$

ii $f(-2) = \dfrac{1}{(-2)^5} = -\dfrac{1}{32}$

iii $f(\frac{1}{2}) = \dfrac{1}{(\frac{1}{2})^5} = 32$

iv $f(-\frac{1}{2}) = \dfrac{1}{(-\frac{1}{2})^5} = -32$

c



Example 30

Let $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = x^{-1}$ and $g\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $g(x) = x^{-3}$.

a Find the values of x for which $f(x) = g(x)$.

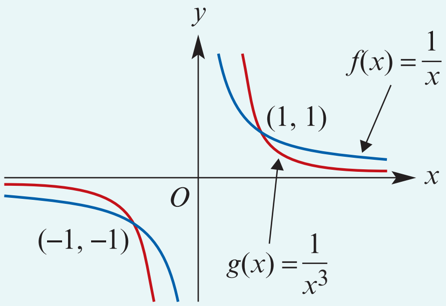
b Sketch the graphs of $y = f(x)$ and $y = g(x)$ on the one set of axes.

Solution

a

$$f(x) = g(x)$$
$$x^{-1} = x^{-3}$$
$$\frac{1}{x} = \frac{1}{x^3}$$
$$x^2 = 1$$
$$\therefore \quad x = 1 \text{ or } x = -1$$

b



Note:

If $x > 1$, then $x^3 > x$ and so $\dfrac{1}{x} > \dfrac{1}{x^3}$.

If $0 < x < 1$, then $x^3 < x$ and so $\dfrac{1}{x} < \dfrac{1}{x^3}$.

If $x < -1$, then $x^3 < x$ and so $\dfrac{1}{x} < \dfrac{1}{x^3}$.

If $-1 < x < 0$, then $x^3 > x$ and so $\dfrac{1}{x} > \dfrac{1}{x^3}$.

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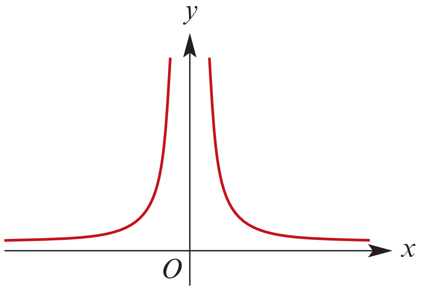
The function $f(x) = x^n$ where n is an even negative integer

Taking $n = -2$, we obtain

$$f(x) = x^{-2} = \frac{1}{x^2}$$

The graph of this function is shown on the right. The graphs of functions of this type are all similar to this one.



In general, we consider the functions $f \colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = x^{-k}$ for $k = 2, 4, 6, \ldots$

- the maximal domain $\mathbb{R} \setminus \{0\}$ and the range is $\mathbb{R}^+$
- f is an even function
- there is a horizontal asymptote with equation $y = 0$
- there is a vertical asymptote with equation $x = 0$.

▶ The function $f(x) = x^{\frac{1}{n}}$ where n is a positive integer

Let a be a positive real number and let $n \in \mathbb{N}$. Then $a^{\frac{1}{n}}$ is defined to be the nth root of a. That is, $a^{\frac{1}{n}}$ is the positive number whose nth power is a. We can also write this as $a^{\frac{1}{n}} = \sqrt[n]{a}$. For example: $9^{\frac{1}{2}} = 3$, since $3^2 = 9$.

We define $0^{\frac{1}{n}} = 0$, for each natural number n, since $0^n = 0$.

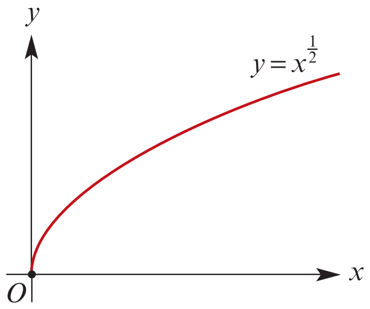
If n is odd, then we can also define $a^{\frac{1}{n}}$ when a is negative. If a is negative and n is odd, define $a^{\frac{1}{n}}$ to be the number whose nth power is a. For example: $(-8)^{\frac{1}{3}} = -2$, as $(-2)^3 = -8$.

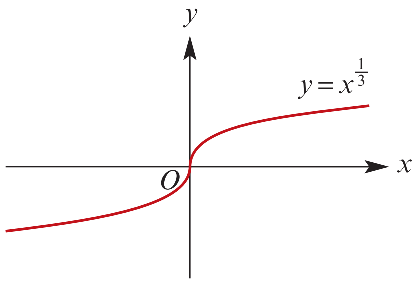
In all three cases we can write:

$$a^{\frac{1}{n}} = \sqrt[n]{a} \quad \text{with} \quad \left(a^{\frac{1}{n}}\right)^n = a$$

In particular, $x^{\frac{1}{2}} = \sqrt{x}$.

Let $f(x) = x^{\frac{1}{n}}$. When n is even the maximal domain is $\mathbb{R}^+ \cup \{0\}$ and when n is odd the maximal domain is $\mathbb{R}$. The graphs of $f(x) = \sqrt{x} = x^{\frac{1}{2}}$ and $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$ are as shown.





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Example 31

Let $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^{\frac{1}{3}}$ and $g\colon \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$, $g(x) = x^{\frac{1}{2}}$.

a Find the values of x for which $f(x) = g(x)$.

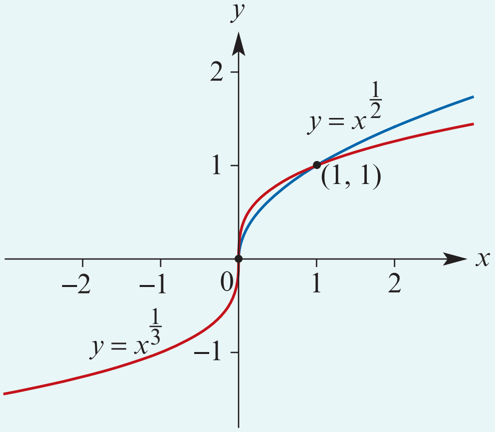
b Sketch the graphs of $y = f(x)$ and $y = g(x)$ on the one set of axes.

Solution

a

$$\begin{aligned} f(x) &= g(x) \\ x^{\frac{1}{3}} &= x^{\frac{1}{2}} \\ x^{\frac{1}{3}} - x^{\frac{1}{2}} &= 0 \\ x^{\frac{1}{3}}\left(1 - x^{\frac{1}{6}}\right) &= 0 \\ \therefore\quad x = 0 \ \text{ or } \ 1 - x^{\frac{1}{6}} &= 0 \\ \therefore\quad x = 0 \ \text{ or } \ x &= 1 \end{aligned}$$

b



Inverses of power functions

We prove the following result in the special case when $n = 5$. The general proof is similar.

If n is an odd positive integer, then $f(x) = x^n$ is strictly increasing for $\mathbb{R}$.

Proof Let $f(x) = x^5$ and let $a > b$. To show that $f(a) > f(b)$, we consider five cases.

Case 1: $a > b > 0$ We have

$$\begin{aligned} f(a) - f(b) &= a^5 - b^5 \\ &= (a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4) \qquad \text{(Show by expanding.)} \end{aligned}$$

Since $a > b$, we have $a - b > 0$. Since we are assuming that a and b are positive in this case, all the terms of $a^4 + a^3b + a^2b^2 + ab^3 + b^4$ are positive. Therefore $f(a) - f(b) > 0$ and so $f(a) > f(b)$.

Case 2: $a > 0$ and $b < 0$ In this case, we have $f(a) = a^5 > 0$ and $f(b) = b^5 < 0$ (an odd power of a negative number). Thus $f(a) > f(b)$.

Case 3: $a = 0$ and $b < 0$ We have $f(a) = 0$ and $f(b) < 0$. Thus $f(a) > f(b)$.

Case 4: $b = 0$ and $a > 0$ We have $f(a) > 0$ and $f(b) = 0$. Thus $f(a) > f(b)$.

Case 5: $0 > a > b$ Let $a = -c$ and $b = -d$, where c and d are positive. Then $a > b$ implies $-c > -d$ and so $c < d$. Hence $f(c) < f(d)$ by Case 1 and thus $f(-a) < f(-b)$. But f is an odd function and so $-f(a) < -f(b)$. Finally, we have $f(a) > f(b)$.

Note: For the general proof, use the identity

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + a^2b^{n-3} + ab^{n-2} + b^{n-1})$$

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If f is a strictly increasing function on $\mathbb{R}$, then it is a one-to-one function and so has an inverse. Thus $f(x) = x^n$ has an inverse function, where n is an odd positive integer.

Similar results can be achieved for restrictions of functions with rules $f(x) = x^n$, where n is an even positive integer. For example, $g\colon \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$, $g(x) = x^6$ is a strictly increasing function and $h\colon \mathbb{R}^- \cup \{0\} \to \mathbb{R}$, $h(x) = x^6$ is a strictly decreasing function. In both cases, these restricted functions are one-to-one.

If f is an odd one-to-one function, then f^{-1} is also an odd function.

Proof Let $x \in \operatorname{dom} f^{-1}$ and let $y = f^{-1}(x)$. Then $f(y) = x$. Since f is an odd function, we have $f(-y) = -x$, which implies that $f^{-1}(-x) = -y$. Hence $f^{-1}(-x) = -f^{-1}(x)$.

By this result we see that, if n is odd, then $f(x) = x^{\frac{1}{n}}$ is an odd function. It can also be shown that, if f is a strictly increasing function, then f^{-1} is strictly increasing.

Example 32

Find the inverse of each of the following functions:

a $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^5$

b $f\colon (-\infty, 0] \to \mathbb{R}$, $f(x) = x^4$

c $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 8x^3$

d $f\colon (1, \infty) \to \mathbb{R}$, $f(x) = 64x^6$

Solution

a $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^5$

Write $y = x^5$. Interchange x and y and then solve for y:

$$x = y^5$$
$$\therefore \quad y = x^{\frac{1}{5}}$$

Thus $f^{-1}\colon \mathbb{R} \to \mathbb{R}$, $f^{-1}(x) = x^{\frac{1}{5}}$

b $f\colon (-\infty, 0] \to \mathbb{R}$, $f(x) = x^4$

Note that f has range $[0, \infty)$. Therefore f^{-1} has domain $[0, \infty)$ and range $(-\infty, 0]$. Write $y = x^4$. Interchange x and y and then solve for y:

$$x = y^4$$
$$\therefore \quad y = \pm x^{\frac{1}{4}}$$

Thus $f^{-1}\colon [0, \infty) \to \mathbb{R}$, $f^{-1}(x) = -x^{\frac{1}{4}}$

c $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 8x^3$

Write $y = 8x^3$. Interchange x and y and then solve for y:

$$x = 8y^3$$
$$y^3 = \frac{x}{8}$$
$$\therefore \quad y = \frac{1}{2}x^{\frac{1}{3}}$$

Thus $f^{-1}\colon \mathbb{R} \to \mathbb{R}$, $f^{-1}(x) = \frac{1}{2}x^{\frac{1}{3}}$

d $f\colon (1, \infty) \to \mathbb{R}$, $f(x) = 64x^6$

Note that f has range $(64, \infty)$. Therefore f^{-1} has domain $(64, \infty)$ and range $(1, \infty)$. Write $y = 64x^6$. Interchange x and y and then solve for y:

$$x = 64y^6$$
$$y^6 = \frac{x}{64}$$
$$\therefore \quad y = \pm\frac{1}{2}x^{\frac{1}{6}}$$

Thus $f^{-1}\colon (64, \infty) \to \mathbb{R}$, $f^{-1}(x) = \frac{1}{2}x^{\frac{1}{6}}$

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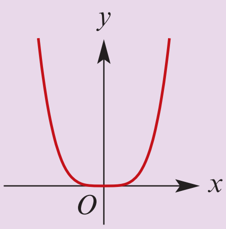
Section summary

- A function f is **strictly increasing** on an interval if $x_2 > x_1$ implies $f(x_2) > f(x_1)$.
- A function f is **strictly decreasing** on an interval if $x_2 > x_1$ implies $f(x_2) < f(x_1)$.
- A **power function** is a function f with rule $f(x) = x^r$, where r is a rational number.
- For a power function $f(x) = x^n$, where n is a non-zero integer, the general shape of the graph depends on whether n is positive or negative and whether n is even or odd:

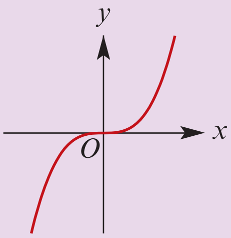
Even positive

$f(x) = x^4$



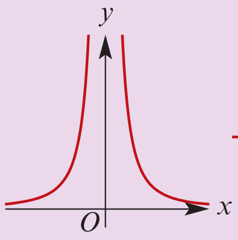
Odd positive

$f(x) = x^3$



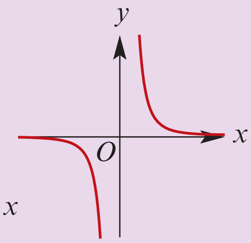
Even negative

$f(x) = x^{-2}$



Odd negative

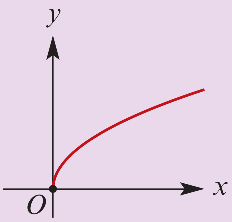
$f(x) = x^{-3}$



- For a power function $f(x) = x^{\frac{1}{n}}$, where n is a positive integer, the general shape of the graph depends on whether n is even or odd:

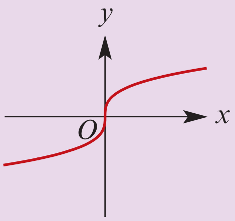
Even

$f(x) = x^{\frac{1}{2}}$



Odd

$f(x) = x^{\frac{1}{3}}$



Exercise 1G

Example 29

1 For the function f with rule $f(x) = \dfrac{1}{x^4}$:

- **a** State the maximal domain and the corresponding range.
- **b** Evaluate each of the following:
  - **i** $f(2)$
  - **ii** $f(-2)$
  - **iii** $f(\frac{1}{2})$
  - **iv** $f(-\frac{1}{2})$
- **c** Sketch the graph without using your calculator.

2 For each of the following, state whether the function is odd, even or neither:

- **a** $f(x) = 2x^5$
- **b** $f(x) = x^2 + 3$
- **c** $f(x) = x^{\frac{1}{5}}$
- **d** $f(x) = \dfrac{1}{x}$
- **e** $f(x) = \dfrac{1}{x^2}$
- **f** $f(x) = \sqrt[3]{x}$

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Example 30 **3** Let $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = x^{-2}$ and $g\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $g(x) = x^{-4}$.

a Find the values of x for which $f(x) = g(x)$.

b Sketch the graphs of $y = f(x)$ and $y = g(x)$ on the one set of axes.

Example 31 **4** Let $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^{\frac{1}{3}}$ and $g\colon \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$, $g(x) = x^{\frac{1}{4}}$.

a Find the values of x for which $f(x) = g(x)$.

b Sketch the graphs of $y = f(x)$ and $y = g(x)$ on the one set of axes.

Example 32 **5** Find the inverse of each of the following functions:

a $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^7$

b $f\colon (-\infty, 0] \to \mathbb{R}$, $f(x) = x^6$

c $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 27x^3$

d $f\colon (1, \infty) \to \mathbb{R}$, $f(x) = 16x^4$

1H Applications of functions

In this section we use function notation in the solution of some problems.

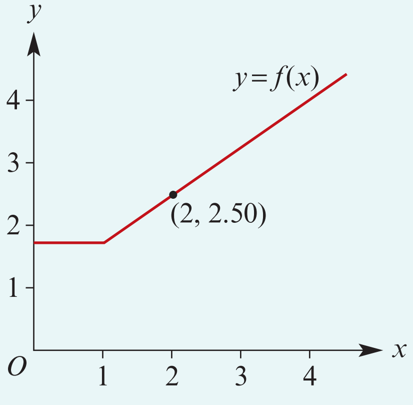
Example 33

The cost of a taxi trip in a particular city is \$1.75 up to and including 1 km. After 1 km the passenger pays an additional 75 cents per kilometre. Find the function f which describes this method of payment and sketch the graph of $y = f(x)$.

Solution

Let x denote the length of the trip in kilometres.
Then the cost in dollars is given by

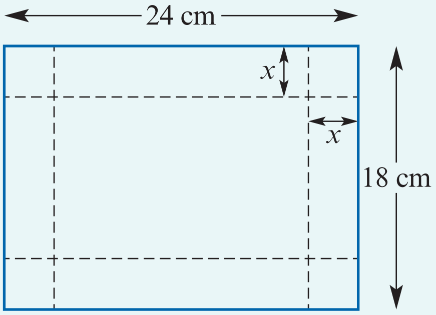
$$f(x) = \begin{cases} 1.75 & \text{for } 0 \le x \le 1 \\ 1.75 + 0.75(x - 1) & \text{for } x > 1 \end{cases}$$



Example 34

A rectangular piece of cardboard has dimensions 18 cm by 24 cm. Four squares each x cm by x cm are cut from the corners. An open box is formed by folding up the flaps.

Find a function V which gives the volume of the box in terms of x, and state the domain of the function.



Solution

The dimensions of the box will be $24 - 2x$, $18 - 2x$ and x.

Thus the volume of the box is determined by the function

$$V(x) = (24 - 2x)(18 - 2x)x$$

For the box to be formed:

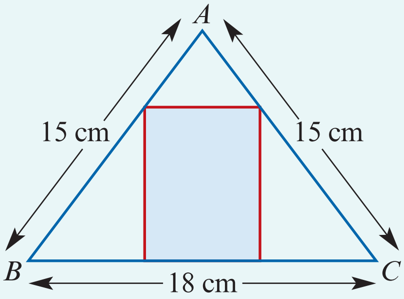
$$24 - 2x \geq 0 \quad \text{and} \quad 18 - 2x \geq 0 \quad \text{and} \quad x \geq 0$$

Therefore $x \leq 12$ and $x \leq 9$ and $x \geq 0$. The domain of V is $[0, 9]$.

Example 35

A rectangle is inscribed in an isosceles triangle with the dimensions as shown.

Find an area-of-the-rectangle function and state the domain.

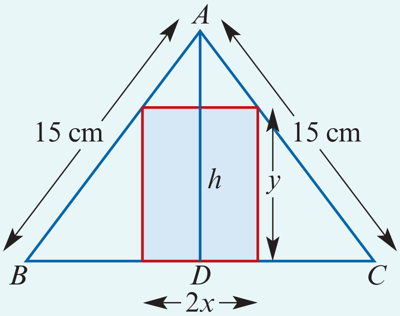


Solution

Let the height of the rectangle be y cm and the width $2x$ cm.

The height (h cm) of the triangle can be determined by Pythagoras' theorem:

$$h = \sqrt{15^2 - 9^2} = 12$$

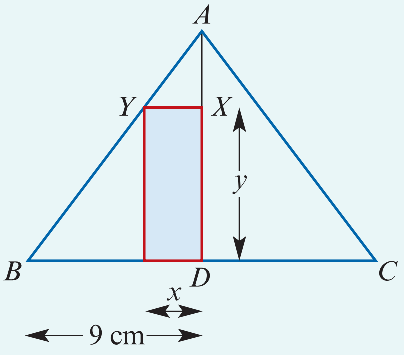


In the diagram opposite, the triangle AYX is similar to the triangle ABD. Therefore

$$\frac{x}{9} = \frac{12-y}{12}$$

$$\frac{12x}{9} = 12 - y$$

$$\therefore \quad y = 12 - \frac{12x}{9}$$



The area of the rectangle is $A = 2xy$, and so

$$A(x) = 2x\left(12 - \frac{12x}{9}\right) = \frac{24x}{9}(9-x)$$

For the rectangle to be formed, we need

$$x \geq 0 \quad \text{and} \quad 12 - \frac{12x}{9} \geq 0$$

$$\therefore \quad x \geq 0 \quad \text{and} \quad x \leq 9$$

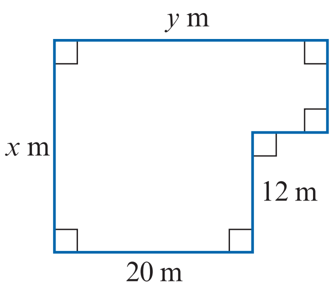
The domain is $[0, 9]$, and so the function is $A\colon [0,9] \to \mathbb{R}$, $A(x) = \dfrac{24x}{9}(9-x)$

Exercise 1H

Example 33 **1** The cost of a taxi trip in a particular city is \$4.00 up to and including 2 km. After 2 km the passenger pays an additional \$2.00 per kilometre. Find the function f which describes this method of payment and sketch the graph of $y = f(x)$, where x is the number of kilometres travelled. (Use a continuous model.)

Example 34 **2** A rectangular piece of cardboard has dimensions 20 cm by 36 cm. Four squares each x cm by x cm are cut from the corners. An open box is formed by folding up the flaps. Find a function V which gives the volume of the box in terms of x, and state the domain for the function.

3 The dimensions of an enclosure are shown. The perimeter of the enclosure is 160 m.



- **a** Find a rule for the area, A m^2, of the enclosure in terms of x.
- **b** State a suitable domain of the function $A(x)$.
- **c** Sketch the graph of A against x.
- **d** Find the maximum possible area of the enclosure and state the corresponding values of x and y.

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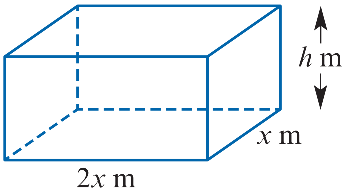
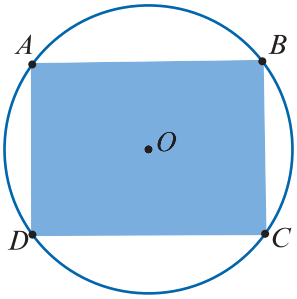
4 A cuboid tank is open at the top and the internal dimensions of its base are x m and $2x$ m. The height is h m. The volume of the tank is V m^3 and the volume is fixed. Let S m^2 denote the internal surface area of the tank.



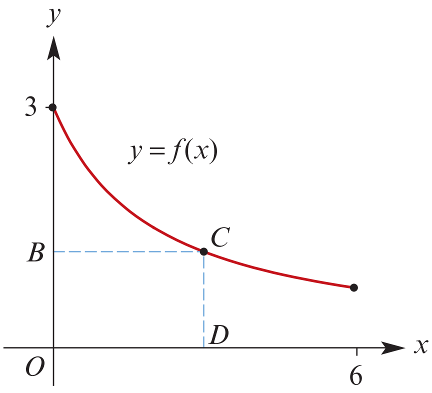
- **a** Find S in terms of:
  - **i** x and h
  - **ii** V and x
- **b** State the maximal domain for the function defined by the rule in part a ii.
- **c** If $2 \le x \le 15$, find the maximum value of S if $V = 1000$ m^3.

Example 35

5 A rectangle $ABCD$ is inscribed in a circle of radius a. Find an area-of-the-rectangle function and state the domain.



6 Let $f: [0, 6] \to \mathbb{R}$, $f(x) = \dfrac{6}{x+2}$.
Rectangle $OBCD$ is formed so that the coordinates of C are $(a, f(a))$.



- **a** Find an expression for the area-of-rectangle function A.
- **b** State the implied domain and range of A.
- **c** State the maximum value of $A(x)$ for $x \in [0, 6]$.
- **d** Sketch the graph of $y = A(x)$ for $x \in [0, 6]$.

7 A man walks at a speed of 2 km/h for 45 minutes and then runs at 4 km/h for 30 minutes. Let S km be the distance the man has travelled after t minutes. The distance travelled can be described by

$$S(t) = \begin{cases} at & \text{if } 0 \le t \le c \\ bt + d & \text{if } c < t \le e \end{cases}$$

- **a** Find the values a, b, c, d, e.
- **b** Sketch the graph of $S(t)$ against t.
- **c** State the range of the function.

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Chapter summary

Spreadsheet

AS

Nrich

Relations

- A **relation** is a set of ordered pairs.
- The **domain** is the set of all the first coordinates of the ordered pairs in the relation.
- The **range** is the set of all the second coordinates of the ordered pairs in the relation.

Functions

- A **function** is a relation such that no two ordered pairs in the relation have the same first coordinate.
- For each x in the domain of a function f, there is a unique element y in the range such that $(x, y) \in f$. The element y is called the **image** of x under f or the **value** of f at x and is denoted by $f(x)$.
- When the domain of a function is not explicitly stated, it is assumed to consist of all real numbers for which the rule has meaning; this is called the **implied domain** or the **maximal domain** of the function.
- For a function f, the domain is denoted by **dom** $\boldsymbol{f}$ and the range by **ran** $\boldsymbol{f}$.
- Let f and g be functions such that $\text{dom}\, f \cap \text{dom}\, g \neq \emptyset$. Then the **sum**, $f + g$, and the **product**, fg, as functions on $\text{dom}\, f \cap \text{dom}\, g$ are defined by

$$(f + g)(x) = f(x) + g(x) \quad \text{and} \quad (fg)(x) = f(x) \cdot g(x)$$

- The **composition** of functions f and g is denoted by $f \circ g$. The rule is given by

$$f \circ g(x) = f(g(x))$$

The domain of $f \circ g$ is the domain of g. The composition $f \circ g$ is defined only if the range of g is a subset of the domain of f.

One-to-one functions and inverses

- A function f is said to be **one-to-one** if $a \neq b$ implies $f(a) \neq f(b)$, for all $a, b \in \text{dom}\, f$.
- If f is a one-to-one function, then a new function f^{-1}, called the **inverse** of f, may be defined by

$$f^{-1}(x) = y \ \text{ if } \ f(y) = x, \quad \text{for } x \in \text{ran}\, f,\ y \in \text{dom}\, f$$

- For a one-to-one function f and its inverse f^{-1}:

$$\text{dom}\, f^{-1} = \text{ran}\, f$$
$$\text{ran}\, f^{-1} = \text{dom}\, f$$

Types of functions

- A function f is **odd** if $f(-x) = -f(x)$ for all x in the domain of f.
- A function f is **even** if $f(-x) = f(x)$ for all x in the domain of f.
- A function f is **strictly increasing** on an interval if $x_2 > x_1$ implies $f(x_2) > f(x_1)$.
- A function f is **strictly decreasing** on an interval if $x_2 > x_1$ implies $f(x_2) < f(x_1)$.
- A **power function** is a function f with rule $f(x) = x^r$, where r is a rational number.

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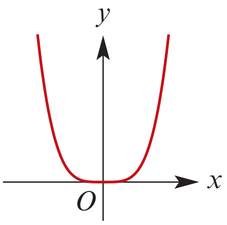
- For a power function $f(x) = x^n$, where n is a non-zero integer, the general shape of the graph depends on whether n is positive or negative and whether n is even or odd:

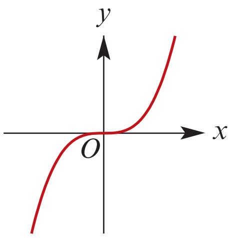
Even positive

$f(x) = x^4$



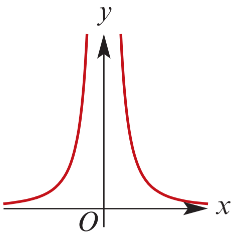
Odd positive

$f(x) = x^3$



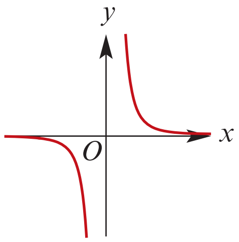
Even negative

$f(x) = x^{-2}$



Odd negative

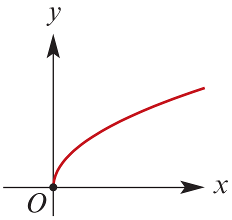
$f(x) = x^{-3}$



- For a power function $f(x) = x^{\frac{1}{n}}$, where n is a positive integer, the general shape of the graph depends on whether n is even or odd:

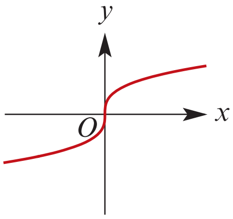
Even

$f(x) = x^{\frac{1}{2}}$



Odd

$f(x) = x^{\frac{1}{3}}$



Technology-free questions

1 Sketch the graph of each of the following relations and state the implied domain and range:

a $f(x) = x^2 + 1$ **b** $f(x) = 2x - 6$ **c** $\{ (x, y) : x^2 + y^2 = 25 \}$

d $\{ (x, y) : y \geq 2x + 1 \}$ **e** $\{ (x, y) : y < x - 3 \}$

2 For the function $g\colon [0, 5] \to \mathbb{R}$, $g(x) = \dfrac{x+3}{2}$:

a Sketch the graph of $y = g(x)$. **b** State the range of g.

c Find g^{-1}, stating the domain and range of g^{-1}.

d Find $\{ x : g(x) = 4 \}$. **e** Find $\{ x : g^{-1}(x) = 4 \}$.

3 For $g(x) = 5x + 1$, find:

a $\{ x : g(x) = 2 \}$ **b** $\{ x : g^{-1}(x) = 2 \}$ **c** $\left\{ x : \dfrac{1}{g(x)} = 2 \right\}$

4 Sketch the graph of the function f for which

$$f(x) = \begin{cases} x + 1 & \text{for } x > 2 \\ x^2 - 1 & \text{for } 0 \leq x \leq 2 \\ -x^2 & \text{for } x < 0 \end{cases}$$

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5 Find the implied domain for each of the following:

a $f(x) = \dfrac{1}{2x - 6}$ **b** $g(x) = \dfrac{1}{\sqrt{x^2 - 5}}$ **c** $h(x) = \dfrac{1}{(x-1)(x+2)}$

d $h(x) = \sqrt{25 - x^2}$ **e** $f(x) = \sqrt{x - 5} + \sqrt{15 - x}$ **f** $h(x) = \dfrac{1}{3x - 6}$

6 For $f(x) = (x + 2)^2$ and $g(x) = x - 3$, find $(f + g)(x)$ and $(fg)(x)$.

7 For $f\colon [1, 5] \to \mathbb{R}$, $f(x) = (x - 1)^2$ and $g\colon \mathbb{R} \to \mathbb{R}$, $g(x) = 2x$, find $f + g$ and fg.

8 For $f\colon [3, \infty) \to \mathbb{R}$, $f(x) = x^2 - 1$, find f^{-1}.

9 For $f(x) = 2x + 3$ and $g(x) = -x^2$, find:

a $(f + g)(x)$ **b** $(fg)(x)$ **c** $\{\, x : (f + g)(x) = 0 \,\}$

10 Let $f\colon (-\infty, 2] \to \mathbb{R}$, $f(x) = 3x - 4$. On the one set of axes, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$.

11 Find the inverse of each of the following functions:

a $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 8x^3$ **b** $f\colon (-\infty, 0] \to \mathbb{R}$, $f(x) = 32x^5$

c $f\colon [0, \infty) \to \mathbb{R}$, $f(x) = 64x^6$ **d** $f\colon (1, \infty) \to \mathbb{R}$, $f(x) = 10\,000x^4$

12 For $f(x) = 2x + 3$ and $g(x) = -x^3$, find:

a $f \circ g(x)$ **b** $g \circ f(x)$ **c** $g \circ g(x)$ **d** $f \circ f(x)$

e $f \circ (f + g)(x)$ **f** $f \circ (f - g)(x)$ **g** $f \circ (f \cdot g)(x)$

13 If the function f has the rule $f(x) = \sqrt{x^2 - 16}$ and the function g has rule $g(x) = x + 5$, find the largest domain for g such that $f \circ g$ is defined.

14 For the function h with rule $h(x) = 2x^5 + 64$, find the rule for the inverse function h^{-1}.

Multiple-choice questions

1 For the function with rule $f(x) = \sqrt{6 - 2x}$, which of the following is the maximal domain?

A $(-\infty, 6]$ **B** $[3, \infty)$ **C** $(-\infty, 6]$ **D** $(3, \infty)$ **E** $(-\infty, 3]$

2 For $f\colon [-1, 3) \to \mathbb{R}$, $f(x) = -x^2$, the range is

A $\mathbb{R}$ **B** $(-9, 0]$ **C** $(-\infty, 0]$ **D** $(-9, -1]$ **E** $[-9, 0]$

3 For $f(x) = 3x^2 + 2x$, $f(2a) =$

A $20a^2 + 4a$ **B** $6a^2 + 2a$ **C** $6a^2 + 4a$ **D** $36a^2 + 4a$ **E** $12a^2 + 4a$

4 For $f(x) = 2x - 3$, $f^{-1}(x) =$

A $2x + 3$ **B** $\dfrac{1}{2}x + 3$ **C** $\dfrac{1}{2}x + \dfrac{3}{2}$ **D** $\dfrac{1}{2x - 3}$ **E** $\dfrac{1}{2}x - 3$

5 For $f\colon (a, b] \to \mathbb{R}$, $f(x) = 10 - x$ where $a < b$, the range is

A $(10 - a, 10 - b)$ **B** $(10 - a, 10 - b]$ **C** $(10 - b, 10 - a)$

D $(10 - b, 10 - a]$ **E** $[10 - b, 10 - a)$

6 For the function with rule

$$f(x) = \begin{cases} x^2 + 5 & x \geq 3 \\ -x + 6 & x < 3 \end{cases}$$

the value of $f(a + 3)$, where a is a negative real number, is

A $a^2 + 6a + 14$ **B** $-a + 9$ **C** $-a + 3$ **D** $a^2 + 14$ **E** $a^2 + 8a + 8$

7 Which one of the following sets is a possible domain for the function with rule $f(x) = (x + 3)^2 - 6$ if the inverse function is to exist?

A $\mathbb{R}$ **B** $[-6, \infty)$ **C** $(-\infty, 3]$ **D** $[6, \infty)$ **E** $(-\infty, 0)$

8 For which one of the following functions does an inverse function not exist?

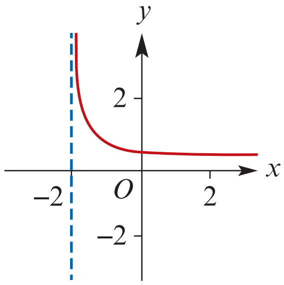
A $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 2x - 4$ **B** $g\colon [-4, 4] \to \mathbb{R}$, $g(x) = \sqrt{16 - x^2}$

C $h\colon [0, \infty) \to \mathbb{R}$, $h(x) = -\dfrac{1}{5}x^2$ **D** $p\colon \mathbb{R}^+ \to \mathbb{R}$, $p(x) = \dfrac{1}{x^2}$

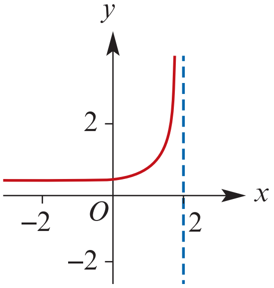
E $q\colon \mathbb{R} \to \mathbb{R}$, $q(x) = 2x^3 - 5$

9 The graph of the function f is shown on the right.

Which one of the following is most likely to be the graph of the inverse function of f?



A



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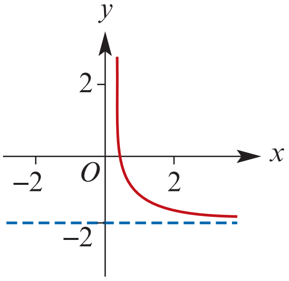

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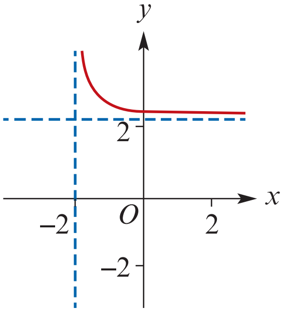

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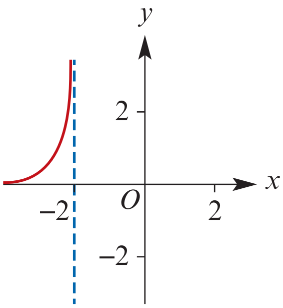
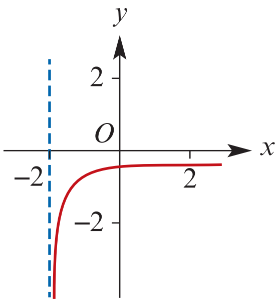


E



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10 The maximal domain and range of $f(x) = \dfrac{2x+1}{x-1}$ are

A $\mathbb{R} \setminus \{0\}, \quad \mathbb{R} \setminus \{2\}$
B $\mathbb{R} \setminus \{1\}, \quad \mathbb{R} \setminus \{-2\}$
C $\mathbb{R} \setminus \{1\}, \quad \mathbb{R} \setminus \{2\}$
D $\mathbb{R} \setminus \{2\}, \quad \mathbb{R} \setminus \{1\}$
E $\mathbb{R} \setminus \{-2\}, \quad \mathbb{R} \setminus \{-1\}$

11 If $f(x) = 3x^2$ and $g(x) = 2x + 1$, then $f(g(a))$ is equal to

A $12a^2 + 3$
B $12a^2 + 12a + 3$
C $6a^2 + 1$
D $6a^2 + 4$
E $4a^2 + 4a + 1$

12 The range of the function $f\colon [-2, 4) \to \mathbb{R}$, $f(x) = x^2 + 2x - 6$ is

A $\mathbb{R}$
B $(-3, 18]$
C $(-6, 18)$
D $[0, 6]$
E $[-7, 18)$

13 Which of the following functions is strictly increasing on the interval $(-\infty, -1]$?

A $f(x) = x^2$
B $f(x) = x^4$
C $f(x) = x^{\frac{1}{5}}$
D $f(x) = \sqrt{4 - x}$
E $f(x) = -x^3$

14 If $f\colon (-1, \infty) \to \mathbb{R}$, $f(x) = \sqrt{x+1}$ and $g\colon (-\infty, 4] \to \mathbb{R}$, $g(x) = \sqrt{4 - x}$, then the maximal domain of the function $f + g$ is

A $\mathbb{R}$
B $(-\infty, -1)$
C $(-1, 4]$
D $(-1, \infty)$
E $[-4, 1)$

15 If $f\colon (2, \infty) \to \mathbb{R}$, $f(x) = \sqrt{2x + 3}$, then the inverse function is

A $f^{-1}\colon (\sqrt{7}, \infty) \to \mathbb{R}$, $f^{-1}(x) = \dfrac{x^2 - 3}{2}$
B $f^{-1}\colon (7, \infty) \to \mathbb{R}$, $f^{-1}(x) = \sqrt{\dfrac{x}{2} - 3}$
C $f^{-1}\colon (\sqrt{7}, \infty) \to \mathbb{R}$, $f^{-1}(x) = \dfrac{x^2 + 3}{2}$
D $f^{-1}\colon (7, \infty) \to \mathbb{R}$, $f^{-1}(x) = \dfrac{x^2 - 3}{2}$
E $f^{-1}\colon (2, \infty) \to \mathbb{R}$, $f^{-1}(x) = \dfrac{x^2 - 2}{3}$

16 The linear function $f\colon D \to \mathbb{R}$, $f(x) = 5 - x$ has range $[-2, 3)$. The domain D is

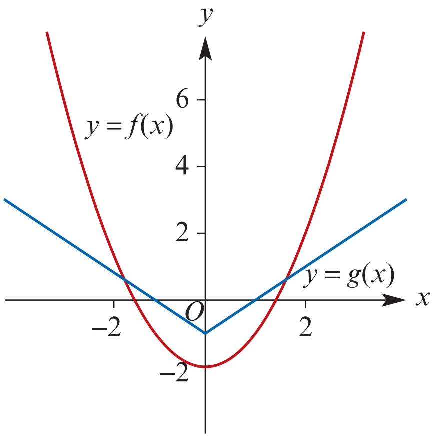
A $[-7, 2)$
B $(2, 7]$
C $\mathbb{R}$
D $[-2, 7)$
E $[2, 7)$

17 The function $g\colon \mathbb{R} \setminus \{3\} \to \mathbb{R}$, where $g(x) = \dfrac{1}{x-3} + 2$, has an inverse g^{-1}. The rule and domain of g^{-1} are

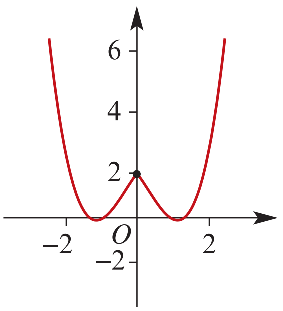
A $g^{-1}(x) = \dfrac{1}{x-2} + 3, \ \operatorname{dom} g^{-1} = \mathbb{R} \setminus \{2\}$
B $g^{-1}(x) = \dfrac{1}{x-2} + 3, \ \operatorname{dom} g^{-1} = \mathbb{R} \setminus \{3\}$
C $g^{-1}(x) = \dfrac{1}{x+2} - 3, \ \operatorname{dom} g^{-1} = \mathbb{R} \setminus \{2\}$
D $g^{-1}(x) = \dfrac{-1}{x+2} - 3, \ \operatorname{dom} g^{-1} = \mathbb{R} \setminus \{3\}$
E $g^{-1}(x) = \dfrac{1}{x-2} + 3, \ \operatorname{dom} g^{-1} = \mathbb{R} \setminus \{-3\}$

18 The graphs of $y = f(x)$ and $y = g(x)$ are as shown on the right.

Which one of the following best represents the graph of $y = f(g(x))$?



A



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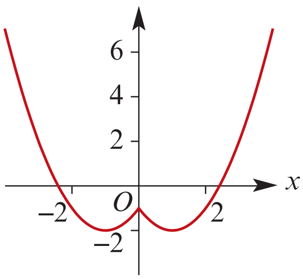

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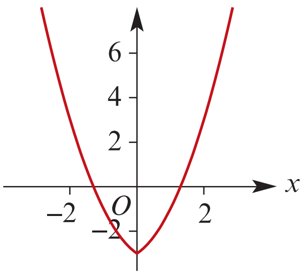

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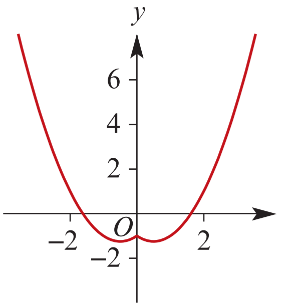

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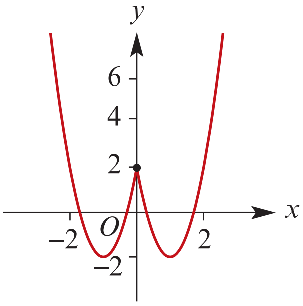

19 Let $g(x) = \dfrac{3}{(x+1)^3} - 2$. The equations of the asymptotes of the inverse function g^{-1} are

A $x = -2,\ y = 1$

B $x = -2,\ y = -1$

C $x = 1,\ y = -2$

D $x = -1,\ y = -2$

E $x = 2,\ y = -1$

20 The equations of the vertical and horizontal asymptotes of the graph with equation $y = \dfrac{-2}{(x+3)^4} - 5$ are

A $x = 3,\ y = -5$

B $x = -5,\ y = -3$

C $x = -3,\ y = -5$

D $x = -2,\ y = -5$

E $x = -3,\ y = 5$

21 Which one of the following functions does not have an inverse function?

A $f\colon [0, \infty) \to \mathbb{R},\ f(x) = (x-2)^2$

B $f\colon \mathbb{R} \to \mathbb{R},\ f(x) = x^3$

C $f\colon [-3, 3] \to \mathbb{R},\ f(x) = \sqrt{9-x}$

D $f\colon \mathbb{R} \to \mathbb{R},\ f(x) = x^{\frac{1}{3}} + 4$

E $f\colon \mathbb{R} \to \mathbb{R},\ f(x) = 3x + 7$

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22 A function with rule $f(x) = \dfrac{1}{x^4}$ can be defined on different domains. Which one of the following does not give the correct range for the given domain?

A $\operatorname{dom} f = [-1, -0.5]$, $\operatorname{ran} f = [1, 16]$

B $\operatorname{dom} f = [-0.5, 0.5] \setminus \{0\}$, $\operatorname{ran} f = [16, \infty)$

C $\operatorname{dom} f = (-0.5, 0.5) \setminus \{0\}$, $\operatorname{ran} f = (16, \infty)$

D $\operatorname{dom} f = [-0.5, 1] \setminus \{0\}$, $\operatorname{ran} f = [1, 16]$

E $\operatorname{dom} f = [0.5, 1)$, $\operatorname{ran} f = (1, 16]$

Extended-response questions

1 Self-Travel, a car rental firm, has two methods of charging for car rental:

Method 1 \$64 per day + 25 cents per kilometre

Method 2 \$89 per day with unlimited travel.

a Write a rule for each method if x kilometres per day are travelled and the cost in dollars is C_1 using method 1 and C_2 using method 2.

b Draw the graph of each, using the same axes.

c Determine, from the graph, the distance that must be travelled per day if method 2 is cheaper than method 1.

2 Express the total surface area, S, of a cube as a function of:

a the length x of an edge

b the volume V of the cube.

3 Express the area, A, of an equilateral triangle as a function of:

a the length s of each side

b the altitude h.

4 The base of a 3 m ladder leaning against a wall is x metres from the wall.

a Express the distance, d, from the top of the ladder to the ground as a function of x and sketch the graph of the function.

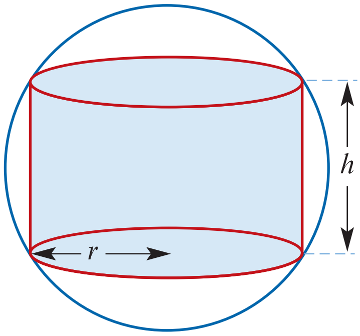
b State the domain and range of the function.

5 A car travels half the distance of a journey at an average speed of 80 km/h and half at an average speed of x km/h. Define a function, S, which gives the average speed for the total journey as a function of x.

6 A cylinder is inscribed in a sphere with a radius of length 6 cm.

a Define a function, V_1, which gives the volume of the cylinder as a function of its height, h. (State the rule and domain.)

b Define a function, V_2, which gives the volume of the cylinder as a function of the radius of the cylinder, r. (State the rule and domain.)



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7 Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$, where $f(x) = x + 1$ and $g(x) = 2 + x^3$.

a State why $g \circ f$ exists and find $g \circ f(x)$.

b State why $(g \circ f)^{-1}$ exists and find $(g \circ f)^{-1}(10)$.

8 A function f is defined as follows:

$$f(x) = \begin{cases} x^2 - 4 & \text{for } x \in (-\infty, 2) \\ x & \text{for } x \in [2, \infty) \end{cases}$$

a Sketch the graph of f.

b Find the value of:

i $f(-1)$ **ii** $f(3)$

c Given $g: S \to \mathbb{R}$ where $g(x) = f(x)$, find the largest set S such that the inverse of g exists and $-1 \in S$.

d If $h(x) = 2x$, find $f(h(x))$ and $h(f(x))$.

9 Find the rule for the area, $A(t)$, enclosed by the graph of the function

$$f(x) = \begin{cases} 3x, & 0 \le x \le 1 \\ 3, & x > 1 \end{cases}$$

the x-axis, the y-axis and the vertical line $x = t$ (for $t > 0$). State the domain and range of the function A.

10 Let $f: \mathbb{R} \setminus \left\{\frac{-d}{c}\right\} \to \mathbb{R}$, $f(x) = \dfrac{ax + b}{cx + d}$.

a Find the inverse function f^{-1}.

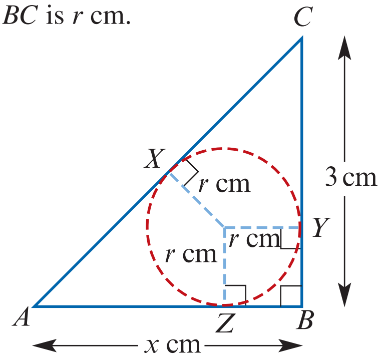
b Find the inverse function when:

i $a = 3,\ b = 2,\ c = 3,\ d = 1$ **ii** $a = 3,\ b = 2,\ c = 2,\ d = -3$

iii $a = 1,\ b = -1,\ c = -1,\ d = -1$ **iv** $a = -1,\ b = 1,\ c = 1,\ d = 1$

c Determine the possible values of a, b, c and d if $f = f^{-1}$.

11 The radius of the incircle of the right-angled triangle ABC is r cm.



a Find:

i YB in terms of r

ii ZB in terms of r

iii AZ in terms of r and x

iv CY

b Use the geometric results $CY = CX$ and $AX = AZ$ to find an expression for r in terms of x.

c **i** Find r when $x = 4$.

ii Find x when $r = 0.5$.

d Use a CAS calculator to investigate the possible values r can take.

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12 Let $f(x) = \dfrac{px + q}{x + r}$ where $x \in \mathbb{R} \setminus \{-r, r\}$.

- **a** If $f(x) = f(-x)$ for all x, show that $f(x) = p$ for $x \in \mathbb{R} \setminus \{-r, r\}$.
- **b** If $f(-x) = -f(x)$ for $x \neq 0$, find the rule for $f(x)$ in terms of q.
- **c** If $p = 3$, $q = 8$ and $r = -3$:
  - **i** find the inverse function of f
  - **ii** find the values of x for which $f(x) = x$.

13
- **a** Let $f(x) = \dfrac{x + 1}{x - 1}$.
  - **i** Find $f(2)$, $f(f(2))$ and $f(f(f(2)))$.
  - **ii** Find $f(f(x))$.
- **b** Let $f(x) = \dfrac{x - 3}{x + 1}$. Find $f(f(x))$ and $f(f(f(x)))$.

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2

Coordinate geometry and matrices

Objectives

- To revise:
    - methods for solving **linear equations**
    - methods for solving **simultaneous linear equations**
    - finding the **distance** between two points
    - finding the **midpoint** of a line segment
    - calculating the **gradient** of a straight line
    - interpreting and using different forms of the **equation of a straight line**
    - finding the **angle of slope** of a straight line
    - determining the gradient of a line **perpendicular** to a given line
    - **matrix** arithmetic.
- To apply a knowledge of **linear functions** to solving problems.

Much of the material presented in this chapter has been covered in Mathematical Methods Units 1 & 2. The chapter provides a framework for revision with worked examples and practice exercises.

There is also a section on the solution of simultaneous linear equations with more than two variables. The use of a CAS calculator to solve such systems of equations is emphasised.

The use of matrices in this course is confined to the description of transformations of the plane, which is covered in Chapter 3. Additional material on matrices is available in the Interactive Textbook.

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2A Linear equations

This section contains exercises in linear equations. The worded problems provide an opportunity to practise the important skill of going from a problem expressed in English to a mathematical formulation of the problem.

Section summary

- An equation is solved by finding the value or values of the variables that would make the statement true.
- A linear equation is one in which the variable is to the first power.
- There are often several different ways to solve a linear equation. The following steps provide some suggestions:
  - Expand brackets and, if the equation involves fractions, multiply through by the lowest common denominator of the terms.
  - Group all of the terms containing a variable on one side of the equation and the terms without the variable on the other side.
- Steps for solving a word problem with a linear equation:
  - Read the question carefully and write down the known information clearly.
  - Identify the unknown quantity that is to be found.
  - Assign a variable to this quantity.
  - Form an expression in terms of x (or the variable being used) and use the other relevant information to form the equation.
  - Solve the equation.
  - Write a sentence answering the initial question.

Exercise 2A

Skillsheet

1 Solve the following linear equations:

a $3x - 4 = 2x + 6$
b $8x - 4 = 3x + 1$
c $3(2 - x) - 4(3 - 2x) = 14$
d $\dfrac{3x}{4} - 4 = 17$
e $6 - 3y = 5y - 62$
f $\dfrac{2}{3x - 1} = \dfrac{3}{7}$
g $\dfrac{2x - 1}{3} = \dfrac{x + 1}{4}$
h $\dfrac{2(x - 1)}{3} - \dfrac{x + 4}{2} = \dfrac{5}{6}$
i $4y - \dfrac{3y + 4}{2} + \dfrac{1}{3} = \dfrac{5(4 - y)}{3}$
j $\dfrac{x + 1}{2x - 1} = \dfrac{3}{4}$

2 Solve each of the following pairs of simultaneous linear equations:

a $x - 4 = y$
$4y - 2x = 8$

b $9x + 4y = 13$
$2x + y = 2$

c $7x = 18 + 3y$
$2x + 5y = 11$

d $5x + 3y = 13$
$7x + 2y = 16$

e $19x + 17y = 0$
$2x - y = 53$

f $\dfrac{x}{5} + \dfrac{y}{2} = 5$
$x - y = 4$

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3 The length of a rectangle is 4 cm more than the width. If the length were to be decreased by 5 cm and the width decreased by 2 cm, the perimeter would be 18 cm. Calculate the dimensions of the rectangle.

4 In a basketball game, a field goal scores two points and a free throw scores one point. John scored 11 points and David 19 points. David scored the same number of free throws as John, but twice as many field goals. How many field goals did each score?

5 The weekly wage, $\$w$, of a sales assistant consists of a fixed amount of \$800 and then \$20 for each unit he sells.

- **a** If he sells n units a week, find a rule for his weekly wage, w, in terms of the number of units sold.
- **b** Find his wage if he sells 30 units.
- **c** How many units does he sell if his weekly wage is \$1620?

6 Water flows into a tank at a rate of 15 litres per minute. At the beginning, the tank contained 250 litres.

- **a** Write an expression for the volume, V litres, of water in the tank at time t minutes.
- **b** How many litres of water are there in the tank after an hour?
- **c** The tank has a capacity of 5000 litres. How long does it take to fill?

7 A tank contains 10 000 litres of water. Water flows out at a rate of 10 litres per minute.

- **a** Write an expression for the volume, V litres, of water in the tank at time t minutes.
- **b** How many litres of water are there in the tank after an hour?
- **c** How long does it take for the tank to empty?

8 An aircraft, used for fire spotting, flies from its base to locate a fire at an unknown distance, x km away. It travels straight to the fire and back, averaging 240 km/h for the outward trip and 320 km/h for the return trip. If the plane was away for 35 minutes, find the distance, x km.

9 A group of hikers is to travel x km by bus at an average speed of 48 km/h to an unknown destination. They then plan to walk back along the same route at an average speed of 4.8 km/h and to arrive back 24 hours after setting out in the bus. If they allow 2 hours for lunch and rest, how far must the bus take them?

10 The cost of hiring diving equipment is \$100 plus \$25 per hour.

- **a** Write a rule which gives the total charge, $\$C$, of hiring the equipment for t hours (assume that parts of hours are paid for proportionately).
- **b** Find the cost of hiring the equipment for:
  - **i** 2 hours **ii** 2 hours 30 minutes
- **c** For how many hours can the equipment be hired if the following amounts are available?
  - **i** \$375 **ii** \$400

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2B Linear literal equations and simultaneous linear literal equations

A literal equation in x is an equation whose solution will be expressed in terms of pronumerals rather than numbers.

For the equation $2x + 5 = 7$, the solution is the number 1.

For the literal equation $ax + b = c$, the solution is $x = \dfrac{c - b}{a}$.

Literal equations are solved in the same way as numerical equations. Essentially, the literal equation is transposed to make x the subject.

Example 1

Solve the following for x:

a $px - q = r$ **b** $ax + b = cx + d$ **c** $\dfrac{a}{x} = \dfrac{b}{2x} + c$

Solution

a
$$\begin{aligned} px - q &= r \\ px &= r + q \\ \therefore \quad x &= \frac{r + q}{p} \end{aligned}$$

b
$$\begin{aligned} ax + b &= cx + d \\ ax - cx &= d - b \\ x(a - c) &= d - b \\ \therefore \quad x &= \frac{d - b}{a - c} \end{aligned}$$

c Multiply both sides of the equation by $2x$:
$$\begin{aligned} 2a &= b + 2xc \\ 2a - b &= 2xc \\ \therefore \quad x &= \frac{2a - b}{2c} \end{aligned}$$

Simultaneous literal equations are solved by the usual methods of solution of simultaneous equations: substitution and elimination.

Example 2

Solve the following simultaneous equations for x and y:

$$y = ax + c$$
$$y = bx + d$$

Solution

$$\begin{aligned} ax + c &= bx + d \qquad \text{(Equate the two expressions for } y\text{.)} \\ ax - bx &= d - c \\ x(a - b) &= d - c \end{aligned}$$

Thus $\quad x = \dfrac{d - c}{a - b}$

and
$$\begin{aligned} y &= a\left(\frac{d - c}{a - b}\right) + c \\ &= \frac{ad - ac + ac - bc}{a - b} = \frac{ad - bc}{a - b} \end{aligned}$$

Example 3

Solve the simultaneous equations $ax - y = c$ and $x + by = d$ for x and y.

Solution

$$ax - y = c \qquad (1)$$

$$x + by = d \qquad (2)$$

Multiply (1) by b:

$$abx - by = bc \qquad (1')$$

Add (1′) and (2):

$$\begin{aligned} abx + x &= bc + d \\ x(ab + 1) &= bc + d \\ \therefore \quad x &= \frac{bc + d}{ab + 1} \end{aligned}$$

Using equation (1):

$$\begin{aligned} y &= ax - c \\ &= a\left(\frac{bc + d}{ab + 1}\right) - c = \frac{ad - c}{ab + 1} \end{aligned}$$

Section summary

- An equation for the variable x in which all the coefficients of x, including the constants, are pronumerals is known as a **literal equation**.
- The methods for solving linear literal equations or pairs of simultaneous linear literal equations are exactly the same as when the coefficients are given numbers.

Exercise 2B

Example 1

1 Solve each of the following for x:

a $ax + n = m$

b $ax + b = bx$

c $\frac{ax}{b} + c = 0$

d $px = qx + 5$

e $mx + n = nx - m$

f $\frac{1}{x + a} = \frac{b}{x}$

g $\frac{b}{x - a} = \frac{2b}{x + a}$

h $\frac{x}{m} + n = \frac{x}{n} + m$

i $-b(ax + b) = a(bx - a)$

j $p^2(1 - x) - 2pqx = q^2(1 + x)$

k $\frac{x}{a} - 1 = \frac{x}{b} + 2$

l $\frac{x}{a - b} + \frac{2x}{a + b} = \frac{1}{a^2 - b^2}$

m $\frac{p - qx}{t} + p = \frac{qx - t}{p}$

n $\frac{1}{x + a} + \frac{1}{x + 2a} = \frac{2}{x + 3a}$

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Example 2, 3

2 Solve each of the following pairs of simultaneous equations for x and y:

a $ax + y = c$
$x + by = d$

b $ax - by = a^2$
$bx - ay = b^2$

c $ax + by = t$
$ax - by = s$

d $ax + by = a^2 + 2ab - b^2$
$bx + ay = a^2 + b^2$

e $(a + b)x + cy = bc$
$(b + c)y + ax = -ab$

f $3(x - a) - 2(y + a) = 5 - 4a$
$2(x + a) + 3(y - a) = 4a - 1$

3 For each of the following pairs of equations, write s in terms of a only:

a $s = ah$
$h = 2a + 1$

b $s = ah$
$h = a(2 + h)$

c $as = a + h$
$h + ah = 1$

d $as = s + h$
$ah = a + h$

e $s = h^2 + ah$
$h = 3a^2$

f $as = a + 2h$
$h = a - s$

g $s = 2 + ah + h^2$
$h = a - \frac{1}{a}$

h $3s - ah = a^2$
$as + 2h = 3a$

4 For the simultaneous equations $ax + by = p$ and $bx - ay = q$, show that $x = \dfrac{ap + bq}{a^2 + b^2}$ and $y = \dfrac{bp - aq}{a^2 + b^2}$.

5 For the simultaneous equations $\dfrac{x}{a} + \dfrac{y}{b} = 1$ and $\dfrac{x}{b} + \dfrac{y}{a} = 1$, show that $x = y = \dfrac{ab}{a + b}$.

2C Linear coordinate geometry

In this section we revise the concepts of linear coordinate geometry.

Example 4

A straight line passes through the points $A(-2, 6)$ and $B(4, 7)$. Find:

a the distance AB

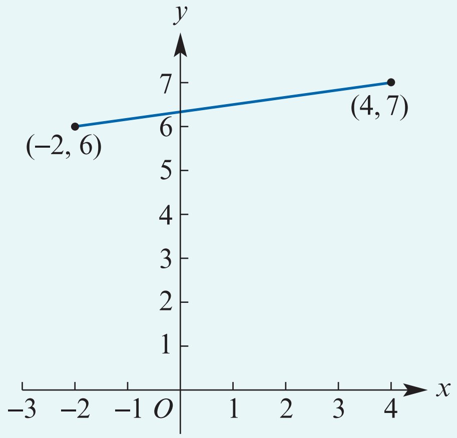
b the midpoint of line segment AB

c the gradient of line AB

d the equation of line AB

e the equation of the line parallel to AB which passes through the point $(1, 5)$

f the equation of the line perpendicular to AB which passes through the midpoint of AB.



Solution

a The distance AB is

$$\sqrt{(4 - (-2))^2 + (7 - 6)^2} = \sqrt{37}$$

Explanation

The distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

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b The midpoint of AB is

$$\left(\frac{-2+4}{2}, \frac{6+7}{2}\right) = \left(1, \frac{13}{2}\right)$$

The line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$ has midpoint $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

c The gradient of line AB is

$$\frac{7-6}{4-(-2)} = \frac{1}{6}$$

Gradient

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

d The equation of line AB is

$$y - 6 = \frac{1}{6}(x - (-2))$$

which simplifies to $6y - x - 38 = 0$.

Equation of a straight line passing through a given point (x_1, y_1) and having gradient m is $y - y_1 = m(x - x_1)$.

e Gradient $m = \frac{1}{6}$ and $(x_1, y_1) = (1, 5)$.

The line has equation

$$y - 5 = \frac{1}{6}(x - 1)$$

which simplifies to $6y - x - 29 = 0$.

Parallel lines have the same gradient.

f A perpendicular line has gradient -6.

Thus the equation is

$$y - \frac{13}{2} = -6(x - 1)$$

which simplifies to $2y + 12x - 25 = 0$.

If two straight lines are perpendicular to each other, then the product of their gradients is -1.

Example 5

A fruit and vegetable wholesaler sells 30 kg of hydroponic tomatoes for \$148.50 and sells 55 kg of hydroponic tomatoes for \$247.50. Find a linear model for the cost, \$$C$, of x kg of hydroponic tomatoes. How much would 20 kg of tomatoes cost?

Solution

Let $(x_1, C_1) = (30, 148.5)$ and $(x_2, C_2) = (55, 247.5)$.

The equation of the straight line is given by

$$C - C_1 = m(x - x_1) \quad \text{where } m = \frac{C_2 - C_1}{x_2 - x_1}$$

Now $m = \dfrac{247.5 - 148.5}{55 - 30} = 3.96$ and so

$$C - 148.5 = 3.96(x - 30)$$

Therefore the straight line has equation $C = 3.96x + 29.7$.

Substitute $x = 20$:

$$C = 3.96 \times 20 + 29.7 = 108.9$$

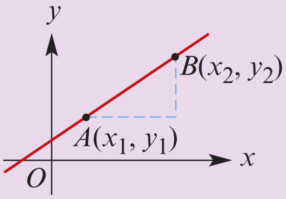
Hence it would cost \$108.90 to buy 20 kg of tomatoes.

The following is a summary of the material that is assumed to have been covered in Mathematical Methods Units 1 & 2.

Section summary

- **Distance between two points**

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



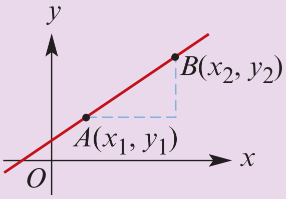
- **Midpoint of a line segment**

The midpoint of the line segment joining two points (x_1, y_1) and (x_2, y_2) is the point with coordinates

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

- **Gradient of a straight line**

Gradient $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

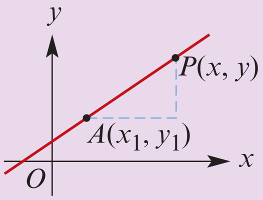


- **Equation of a straight line**
  - Gradient–intercept form: A straight line with gradient m and y-axis intercept c has equation

$$y = mx + c$$

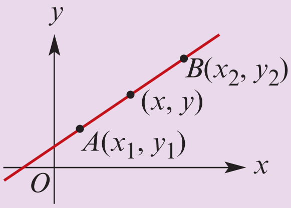
  - The equation of a straight line passing through a given point (x_1, y_1) and having gradient m is

$$y - y_1 = m(x - x_1)$$



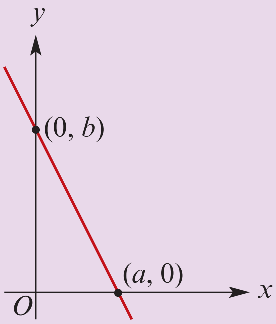
  - The equation of a straight line passing through two given points (x_1, y_1) and (x_2, y_2) is

$$y - y_1 = m(x - x_1) \quad \text{where } m = \frac{y_2 - y_1}{x_2 - x_1}$$



  - Intercept form: The straight line passing through the two points $(a, 0)$ and $(0, b)$ has equation

$$\frac{x}{a} + \frac{y}{b} = 1$$



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- **Tangent of the angle of slope**
  For a straight line with gradient m, the angle of slope is found using
  $$m = \tan\theta$$
  where θ is the angle that the line makes with the positive direction of the x-axis.
- **Perpendicular straight lines**
  If two straight lines are perpendicular to each other, the product of their gradients is -1, i.e. $m_1 m_2 = -1$. (Unless one line is vertical and the other horizontal.)

Exercise 2C

Example 4

Skillsheet

1 A straight line passes through the points $A(-2, 6)$ and $B(4, -7)$. Find:

- **a** the distance AB
- **b** the midpoint of line segment AB
- **c** the gradient of line AB
- **d** the equation of line AB
- **e** the equation of the line parallel to AB which passes through the point $(1, 5)$
- **f** the equation of the line perpendicular to AB which passes through the midpoint of AB.

2 Find the coordinates of M, the midpoint of AB, where A and B have the following coordinates:

a $A(1, 4),\ B(5, 11)$　**b** $A(-6, 4),\ B(1, -8)$　**c** $A(-1, -6),\ B(4, 7)$

3 If M is the midpoint of XY, find the coordinates of Y when X and M have the following coordinates:

a $X(-4, 5),\ M(0, 6)$　**b** $X(-1, -4),\ M(2, -3)$

c $X(6, -3),\ M(4, 8)$　**d** $X(2, -3),\ M(0, -6)$

4 Use $y = mx + c$ to sketch the graph of each of the following:

a $y = 3x - 3$　**b** $y = -3x + 4$　**c** $3y + 2x = 12$

d $4x + 6y = 12$　**e** $3y - 6x = 18$　**f** $8x - 4y = 16$

5 Find the equations of the following straight lines:

- **a** gradient $+2$, passing through $(4, 2)$
- **b** gradient -3, passing through $(-3, 4)$
- **c** passing through the points $(1, 3)$ and $(4, 7)$
- **d** passing through the points $(-2, -3)$ and $(2, 5)$

6 Use the intercept method to find the equations of the straight lines passing through:

a $(-3, 0)$ and $(0, 2)$　**b** $(4, 0)$ and $(0, 6)$

c $(-4, 0)$ and $(0, -3)$　**d** $(0, -2)$ and $(6, 0)$

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7 Write the following in intercept form and hence draw their graphs:

a $3x + 6y = 12$ **b** $4y - 3x = 12$ **c** $4y - 2x = 8$ **d** $\frac{3}{2}x - 3y = 9$

Example 5 **8** A printing firm charges \$35 for printing 600 sheets of headed notepaper and \$46 for printing 800 sheets. Find a linear model for the charge, \$$C$, for printing n sheets. How much would they charge for printing 1000 sheets?

9 An electronic bank teller registered \$775 after it had counted 120 notes and \$975 after it had counted 160 notes.

a Find a formula for the sum registered (\$$C$) in terms of the number of notes (n) counted.

b Was there a sum already on the register when counting began?

c If so, how much?

10 Find the distance between each of the following pairs of points:

a $(2, 6)$, $(3, 4)$ **b** $(5, 1)$, $(6, 2)$ **c** $(-1, 3)$, $(4, 5)$

d $(-1, 7)$, $(1, -11)$ **e** $(-2, -6)$, $(2, -8)$ **f** $(0, 4)$, $(3, 0)$

11 **a** Find the equation of the straight line which passes through the point $(1, 6)$ and is:

- **i** parallel to the line with equation $y = 2x + 3$
- **ii** perpendicular to the line with equation $y = 2x + 3$.

b Find the equation of the straight line which passes through the point $(2, 3)$ and is:

- **i** parallel to the line with equation $4x + 2y = 10$
- **ii** perpendicular to the line with equation $4x + 2y = 10$.

12 Find the equation of the line which passes through the point of intersection of the lines $y = x$ and $x + y = 6$ and which is perpendicular to the line with equation $3x + 6y = 12$.

13 The length of the line segment joining $A(2, -1)$ and $B(5, y)$ is 5 units. Find y.

14 The length of the line segment joining $A(2, 6)$ and $B(10, y)$ is 10 units. Find y.

15 The length of the line segment joining $A(2, 8)$ and $B(12, y)$ is 26 units. Find y.

16 Find the equation of the line passing through the point $(-1, 3)$ which is:

a **i** parallel to the line with equation $2x + 5y - 10 = 0$

ii parallel to the line with equation $4x + 5y + 3 = 0$

b **i** perpendicular to the line with equation $2x + 5y - 10 = 0$

ii perpendicular to the line with equation $4x + 5y + 3 = 0$.

17 For each of the following, find the angle that the line joining the given points makes with the positive direction of the x-axis:

a $(-4, 1)$, $(4, 6)$ **b** $(2, 3)$, $(-4, 6)$ **c** $(5, 1)$, $(-1, -8)$ **d** $(-4, 2)$, $(2, -8)$

18 Find the acute angle between the lines $y = 2x + 4$ and $y = -3x + 6$.

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19 Given the points $A(a, 3)$, $B(-2, 1)$ and $C(3, 2)$, find the possible values of a if the length of AB is twice the length of BC.

20 Three points have coordinates $A(1, 7)$, $B(7, 5)$ and $C(0, -2)$. Find:

a the equation of the perpendicular bisector of AB

b the point of intersection of this perpendicular bisector and BC.

21 The point (h, k) lies on the line $y = x + 1$ and is 5 units from the point $(0, 2)$. Write down two equations connecting h and k and hence find the possible values of h and k.

22 P and Q are the points of intersection of the line $\frac{y}{2} + \frac{x}{3} = 1$ with the x- and y-axes respectively. The gradient of QR is $\frac{1}{2}$ and the point R has x-coordinate $2a$, where $a > 0$.

a Find the y-coordinate of R in terms of a.

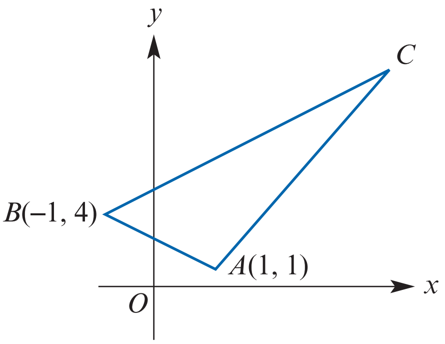
b Find the value of a if the gradient of PR is -2.

23 The figure shows a triangle ABC with $A(1, 1)$ and $B(-1, 4)$. The gradients of AB, AC and BC are $-3m$, $3m$ and m respectively.

a Find the value of m.

b Find the coordinates of C.

c Show that $AC = 2AB$.



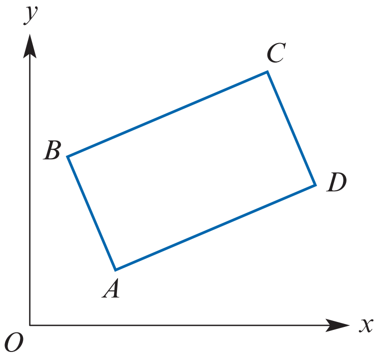
24 In the rectangle $ABCD$, the points A and B are $(4, 2)$ and $(2, 8)$ respectively. Given that the equation of AC is $y = x - 2$, find:

a the equation of BC

b the coordinates of C

c the coordinates of D

d the area of rectangle $ABCD$.



25 $ABCD$ is a parallelogram, with vertices labelled anticlockwise, such that A and C are the points $(-1, 5)$ and $(5, 1)$ respectively.

a Find the coordinates of the midpoint of AC.

b Given that BD is parallel to the line with equation $y + 5x = 2$, find the equation of BD.

c Given that BC is perpendicular to AC, find:

- **i** the equation of BC
- **ii** the coordinates of B
- **iii** the coordinates of D.

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2D Applications of linear functions

In this section, we revise applications of linear functions.

Example 6

There are two possible methods for paying gas bills:

Method A A fixed charge of \$25 per quarter + 50c per unit of gas used

Method B A fixed charge of \$50 per quarter + 25c per unit of gas used

Determine the number of units which must be used before method B becomes cheaper than method A.

Solution

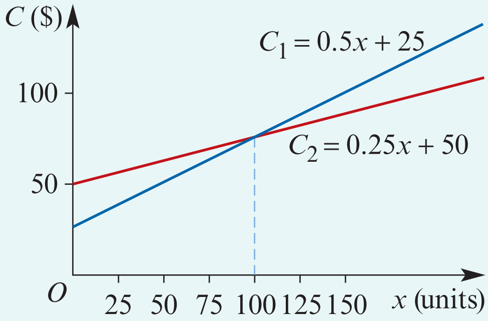
Let C_1 = charge (\$) using method A

C_2 = charge (\$) using method B

x = number of units of gas used

Then $C_1 = 25 + 0.5x$

$C_2 = 50 + 0.25x$



From the graph, we see that method B is cheaper if the number of units exceeds 100.

The solution can also be obtained by solving simultaneous linear equations:

$$\begin{aligned} C_1 &= C_2 \\ 25 + 0.5x &= 50 + 0.25x \\ 0.25x &= 25 \\ x &= 100 \end{aligned}$$

Exercise 2D

Example 6 Skillsheet

1 On a small island two rival taxi firms have the following fare structures:

Firm A Fixed charge of \$1 plus 40 cents per kilometre

Firm B 60 cents per kilometre, no fixed charge

a Find an expression for C_A, the charge of firm A, in terms of n, the number of kilometres travelled, and an expression for C_B, the charge of firm B, in terms of the number of kilometres travelled.

b On the one set of axes, sketch the graphs of the charge of each firm against the number of kilometres travelled.

c Find the distance for which the two firms charge the same amount.

d On a new set of axes, sketch the graph of $D = C_A - C_B$ against n, and explain what this graph represents.

2 A car journey of 300 km lasts 4 hours. Part of this journey is on a freeway at an average speed of 90 km/h. The rest is on country roads at an average speed of 70 km/h. Let T be the time (in hours) spent on the freeway.

a In terms of T, state the number of hours travelling on country roads.

b **i** State the distance travelled on the freeway in terms of T.

ii State the distance travelled on country roads in terms of T.

c **i** Find T.

ii Find the distance travelled on each type of road.

3 A farmer measured the quantity of water in a storage tank 20 days after it was filled and found it contained 3000 litres. After a further 15 days it was measured again and found to contain 1200 litres of water. Assume that the amount of water in the tank decreases at a constant rate.

a Find the relation between L, the number of litres of water in the tank, and t, the number of days after the tank was filled.

b How much water does the tank hold when it is full?

c Sketch the graph of L against t for a suitable domain.

d State this domain.

e How long does it take for the tank to empty?

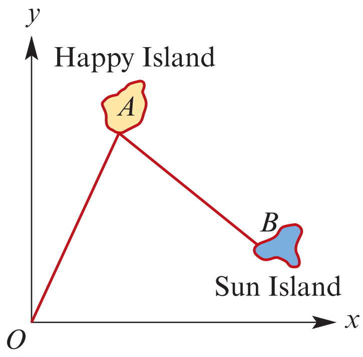
f At what rate does the water leave the tank?

4 A boat leaves from O to sail to two islands. The boat arrives at a point A on Happy Island with coordinates $(10, 22.5)$, where units are in kilometres.

a Find the equation of the line through points O and A.

b Find the distance OA to the nearest metre.

The boat arrives at Sun Island at point B. The coordinates of point B are $(23, 9)$.



c Find the equation of line AB.

d A third island lies on the perpendicular bisector of line segment AB. Its port is denoted by C. It is known that the x-coordinate of C is 52. Find the y-coordinate of the point C.

5 $ABCD$ is a parallelogram with vertices $A(2, 2)$, $B(1.5, 4)$ and $C(6, 6)$.

a Find the gradient of:

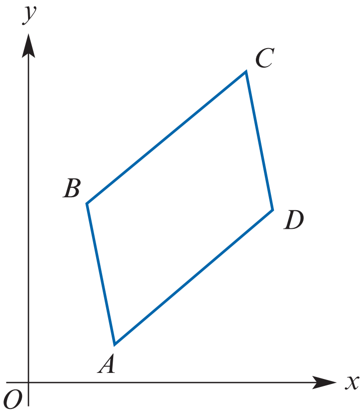
i line AB **ii** line AD

b Find the equation of:

i line BC **ii** line CD

c Find the equations of the diagonals AC and BD.

d Find the coordinates of the point of intersection of the diagonals.



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6 The triangle ABC is isosceles. The vertices are $A(5, 0)$, $B(13, 0)$ and $C(9, 10)$.

a Find the coordinates of the midpoints M and N of AC and BC respectively.

b Find the equation of the lines:

- **i** AC
- **ii** BC
- **iii** MN

c Find the equations of the lines perpendicular to AC and BC, passing through the points M and N respectively, and find the coordinates of their intersection point.

2E Matrices

This section provides a brief introduction to matrices. In Chapter 3, we will see that the transformations we consider in this course can be determined through matrix arithmetic. We will consider the inverse of a 2×2 matrix only in the context of transformations; this is done in Section 3J. Additional information and exercises on matrices are available in the Interactive Textbook.

Matrix notation

A **matrix** is a rectangular array of numbers. The numbers in the array are called the **entries** of the matrix. The following are examples of matrices:

$$\begin{bmatrix} -3 & 4 \\ 5 & 6 \end{bmatrix} \qquad \begin{bmatrix} 6 \\ 7 \end{bmatrix} \qquad \begin{bmatrix} \sqrt{2} & \pi & 3 \\ 0 & 0 & 1 \\ \sqrt{2} & 0 & \pi \end{bmatrix} \qquad \begin{bmatrix} 5 \end{bmatrix}$$

The size, or **dimension**, of a matrix is described by specifying the number of **rows** (horizontal lines) and **columns** (vertical lines).

The dimensions of the above matrices are, in order:

$$2 \times 2, \qquad 2 \times 1, \qquad 3 \times 3, \qquad 1 \times 1$$

The first number represents the number of rows, and the second the number of columns.

In this book we are only interested in 2×2 matrices and 2×1 matrices.

If $\mathbf{A}$ is a matrix, then a_{ij} will be used to denote the entry that occurs in row i and column j of $\mathbf{A}$. Thus a 2×2 matrix may be written as

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

A general 2×1 matrix may be written as

$$\mathbf{B} = \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix}$$

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A matrix is, then, a way of recording a set of numbers, arranged in a particular way. As in Cartesian coordinates, the order of the numbers is significant. Although the matrices

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

have the same numbers and the same number of entries, they are different matrices (just as (2, 1) and (1, 2) are the coordinates of different points).

Two matrices $\mathbf{A}$ and $\mathbf{B}$ are **equal**, and we can write $\mathbf{A} = \mathbf{B}$, when:

- they have the same number of rows and the same number of columns, and
- they have the same number or entry at corresponding positions.

Addition, subtraction and multiplication by a scalar

Addition is defined for two matrices only when they have the same dimension. In this case, the sum of the two matrices is found by adding corresponding entries.

For example,

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 4 & 3 \end{bmatrix}$$

and $\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} \\ a_{21} + b_{21} \end{bmatrix}$

Subtraction is defined in a similar way: If two matrices have the same dimension, then their difference is found by subtracting corresponding entries.

Example 7

Find:

a $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -4 & 1 \end{bmatrix}$

b $\begin{bmatrix} 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Solution

a $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$

b $\begin{bmatrix} 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

It is useful to define **multiplication of a matrix by a real number**. If $\mathbf{A}$ is an $m \times n$ matrix and k is a real number (also called a **scalar**), then $k\mathbf{A}$ is an $m \times n$ matrix whose entries are k times the corresponding entries of $\mathbf{A}$. Thus

$$3\begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -6 \\ 0 & 3 \end{bmatrix}$$

These definitions have the helpful consequence that, if a matrix is added to itself, the result is twice the matrix, i.e. $\mathbf{A} + \mathbf{A} = 2\mathbf{A}$. Similarly, the sum of n matrices each equal to $\mathbf{A}$ is $n\mathbf{A}$ (where n is a natural number).

The $m \times n$ matrix with all entries equal to zero is called the **zero matrix**.

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Example 8

If $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & -4 \\ -2 & 8 \end{bmatrix}$, find the matrix $\mathbf{X}$ such that $2\mathbf{A} + \mathbf{X} = \mathbf{B}$.

Solution

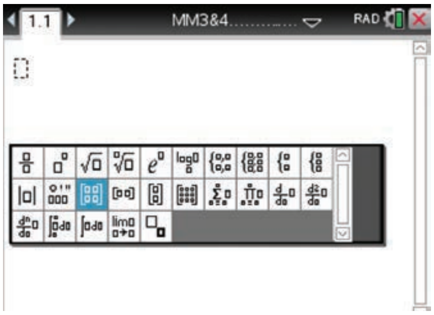
If $2\mathbf{A} + \mathbf{X} = \mathbf{B}$, then $\mathbf{X} = \mathbf{B} - 2\mathbf{A}$. Therefore

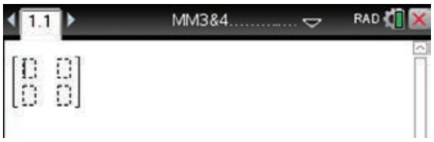
$$\begin{aligned}\mathbf{X} &= \begin{bmatrix} 0 & -4 \\ -2 & 8 \end{bmatrix} - 2\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 - 2 \times 3 & -4 - 2 \times 2 \\ -2 - 2 \times (-1) & 8 - 2 \times 1 \end{bmatrix} \\ &= \begin{bmatrix} -6 & -8 \\ 0 & 6 \end{bmatrix}\end{aligned}$$

Using the TI-Nspire

The matrix template

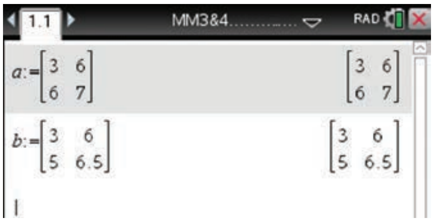
- The simplest way to enter a 2 × 2 matrix is using the 2 × 2 matrix template as shown. (Access the templates using either ⊞ or ctrl menu > **Math Templates**.)
- Notice that there is also a template for entering $m \times n$ matrices.





- Define the matrix $\mathbf{A} = \begin{bmatrix} 3 & 6 \\ 6 & 7 \end{bmatrix}$ as shown.

  The assignment symbol := is accessed using ctrl ⊞. Use the touchpad arrows to move between the entries of the matrix.
- Define the matrix $\mathbf{B} = \begin{bmatrix} 3 & 6 \\ 5 & 6.5 \end{bmatrix}$ similarly.



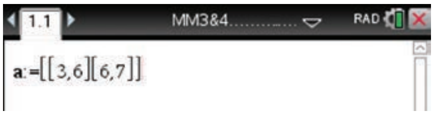
$a:=\begin{bmatrix} 3 & 6 \\ 6 & 7 \end{bmatrix}$ $\begin{bmatrix} 3 & 6 \\ 6 & 7 \end{bmatrix}$

$b:=\begin{bmatrix} 3 & 6 \\ 5 & 6.5 \end{bmatrix}$ $\begin{bmatrix} 3 & 6 \\ 5 & 6.5 \end{bmatrix}$

Note: All variables will be changed to lower case.
Alternatively, you can store (ctrl var) the matrices if preferred.

Entering matrices directly

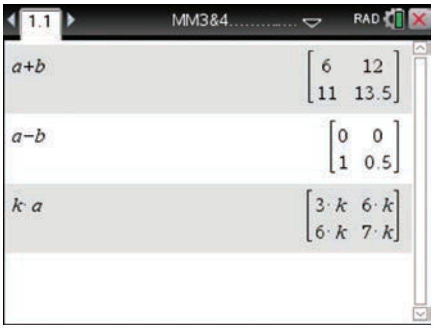
- To enter matrix **A** without using the template, enter the matrix row by row as [[3, 6][6, 7]].



```
a:=[[3,6][6,7]]
```

Addition, subtraction and multiplication by a scalar

- Once **A** and **B** are defined as above, the matrices **A** + **B**, **A** − **B** and k**A** can easily be determined.



$a+b$ $\begin{bmatrix} 6 & 12 \\ 11 & 13.5 \end{bmatrix}$

$a-b$ $\begin{bmatrix} 0 & 0 \\ 1 & 0.5 \end{bmatrix}$

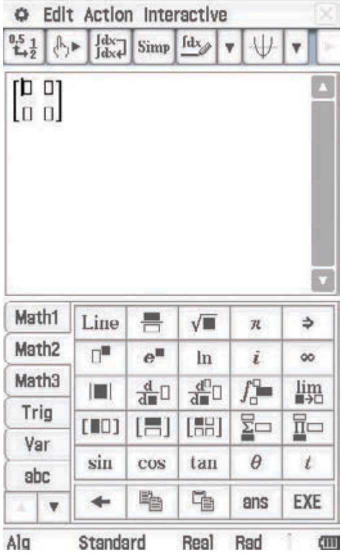
$k\cdot a$ $\begin{bmatrix} 3\cdot k & 6\cdot k \\ 6\cdot k & 7\cdot k \end{bmatrix}$

Using the Casio ClassPad

- Matrices are accessed through the Math2 keyboard.
- Select [▦] and tap on each of the entry boxes to enter the matrix values.

Notes:

- To expand the 2 × 2 matrix to a 3 × 3 matrix, tap on the [▦] button twice.
- To increase the number of rows, tap on the [▤] button. To increase the number of columns, tap on the [▥] button.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tan
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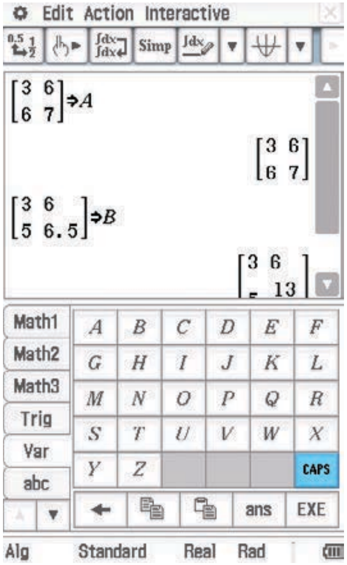
EXE
Alg
Standard
Real
Rad

$\begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}$

- Matrices can be stored as a variable for later use in operations by selecting the store button ⇒ located in Math1 followed by the variable name (usually a capital letter).
- Once **A** and **B** are defined as shown, the matrices **A** + **B**, **A** − **B** and k**A** can be found. (Use the Var keyboard to enter the variable names.)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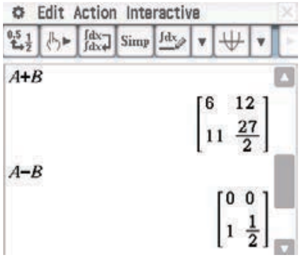
Alg
Standard
Real
Rad

$\begin{bmatrix} 3 & 6 \\ 6 & 7 \end{bmatrix} \Rightarrow A$

$\begin{bmatrix} 3 & 6 \\ 6 & 7 \end{bmatrix}$

$\begin{bmatrix} 3 & 6 \\ 5 & 6.5 \end{bmatrix} \Rightarrow B$

$\begin{bmatrix} 3 & 6 \\ & 13 \end{bmatrix}$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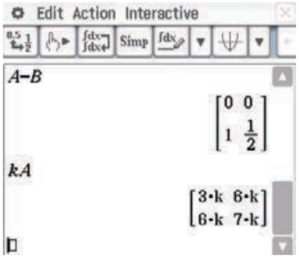
Simp

$A+B$

$\begin{bmatrix} 6 & 12 \\ 11 & \frac{27}{2} \end{bmatrix}$

$A-B$

$\begin{bmatrix} 0 & 0 \\ 1 & \frac{1}{2} \end{bmatrix}$


Edit Action Interactive
Simp

$A-B$

$\begin{bmatrix} 0 & 0 \\ 1 & \frac{1}{2} \end{bmatrix}$

kA

$\begin{bmatrix} 3\cdot k & 6\cdot k \\ 6\cdot k & 7\cdot k \end{bmatrix}$

Multiplication of matrices

Multiplication of a matrix by a real number has been discussed in the previous subsection. The definition for multiplication of matrices is less natural. The procedure for multiplying two 2×2 matrices is shown first.

Let $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 & 1 \\ 6 & 3 \end{bmatrix}$.

$$\begin{aligned} \text{Then} \quad \mathbf{AB} &= \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 6 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 \times 5 + 3 \times 6 & 1 \times 1 + 3 \times 3 \\ 4 \times 5 + 2 \times 6 & 4 \times 1 + 2 \times 3 \end{bmatrix} \\ &= \begin{bmatrix} 23 & 10 \\ 32 & 10 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \text{and} \quad \mathbf{BA} &= \begin{bmatrix} 5 & 1 \\ 6 & 3 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 5 \times 1 + 1 \times 4 & 5 \times 3 + 1 \times 2 \\ 6 \times 1 + 3 \times 4 & 6 \times 3 + 3 \times 2 \end{bmatrix} \\ &= \begin{bmatrix} 9 & 17 \\ 18 & 24 \end{bmatrix} \end{aligned}$$

Note that $\mathbf{AB} \neq \mathbf{BA}$.

If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is an $n \times r$ matrix, then the product $\mathbf{AB}$ is the $m \times r$ matrix whose entries are determined as follows:

To find the entry in row i and column j of $\mathbf{AB}$, single out row i in matrix $\mathbf{A}$ and column j in matrix $\mathbf{B}$. Multiply the corresponding entries from the row and column and then add up the resulting products.

Note: The product $\mathbf{AB}$ is defined only if the number of columns of $\mathbf{A}$ is the same as the number of rows of $\mathbf{B}$.

Example 9

For $\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$, find $\mathbf{AB}$.

Solution

$\mathbf{A}$ is a 2×2 matrix and $\mathbf{B}$ is a 2×1 matrix. Therefore $\mathbf{AB}$ is defined and will be a 2×1 matrix.

$$\mathbf{AB} = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \times 5 + 4 \times 3 \\ 3 \times 5 + 6 \times 3 \end{bmatrix} = \begin{bmatrix} 22 \\ 33 \end{bmatrix}$$

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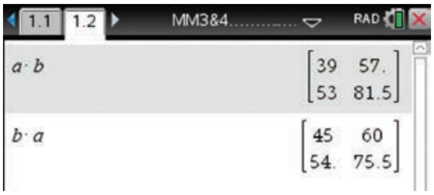
Using the TI-Nspire

Multiplication of

$$\mathbf{A} = \begin{bmatrix} 3 & 6 \\ 6 & 7 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 3 & 6 \\ 5 & 6.5 \end{bmatrix}$$

The products **AB** and **BA** are shown.



$a \cdot b$ $\begin{bmatrix} 39 & 57. \\ 53 & 81.5 \end{bmatrix}$

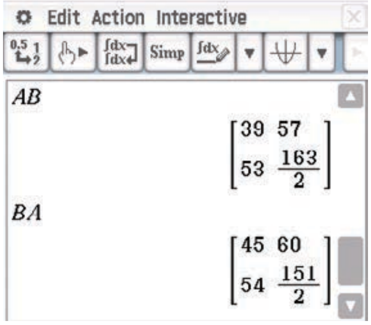
$b \cdot a$ $\begin{bmatrix} 45 & 60 \\ 54. & 75.5 \end{bmatrix}$

Using the Casio ClassPad

Multiplication of

$$\mathbf{A} = \begin{bmatrix} 3 & 6 \\ 6 & 7 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 3 & 6 \\ 5 & 6.5 \end{bmatrix}$$

The products **AB** and **BA** are shown.



AB

$\begin{bmatrix} 39 & 57 \\ 53 & \frac{163}{2} \end{bmatrix}$

BA

$\begin{bmatrix} 45 & 60 \\ 54 & \frac{151}{2} \end{bmatrix}$

A matrix with the same number of rows and columns is called a **square matrix**.

For 2 × 2 matrices, the **identity matrix** is

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This matrix has the property that $\mathbf{AI} = \mathbf{A} = \mathbf{IA}$, for any 2×2 matrix **A**.

In general, for the family of $n \times n$ matrices, the multiplicative identity **I** is the matrix that has ones in the 'top left' to 'bottom right' diagonal and has zeroes in all other positions.

Section summary

- A **matrix** is a rectangular array of numbers.
- Two matrices **A** and **B** are equal when:
  - they have the same number of rows and the same number of columns, and
  - they have the same number or entry at corresponding positions.
- The size or **dimension** of a matrix is described by specifying the number of rows (m) and the number of columns (n). The dimension is written $m \times n$.
- Addition is defined for two matrices only when they have the same dimension. The sum is found by adding corresponding entries.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

Subtraction is defined in a similar way.

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- If $\mathbf{A}$ is an $m \times n$ matrix and k is a real number, then $k\mathbf{A}$ is defined to be an $m \times n$ matrix whose entries are k times the corresponding entries of $\mathbf{A}$.

$$k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

- If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is an $n \times r$ matrix, then the product $\mathbf{AB}$ is the $m \times r$ matrix whose entries are determined as follows:

  To find the entry in row i and column j of $\mathbf{AB}$, single out row i in matrix $\mathbf{A}$ and column j in matrix $\mathbf{B}$. Multiply the corresponding entries from the row and column and then add up the resulting products.

Exercise 2E

Example 7

1 Find:

a $\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}$ **b** $\begin{bmatrix} 4 \\ -2 \end{bmatrix} - \begin{bmatrix} 6 \\ -2 \end{bmatrix}$

Example 8

2 If $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & -4 \\ -3 & 6 \end{bmatrix}$, find the matrix $\mathbf{X}$ such that $2\mathbf{A} + \mathbf{X} = \mathbf{B}$.

Example 9

3 For $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, find $\mathbf{AB}$.

4 For the matrices $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -2 & -2 \\ 3 & 2 \end{bmatrix}$, find:

a $\mathbf{A} + \mathbf{B}$ **b** $\mathbf{AB}$ **c** $\mathbf{BA}$ **d** $\mathbf{A} - \mathbf{B}$
e $k\mathbf{A}$ **f** $2\mathbf{A} + 3\mathbf{B}$ **g** $\mathbf{A} - 2\mathbf{B}$

5 $\mathbf{A} = \begin{bmatrix} 3 & 4 \\ -3 & -3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & -4 \\ 5 & 1 \end{bmatrix}$

Calculate:

a $2\mathbf{A}$ **b** $3\mathbf{B}$ **c** $2\mathbf{A} + 3\mathbf{B}$ **d** $3\mathbf{B} - 2\mathbf{A}$

6 $\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}$, $\mathbf{Q} = \begin{bmatrix} -1 & -4 \\ 5 & 0 \end{bmatrix}$, $\mathbf{R} = \begin{bmatrix} 0 & -4 \\ 1 & 1 \end{bmatrix}$

Calculate:

a $\mathbf{P} + \mathbf{Q}$ **b** $\mathbf{P} + 3\mathbf{Q}$ **c** $2\mathbf{P} - \mathbf{Q} + \mathbf{R}$

7 If $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ -3 & -4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & -5 \\ -2 & 1 \end{bmatrix}$, find matrices $\mathbf{X}$ and $\mathbf{Y}$ such that $2\mathbf{A} - 3\mathbf{X} = \mathbf{B}$ and $3\mathbf{A} + 2\mathbf{Y} = 2\mathbf{B}$.

8 If $\mathbf{X} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, find the products $\mathbf{AX}$, $\mathbf{BX}$, $\mathbf{IX}$, $\mathbf{AI}$, $\mathbf{IB}$, $\mathbf{AB}$, $\mathbf{BA}$, $\mathbf{A}^2$ and $\mathbf{B}^2$.

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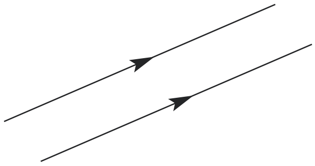
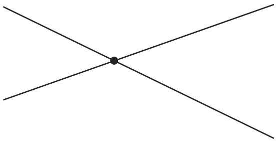
2F The geometry of simultaneous linear equations with two variables

Two distinct straight lines are either parallel or meet at a point.





There are three cases for a system of two linear equations with two variables.

	Example	Solutions	Geometry
Case 1	$2x + y = 5$ $x - y = 4$	Unique solution: $x = 3,\ y = -1$	Two lines meeting at a point
Case 2	$2x + y = 5$ $2x + y = 7$	No solutions	Distinct parallel lines
Case 3	$2x + y = 5$ $4x + 2y = 10$	Infinitely many solutions	Two copies of the same line

Example 10

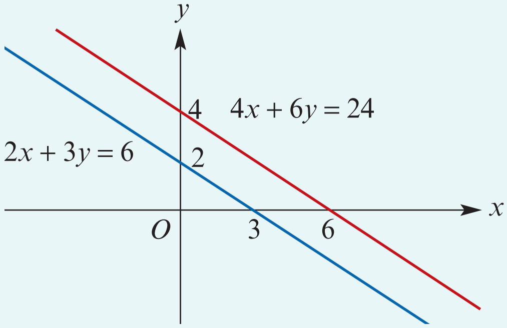
Explain why the simultaneous equations $2x + 3y = 6$ and $4x + 6y = 24$ have no solution.

Solution

First write the two equations in the form $y = mx + c$. They become

$$y = -\tfrac{2}{3}x + 2 \quad \text{and} \quad y = -\tfrac{2}{3}x + 4$$

Both lines have gradient $-\frac{2}{3}$. The y-axis intercepts are 2 and 4 respectively. The equations have no solution as they correspond to distinct parallel lines.



Example 11

The simultaneous equations $2x + 3y = 6$ and $4x + 6y = 12$ have infinitely many solutions. Describe these solutions through the use of a parameter.

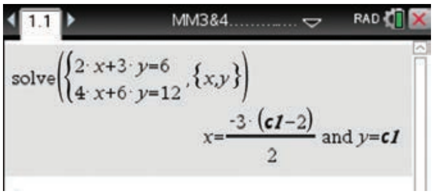
Solution

The two lines coincide, and so the solutions are all points on this line. We make use of a third variable λ as the parameter. If $y = \lambda$, then $x = \dfrac{6 - 3\lambda}{2}$. The points on the line are all points of the form $\left(\dfrac{6 - 3\lambda}{2}, \lambda\right)$.

Using the TI-Nspire

Simultaneous equations can be solved in a **Calculator** application.

- Use menu > **Algebra** > **Solve System of Equations** > **Solve System of Equations**.
- Complete the pop-up screen.



$$\text{solve}\left(\begin{cases}2\cdot x+3\cdot y=6\\4\cdot x+6\cdot y=12\end{cases},\{x,y\}\right)$$

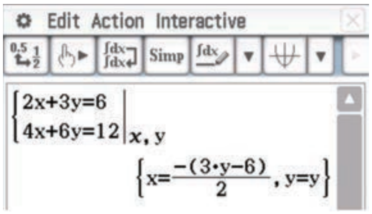
$$x=\frac{-3\cdot(c1-2)}{2} \text{ and } y=c1$$

The solution to this system of equations is given by the calculator as shown. The variable $c1$ takes the place of λ.

Using the Casio ClassPad

To solve the simultaneous equations $2x + 3y = 6$ and $4x + 6y = 12$:

- Open the Math1 keyboard.
- Select the simultaneous equations icon {▤.
- Enter the two equations into the two lines and type x, y in the bottom-right square to indicate the variables.
- Select EXE.



$$\begin{cases}2x+3y=6\\4x+6y=12\end{cases}\Big|_{x,\,y}$$

$$\left\{x=\frac{-(3\cdot y-6)}{2},\ y=y\right\}$$

Choose $y = \lambda$ to obtain the solution $x = \dfrac{6-3\lambda}{2}$, $y = \lambda$ where λ is any real number.

Example 12

Consider the simultaneous linear equations

$$(m-2)x + y = 2 \qquad \text{and} \qquad mx + 2y = k$$

Find the values of m and k such that the system of equations has:

a a unique solution **b** no solution **c** infinitely many solutions.

Solution

Use a CAS calculator to find the solution:

$$x = \frac{4-k}{m-4} \quad \text{and} \quad y = \frac{k(m-2)-2m}{m-4}, \qquad \text{for } m \neq 4$$

a There is a unique solution if $m \neq 4$ and k is any real number.

b If $m = 4$, the equations become

$$2x + y = 2 \quad \text{and} \quad 4x + 2y = k$$

There is no solution if $m = 4$ and $k \neq 4$.

c If $m = 4$ and $k = 4$, there are infinitely many solutions as the equations are the same.

Section summary

There are three cases for a system of two linear equations in two variables:

- unique solution (lines intersect at a point), e.g. $y = 2x + 3$ and $y = 3x + 3$
- infinitely many solutions (lines coincide), e.g. $y = 2x + 3$ and $2y = 4x + 6$
- no solution (lines are parallel), e.g. $y = 2x + 3$ and $y = 2x + 4$.

Exercise 2F

1 Solve each of the following pairs of simultaneous linear equations:

a $3x + 2y = 6$, $x - y = 7$

b $2x + 6y = 0$, $y - x = 2$

c $4x - 2y = 7$, $5x + 7y = 1$

d $2x - y = 6$, $4x - 7y = 5$

2 For each of the following, state whether there is no solution, one solution or infinitely many solutions:

a $3x + 2y = 6$, $3x - 2y = 12$

b $x + 2y = 6$, $2x + 4y = 12$

c $x - 2y = 3$, $2x - 4y = 12$

Example 10 **3** Explain why the simultaneous equations $2x + 3y = 6$ and $4x + 6y = 10$ have no solution.

Example 11 **4** The simultaneous equations $x - y = 6$ and $2x - 2y = 12$ have infinitely many solutions. Describe these solutions through the use of a parameter.

Example 12 **5** Find the value of m for which the simultaneous equations

$$\begin{aligned} 3x + my &= 5 \\ (m+2)x + 5y &= m \end{aligned}$$

a have infinitely many solutions

b have no solution.

6 Find the value of m for which the simultaneous equations

$$\begin{aligned} (m+3)x + my &= 12 \\ (m-1)x + (m-3)y &= 7 \end{aligned}$$

have no solution.

7 Consider the simultaneous equations

$$\begin{aligned} mx + 2y &= 8 \\ 4x - (2-m)y &= 2m \end{aligned}$$

a Find the values of m for which there are:

  i no solutions

  ii infinitely many solutions.

b Solve the equations in terms of m, for suitable values of m.

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8 **a** Solve the simultaneous equations $2x - 3y = 4$ and $x + ky = 2$, where k is a constant.

b Find the value of k for which there is not a unique solution.

9 Find the values of b and c for which the equations $x + 5y = 4$ and $2x + by = c$ have:

a a unique solution

b an infinite set of solutions

c no solution.

2G Simultaneous linear equations with more than two variables

Consider the general system of three linear equations in three unknowns:

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

In this section we look at how to solve such systems of simultaneous equations. In some cases, this can be done easily by elimination, as shown in Examples 13 and 14. In these cases, you could be expected to find the solution by hand. We will see that in some cases using a calculator is the best choice.

Example 13

Solve the following system of three equations in three unknowns:

$$\begin{aligned} 2x + y + z &= -1 && (1) \\ 3y + 4z &= -7 && (2) \\ 6x + z &= 8 && (3) \end{aligned}$$

Solution

Subtract (1) from (3):

$$4x - y = 9 \qquad (4)$$

Subtract (2) from 4 × (3):

$$\begin{aligned} 24x - 3y &= 39 \\ 8x - y &= 13 && (5) \end{aligned}$$

Subtract (4) from (5) to obtain $4x = 4$. Hence $x = 1$.

Substitute in (4) to find $y = -5$, and substitute in (3) to find $z = 2$.

Explanation

The aim is first to eliminate z and obtain two simultaneous equations in x and y only.

Having obtained equations (4) and (5), we solve for x and y. Then substitute to find z.

It should be noted that, just as for two equations in two unknowns, there is a geometric interpretation for three equations in three unknowns. There is only a unique solution if the three equations represent three planes intersecting at a point.

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Example 14

Solve the following simultaneous linear equations for x, y and z:

$$x - y + z = 6, \qquad 2x + z = 4, \qquad 3x + 2y - z = 6$$

Solution

$$\begin{aligned} x - y + z &= 6 && (1) \\ 2x + z &= 4 && (2) \\ 3x + 2y - z &= 6 && (3) \end{aligned}$$

Eliminate z to find two simultaneous equations in x and y:

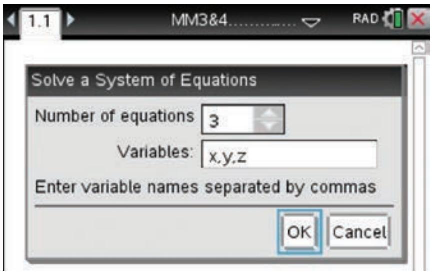
$$\begin{aligned} x + y &= -2 && (4) && \text{subtracted (1) from (2)} \\ 5x + 2y &= 10 && (5) && \text{added (2) to (3)} \end{aligned}$$

Solve to find $x = \dfrac{14}{3}$, $y = -\dfrac{20}{3}$, $z = -\dfrac{16}{3}$.

A CAS calculator can be used to solve a system of three equations in the same way as for solving two simultaneous equations.

Using the TI-Nspire

Use the simultaneous equations template (menu > **Algebra** > **Solve System of Equations** > **Solve System of Equations**) as shown.

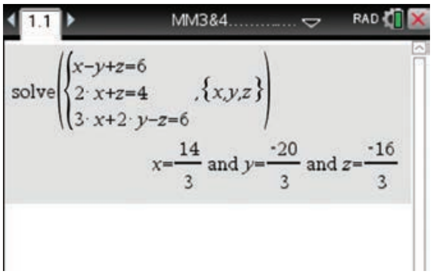


Solve a System of Equations

Number of equations 3

Variables: x,y,z

Enter variable names separated by commas



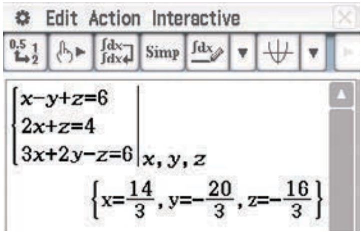
$$\text{solve}\left(\begin{cases} x-y+z=6 \\ 2\cdot x+z=4 \\ 3\cdot x+2\cdot y-z=6 \end{cases}, \{x,y,z\}\right)$$

$$x=\frac{14}{3} \text{ and } y=\frac{-20}{3} \text{ and } z=\frac{-16}{3}$$

Note: The result could also be obtained using:
solve($x - y + z = 6$ and $2x + z = 4$ and $3x + 2y - z = 6$, $\{x, y, z\}$)

Using the Casio ClassPad

- From the Math1 keyboard, tap ⊟ twice to create a template for three simultaneous equations.
- Enter the equations using the Var keyboard.



$$\begin{cases} x-y+z=6 \\ 2x+z=4 \\ 3x+2y-z=6 \end{cases}\Big|_{x,\,y,\,z}$$

$$\left\{x=\frac{14}{3}, y=-\frac{20}{3}, z=-\frac{16}{3}\right\}$$

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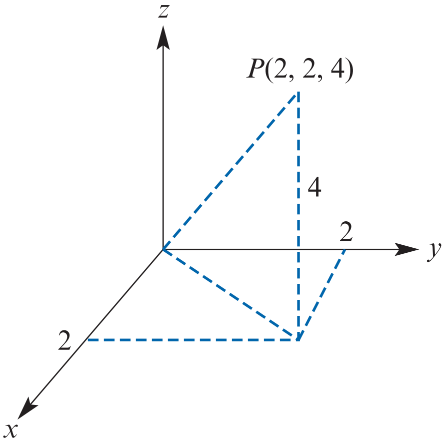
As a linear equation in two variables defines a line, a linear equation in three variables defines a plane.

The coordinate axes in three dimensions are drawn as shown. The point $P(2, 2, 4)$ is marked.



An equation of the form

$$ax + by + cz = d$$

defines a plane. As an example, we will look at the plane

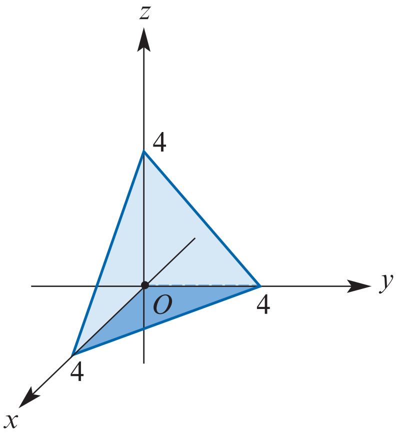
$$x + y + z = 4$$

We get some idea of how the graph sits by considering

- $x = 0,\ y = 0,\ z = 4$
- $x = 0,\ y = 4,\ z = 0$
- $x = 4,\ y = 0,\ z = 0$

and plotting these three points.

This results in being able to sketch the plane $x + y + z = 4$ as shown opposite.



The solution of simultaneous linear equations in three variables can correspond to:

- a point
- a line
- a plane

There also may be no solution. The situations are as shown in the following diagrams. Examples 13 and 14 provide examples of three planes intersecting at a point (Diagram 1).

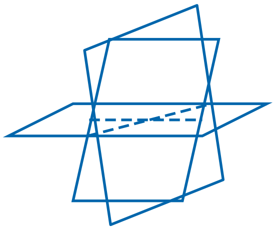


Diagram 1:
Intersection at a point

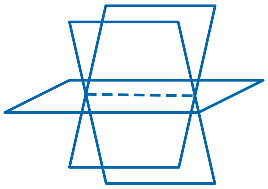


Diagram 2:
Intersection in a line

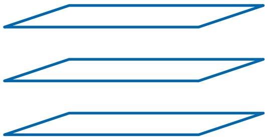


Diagram 3:
No intersection

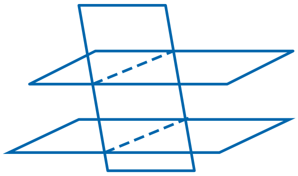


Diagram 4:
No common intersection

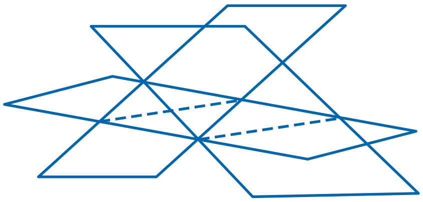


Diagram 5:
No common intersection

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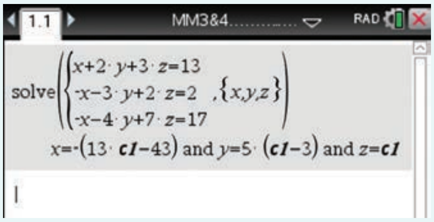
Example 15

The simultaneous equations $x + 2y + 3z = 13$, $-x - 3y + 2z = 2$ and $-x - 4y + 7z = 17$ have infinitely many solutions. Describe these solutions through the use of a parameter.

Solution

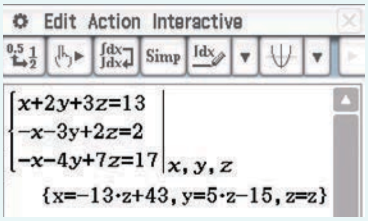
The point $(-9, 5, 4)$ satisfies all three equations, but it is certainly not the only solution.

We can use a CAS calculator to find all the solutions in terms of a parameter λ.



$$\text{solve}\left(\begin{cases}x+2\cdot y+3\cdot z=13\\ -x-3\cdot y+2\cdot z=2\\ -x-4\cdot y+7\cdot z=17\end{cases},\{x,y,z\}\right)$$

$x=-(13\cdot c1-43)$ and $y=5\cdot(c1-3)$ and $z=c1$



$$\begin{cases}x+2y+3z=13\\ -x-3y+2z=2\\ -x-4y+7z=17\end{cases}\Big|_{x,\,y,\,z}$$

$\{x=-13\cdot z+43,\ y=5\cdot z-15,\ z=z\}$

Let $z = \lambda$. Then $x = 43 - 13\lambda$ and $y = 5\lambda - 15$.

For example, if $\lambda = 4$, then $x = -9$, $y = 5$ and $z = 4$.

Note that, as z increases by 1, x decreases by 13 and y increases by 5. All of the points that satisfy the equations lie on a straight line. This is the situation shown in Diagram 2.

Section summary

- A system of simultaneous linear equations in three or more variables can sometimes be solved by hand using elimination (see Example 13). In other cases, using a calculator is the best choice.
- The solution of simultaneous linear equations in three variables can correspond to a point, a line or a plane. There may also be no solution.

Exercise 2G

Example 13, 14

1 Solve each of the following systems of simultaneous equations:

a
$$\begin{aligned}2x + 3y - z &= 12\\ 2y + z &= 7\\ 2y - z &= 5\end{aligned}$$

b
$$\begin{aligned}x + 2y + 3z &= 13\\ -x - y + 2z &= 2\\ -x + 3y + 4z &= 26\end{aligned}$$

c
$$\begin{aligned}x + y &= 5\\ y + z &= 7\\ z + x &= 12\end{aligned}$$

d
$$\begin{aligned}x - y - z &= 0\\ 5x + 20z &= 50\\ 10y - 20z &= 30\end{aligned}$$

Example 15

2 Consider the simultaneous equations $x + 2y - 3z = 4$ and $x + y + z = 6$.

a Subtract the second equation from the first to find y in terms of z.

b Let $z = \lambda$. Solve the equations to give the solution in terms of λ.

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3 Consider the simultaneous equations

$$\begin{aligned} x + 2y + 3z &= 13 &\quad (1)\\ -x - 3y + 2z &= 2 &\quad (2)\\ -x - 4y + 7z &= 17 &\quad (3) \end{aligned}$$

a Add equation (2) to equation (1) and subtract equation (2) from equation (3).

b Comment on the equations obtained in part a.

c Let $z = \lambda$ and find y in terms of λ.

d Substitute for z and y in terms of λ in equation (1) to find x in terms of λ.

4 Solve each of the following pairs of simultaneous equations, giving your answer in terms of a parameter λ. Use the technique introduced in Question 2.

a $$\begin{aligned} x - y + z &= 4\\ -x + y + z &= 6 \end{aligned}$$

b $$\begin{aligned} 2x - y + z &= 6\\ x - z &= 3 \end{aligned}$$

c $$\begin{aligned} 4x - 2y + z &= 6\\ x + y + z &= 4 \end{aligned}$$

5 The system of equations

$$\begin{aligned} x + y + z + w &= 4\\ x + 3y + 3z &= 2\\ x + y + 2z - w &= 6 \end{aligned}$$

has infinitely many solutions. Describe this family of solutions and give the unique solution when $w = 6$.

6 Find all solutions for each of the following systems of equations:

a $$\begin{aligned} 3x - y + z &= 4\\ x + 2y - z &= 2\\ -x + y - z &= -2 \end{aligned}$$

b $$\begin{aligned} x - y - z &= 0\\ 3y + 3z &= -5 \end{aligned}$$

c $$\begin{aligned} 2x - y + z &= 0\\ y + 2z &= 2 \end{aligned}$$

Chapter summary

Spreadsheet AS Nrich

Coordinate geometry

- The **distance** between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- The **midpoint** of the line segment joining (x_1, y_1) and (x_2, y_2) is the point with coordinates

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

- The **gradient** of the straight line joining two points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

- Different forms for the equation of a straight line:

$y = mx + c$	where m is the gradient and c is the y-axis intercept
$y - y_1 = m(x - x_1)$	where m is the gradient and (x_1, y_1) is a point on the line
$\frac{x}{a} + \frac{y}{b} = 1$	where $(a, 0)$ and $(0, b)$ are the axis intercepts

- For a straight line with gradient m, the **angle of slope** is found using

$$m = \tan\theta$$

where θ is the angle that the line makes with the positive direction of the x-axis.

- If two straight lines are **perpendicular** to each other, the product of their gradients is -1, i.e. $m_1 m_2 = -1$. (Unless one line is vertical and the other horizontal.)

Matrices

- A **matrix** is a rectangular array of numbers.
- Two matrices **A** and **B** are equal when:
  - they have the same number of rows and the same number of columns
  - they have the same entry at corresponding positions.
- The size or **dimension** of a matrix is described by specifying the number of rows (m) and the number of columns (n). The dimension is written $m \times n$.
- Addition is defined for two matrices only when they have the same dimension. The sum is found by adding corresponding entries.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

Subtraction is defined in a similar way.

- If $\mathbf{A}$ is an $m \times n$ matrix and k is a real number, then $k\mathbf{A}$ is defined to be an $m \times n$ matrix whose entries are k times the corresponding entries of $\mathbf{A}$.

$$k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

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- If **A** is an $m \times n$ matrix and **B** is an $n \times r$ matrix, then the product **AB** is the $m \times r$ matrix whose entries are determined as follows:

  To find the entry in row i and column j of **AB**, single out row i in matrix **A** and column j in matrix **B**. Multiply the corresponding entries from the row and column and then add up the resulting products.

  The product **AB** is defined only if the number of columns of **A** is the same as the number of rows of **B**.
- The $n \times n$ identity matrix **I** has the property that $\mathbf{AI} = \mathbf{A} = \mathbf{IA}$, for each $n \times n$ matrix **A**.

Simultaneous equations

- There are three cases for a system of two linear equations in two variables:
  - unique solution (lines intersect at a point), e.g. $y = 2x + 3$ and $y = 3x + 3$
  - infinitely many solutions (lines coincide), e.g. $y = 2x + 3$ and $2y = 4x + 6$
  - no solution (lines are parallel), e.g. $y = 2x + 3$ and $y = 2x + 4$.
- The solution of simultaneous linear equations in three variables can correspond to a point, a line or a plane. There may also be no solution.

Technology-free questions

1 Solve the following linear equations:

a $3x - 2 = 4x + 6$ **b** $\dfrac{x+1}{2x-1} = \dfrac{4}{3}$ **c** $\dfrac{3x}{5} - 7 = 11$ **d** $\dfrac{2x+1}{5} = \dfrac{x-1}{2}$

2 Solve each of the following pairs of simultaneous linear equations:

a $y = x + 4$
$5y + 2x = 6$

b $\dfrac{x}{4} - \dfrac{y}{3} = 2$
$y - x = 5$

3 Solve each of the following for x:

a $bx - n = m$ **b** $b - cx = bx$ **c** $\dfrac{cx}{d} - c = 0$ **d** $px = qx - 6$

e $mx - n = nx + m$ **f** $\dfrac{1}{x-a} = \dfrac{a}{x}$

4 Sketch the graphs of the relations:

a $3y + 2x = 5$ **b** $x - y = 6$ **c** $\dfrac{x}{2} + \dfrac{y}{3} = 1$

5 **a** Find the equation of the straight line which passes through $(1, 3)$ and has gradient -2.

b Find the equation of the straight line which passes through $(1, 4)$ and $(3, 8)$.

c Find the equation of the straight line which is perpendicular to the line with equation $y = -2x + 6$ and which passes through the point $(1, 1)$.

d Find the equation of the straight line which is parallel to the line with equation $y = 6 - 2x$ and which passes through the point $(1, 1)$.

6 Find the distance between the points with coordinates $(-1, 6)$ and $(2, 4)$.

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7 Find the midpoint of the line segment AB joining the points $A(4, 6)$ and $B(-2, 8)$.

8 If M is the midpoint of XY, find the coordinates of Y when X and M have the following coordinates:

a $X(-6, 2),\ M(8, 3)$ **b** $X(-1, -4),\ M(2, -8)$

9 For the matrices $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, find:

a $\mathbf{A} + \mathbf{B}$ **b** $\mathbf{AB}$ **c** $\mathbf{AC}$ **d** $\mathbf{BC}$ **e** $3\mathbf{C}$

f $\mathbf{BA}$ **g** $\mathbf{A} - \mathbf{B}$ **h** $k\mathbf{A}$ **i** $2\mathbf{A} + 3\mathbf{B}$ **j** $\mathbf{A} - 2\mathbf{B}$

10 The length of the line segment joining $A(5, 12)$ and $B(10, y)$ is 13 units. Find y.

11 Consider the simultaneous linear equations

$$mx - 4y = m + 3$$
$$4x + (m + 10)y = -2$$

where m is a real constant.

a Find the value of m for which there are infinitely many solutions.

b Find the values of m for which there is a unique solution.

12 Solve the following simultaneous equations. (You will need to use a parameter.)

a $2x - 3y + z = 6$
$-2x + 3y + z = 8$

b $x - z + y = 6$
$2x + z = 4$

Multiple-choice questions

1 A straight line has gradient $-\frac{1}{2}$ and passes through $(1, 4)$. The equation of the line is

A $y = x + 4$ **B** $y = 2x + 2$ **C** $y = 2x + 4$

D $y = -\dfrac{1}{2}x + 4$ **E** $y = -\dfrac{1}{2}x + \dfrac{9}{2}$

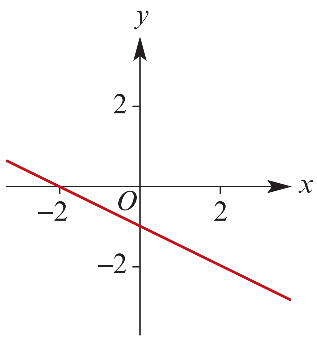
2 The line $y = -2x + 4$ passes through a point $(a, 3)$. The value of a is

A $-\dfrac{1}{2}$ **B** 2 **C** $-\dfrac{7}{2}$ **D** -2 **E** $\dfrac{1}{2}$

3 The gradient of a line that is perpendicular to the line shown could be

A 1 **B** $\frac{1}{2}$ **C** $-\frac{1}{2}$

D 2 **E** -2



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4 The coordinates of the midpoint of AB, where A has coordinates $(1, 7)$ and B has coordinates $(-3, 10)$, are

A $(-2, 3)$ **B** $(-1, 8)$ **C** $(-1, 8.5)$ **D** $(-1, 3)$ **E** $(-2, 8.5)$

5 The solution of the two simultaneous equations $ax - 5by = 11$ and $4ax + 10by = 2$ for x and y, in terms of a and b, is

A $x = -\frac{10}{a},\ y = -\frac{21}{5b}$ **B** $x = \frac{4}{a},\ y = -\frac{4}{5b}$ **C** $x = \frac{13}{5a},\ y = -\frac{42}{25b}$

D $x = \frac{13}{2a},\ y = -\frac{9}{10b}$ **E** $x = -\frac{3}{a},\ y = -\frac{14}{5b}$

6 The gradient of the line passing through $(3, -2)$ and $(-1, 10)$ is

A -3 **B** -2 **C** $-\frac{1}{3}$ **D** 4 **E** 3

7 If two lines $-2x + y - 3 = 0$ and $ax - 3y + 4 = 0$ are parallel, then a equals

A 6 **B** 2 **C** $\frac{1}{3}$ **D** $\frac{2}{3}$ **E** -6

8 A straight line passes through $(-1, -2)$ and $(3, 10)$. The equation of the line is

A $y = 3x - 1$ **B** $y = 3x - 4$ **C** $y = 3x + 1$ **D** $y = \frac{1}{3}x + 9$ **E** $y = 4x - 2$

9 The length of the line segment connecting $(1, 4)$ and $(5, -2)$ is

A 10 **B** $2\sqrt{13}$ **C** 12 **D** 50 **E** $2\sqrt{5}$

10 The function with graph as shown has the rule

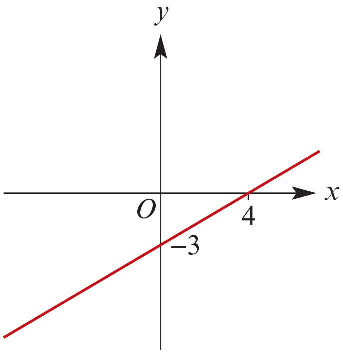
A $f(x) = 3x - 3$

B $f(x) = -\frac{3}{4}x - 3$

C $f(x) = \frac{3}{4}x - 3$

D $f(x) = \frac{4}{3}x - 3$

E $f(x) = 4x - 4$



11 The pair of simultaneous linear equations

$$bx + 3y = 0$$
$$4x + (b + 1)y = 0$$

where b is a real constant, has infinitely many solutions for

A $b \in \mathbb{R}$ **B** $b \in \{-3, 4\}$ **C** $b \in \mathbb{R} \setminus \{-3, 4\}$

D $b \in \{-4, 3\}$ **E** $b \in \mathbb{R} \setminus \{-4, 3\}$

12 The simultaneous equations

$$(a-1)x+5y=7$$
$$3x+(a-3)y=a$$

have a unique solution for

A $a \in \mathbb{R} \setminus \{6, -2\}$
B $a \in \mathbb{R} \setminus \{0\}$
C $a \in \mathbb{R} \setminus \{6\}$
D $a = 6$
E $a = -2$

13 The midpoint of the line segment joining $(0, -6)$ and $(4, d)$ is

A $\left(-2, \dfrac{d+6}{2}\right)$
B $\left(2, \dfrac{d+6}{2}\right)$
C $\left(\dfrac{d+6}{2}, 2\right)$
D $\left(2, \dfrac{d-6}{2}\right)$
E $\dfrac{d+6}{4}$

14 The gradient of a line perpendicular to the line through $(3, 0)$ and $(0, -6)$ is

A $\dfrac{1}{2}$
B -2
C $-\dfrac{1}{2}$
D 2
E 6

Extended-response questions

1 A firm manufacturing jackets finds that it is capable of producing 100 jackets per day, but it can only sell all of these if the charge to wholesalers is no more than \$50 per jacket. On the other hand, at the current price of \$75 per jacket, only 50 can be sold per day.

Assume that the graph of price, $\$P$, against number sold per day, N, is a straight line.

a Sketch the graph of P against N.

b Find the equation of the straight line.

c Use the equation to find:

- **i** the price at which 88 jackets per day could be sold
- **ii** the number of jackets that should be manufactured to sell at \$60 each.

2 A new town was built 10 years ago to house the workers of a woollen mill established in a remote country area. Three years after the town was built, it had a population of 12 000 people. Business in the wool trade steadily grew, and eight years after the town was built the population had swelled to 19 240.

a Assuming the population growth can be modelled by a linear relationship, find a suitable relation for the population, p, in terms of t, the number of years since the town was built.

b Sketch the graph of p against t, and interpret the p-axis intercept.

c Find the current population of the town.

d Calculate the average rate of growth of the town.

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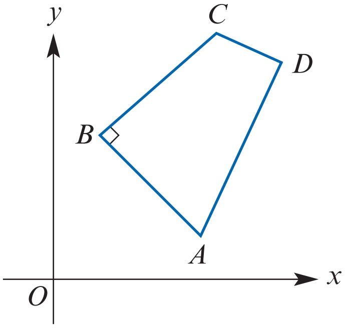
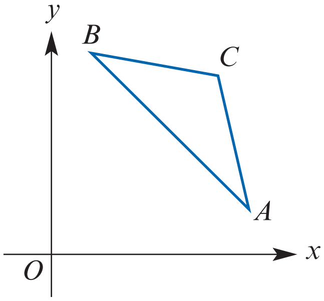
3 *ABCD* is a quadrilateral with angle *ABC* a right angle. The point *D* lies on the perpendicular bisector of *AB*. The coordinates of *A* and *B* are $(7, 2)$ and $(2, 5)$ respectively. The equation of line *AD* is $y = 4x - 26$.



- **a** Find the equation of the perpendicular bisector of line segment *AB*.
- **b** Find the coordinates of point *D*.
- **c** Find the gradient of line *BC*.
- **d** Find the value of the second coordinate c of the point $C(8, c)$.
- **e** Find the area of quadrilateral *ABCD*.

4 Triangle *ABC* is isosceles with $BC = AC$. The coordinates of the vertices are $A(6, 1)$ and $B(2, 8)$.



- **a** Find the equation of the perpendicular bisector of *AB*.
- **b** If the x-coordinate of C is 3.5, find the y-coordinate of C.
- **c** Find the length of *AB*.
- **d** Find the area of triangle *ABC*.

5 If $A = (-4, 6)$ and $B = (6, -7)$, find:

- **a** the coordinates of the midpoint of *AB*
- **b** the length of *AB*
- **c** the distance between *A* and *B*
- **d** the equation of *AB*
- **e** the equation of the perpendicular bisector of *AB*
- **f** the coordinates of the point *P* on the line segment *AB* such that $AP : PB = 3 : 1$
- **g** the coordinates of the point *P* on the line *AB* such that $AP : AB = 3 : 1$ and *P* is closer to point *B* than to point *A*.

6 A chemical manufacturer has an order for 500 litres of a 25% acid solution (i.e. 25% by volume is acid). Solutions of 30% and 18% are available in stock.

- **a** How much acid is required to produce 500 litres of 25% acid solution?
- **b** The manufacturer wishes to make up the 500 litres from a mixture of 30% and 18% solutions.
  Let x denote the amount of 30% solution required.
  Let y denote the amount of 18% solution required.
  Use simultaneous equations in x and y to determine the amount of each solution required.

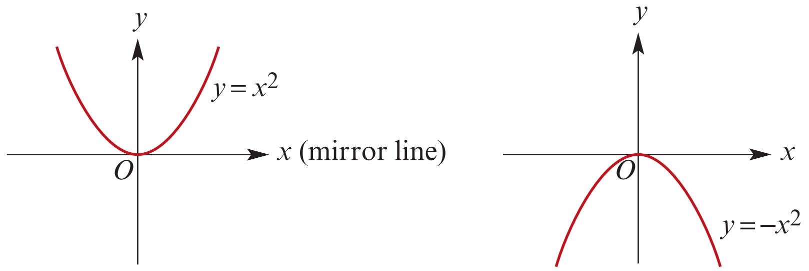
3

Transformations

Objectives

- To introduce a notation for considering transformations of the plane, including **translations**, **reflections** in an axis and **dilations** from an axis.
- To determine a sequence of transformations given the equation of a curve and its image.
- To use transformations to help with graph sketching.
- To consider transformations of power functions.
- To determine the rule for a function given sufficient information.
- To use **matrices** to define transformations.
- To be able to use **matrix equations** in determining the image of a curve under a transformation.

Many graphs of functions can be described as transformations of graphs of other functions, or 'movements' of graphs about the Cartesian plane. For example, the graph of the function $y = -x^2$ can be considered as a reflection in the x-axis of the graph of the function $y = x^2$.



A good understanding of transformations, combined with knowledge of the 'simplest' function and its graph in each family, provides an important tool with which to sketch graphs and identify rules of more complicated functions.

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3A Translations

The **Cartesian plane** is represented by the set $\mathbb{R}^2$ of all ordered pairs of real numbers. That is, $\mathbb{R}^2 = \{ (x, y) : x, y \in \mathbb{R} \}$. The transformations considered in this book associate each ordered pair of $\mathbb{R}^2$ with a unique ordered pair. We can refer to them as examples of **transformations of the plane**.

For example, the translation 3 units in the positive direction of the x-axis (to the right) associates with each ordered pair (x, y) a new ordered pair $(x + 3, y)$. This translation is a transformation of the plane. Each point in the plane is mapped to a unique second point. Furthermore, every point in the plane is an image of another point under this translation.

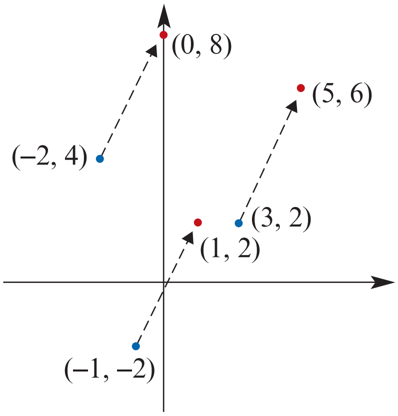
▶ Notation

Consider the translation 2 units in the positive direction of the x-axis (to the right) and 4 units in the positive direction of the y-axis (up). This can be described by the rule $(x, y) \to (x + 2, y + 4)$. This reads as '$(x, y)$ maps to $(x + 2, y + 4)$'.

For example, $(3, 2) \to (3 + 2, 2 + 4)$.

In applying this translation, it is useful to think of every point (x, y) in the plane as being mapped to a new point (x', y'). This point (x, y) is the only point which maps to (x', y'). The following can be written for this translation:

$$x' = x + 2 \quad \text{and} \quad y' = y + 4$$



- A translation of h units in the positive direction of the x-axis and k units in the positive direction of the y-axis is described by the rule

  $$(x, y) \to (x + h, y + k)$$

  or $\quad x' = x + h \quad \text{and} \quad y' = y + k$

  where h and k are positive numbers.
- A translation of h units in the negative direction of the x-axis and k units in the negative direction of the y-axis is described by the rule

  $$(x, y) \to (x - h, y - k)$$

  or $\quad x' = x - h \quad \text{and} \quad y' = y - k$

  where h and k are positive numbers.

Notes:

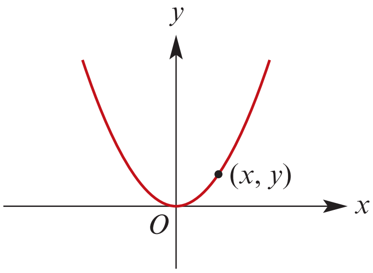
- Under a translation, if $(a', b') = (c', d')$, then $(a, b) = (c, d)$.
- For a translation $(x, y) \to (x + h, y + k)$, for each point $(a, b) \in \mathbb{R}^2$ there is a point (p, q) such that $(p, q) \to (a, b)$. (It is clear that $(p - h, q - k) \to (p, q)$ under this translation.)

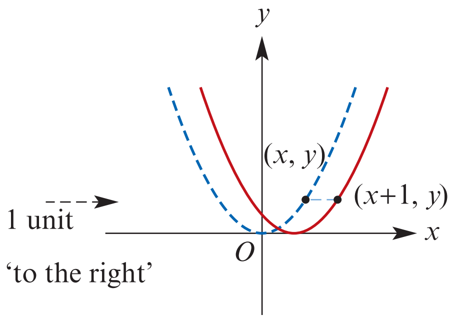
Applying translations to sketch graphs

A translation moves every point on the graph the same distance in the same direction.

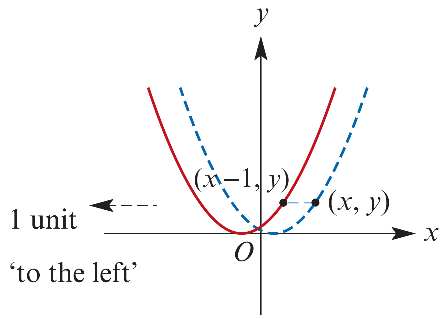
Translations parallel to an axis

We start by looking at the images of the graph of $y = x^2$ shown on the right under translations that are parallel to an axis.



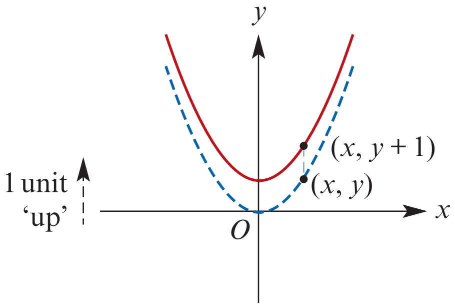
<table>
<tr><th>Translation of 1 unit in the positive direction of the x-axis</th><th>Translation of 1 unit in the negative direction of the x-axis</th></tr>
<tr><td>


The point (x, y) is mapped onto $(x + 1, y)$, i.e. $(x, y) \to (x + 1, y)$.

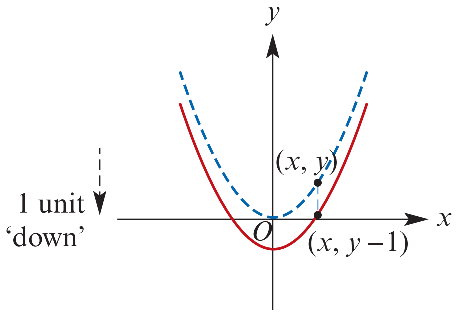
The image has equation $y = (x - 1)^2$.
</td><td>


The point (x, y) is mapped onto $(x - 1, y)$, i.e. $(x, y) \to (x - 1, y)$.

The image has equation $y = (x + 1)^2$.
</td></tr>
</table>

<table>
<tr><th>Translation of 1 unit in the positive direction of the y-axis</th><th>Translation of 1 unit in the negative direction of the y-axis</th></tr>
<tr><td>


The point (x, y) is mapped onto $(x, y + 1)$, i.e. $(x, y) \to (x, y + 1)$.

The image has equation $y = x^2 + 1$.
</td><td>


The point (x, y) is mapped onto $(x, y - 1)$, i.e. $(x, y) \to (x, y - 1)$.

The image has equation $y = x^2 - 1$.
</td></tr>
</table>

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General translations of a curve

Every translation of the plane can be described by giving two components:

- a translation parallel to the x-axis and
- a translation parallel to the y-axis.

Consider a translation of 2 units in the positive direction of the x-axis and 4 units in the positive direction of the y-axis applied to the graph of $y = x^2$.

Translate the set of points defined by the function

$$\{ (x, y) : y = x^2 \}$$

by the translation defined by the rule

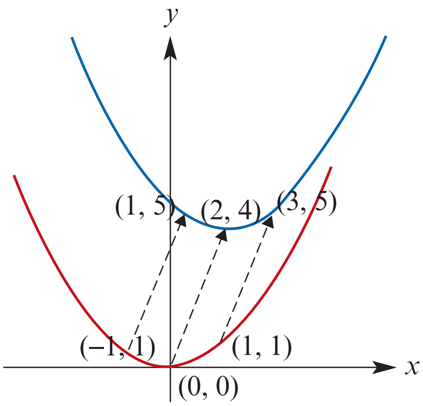
$$(x, y) \to (x + 2, y + 4)$$

$$x' = x + 2 \quad \text{and} \quad y' = y + 4$$

For each point (x, y) there is a unique point (x', y') and vice versa.

We have $x = x' - 2$ and $y = y' - 4$.

This means the points on the curve with equation $y = x^2$ are mapped to the curve with equation $y' - 4 = (x' - 2)^2$.



Hence $\{ (x, y) : y = x^2 \}$ maps to $\{ (x', y') : y' - 4 = (x' - 2)^2 \}$.

For the graph of $y = f(x)$, the following two processes yield the same result:

- Applying the translation $(x, y) \to (x + h, y + k)$ to the graph of $y = f(x)$.
- Replacing x with $x - h$ and y with $y - k$ in the equation to obtain $y - k = f(x - h)$ and graphing the result.

Proof A point (a, b) is on the graph of $y = f(x)$

$$\Leftrightarrow \ f(a) = b$$

$$\Leftrightarrow \ f(a + h - h) = b$$

$$\Leftrightarrow \ f(a + h - h) = b + k - k$$

$$\Leftrightarrow \ (a + h, b + k) \text{ is a point on the graph of } y - k = f(x - h)$$

Note: The double arrows indicate that the steps are reversible.

Example 1

Find the equation for the image of the curve with equation $y = f(x)$, where $f(x) = \dfrac{1}{x}$, under a translation 3 units in the positive direction of the x-axis and 2 units in the negative direction of the y-axis.

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Solution

Let (x', y') be the image of the point (x, y), where (x, y) is a point on the graph of $y = f(x)$.

Then $x' = x + 3$ and $y' = y - 2$.

Hence $x = x' - 3$ and $y = y' + 2$.

The graph of $y = f(x)$ is mapped to the graph of $y' + 2 = f(x' - 3)$

i.e. $y = \frac{1}{x}$ is mapped to $y' + 2 = \frac{1}{x' - 3}$.

The equation of the image can be written as

$$y = \frac{1}{x-3} - 2$$

Explanation

The rule is $(x, y) \to (x + 3, y - 2)$.

Substitute $x = x' - 3$ and $y = y' + 2$ into $y = f(x)$.

Recognising that a transformation has been applied makes it easy to sketch many graphs.

For example, in order to sketch the graph of $y = \sqrt{x-2}$, note that it is of the form $y = f(x-2)$ where $f(x) = \sqrt{x}$. That is, the graph of $y = \sqrt{x}$ is translated 2 units in the positive direction of the x-axis.

Examples of two other functions to which this translation is applied are:

$$f(x) = x^2 \qquad f(x-2) = (x-2)^2$$

$$f(x) = \frac{1}{x} \qquad f(x-2) = \frac{1}{x-2}$$

Section summary

For the graph of $y = f(x)$, the following two processes yield the same result:

- Applying the translation $(x, y) \to (x + h, y + k)$ to the graph of $y = f(x)$.
- Replacing x with $x - h$ and y with $y - k$ in the equation to obtain $y - k = f(x - h)$ and graphing the result.

Exercise 3A

1 Find the image of the point $(-2, 5)$ after a mapping of a translation:

- **a** of 1 unit in the positive direction of the x-axis and 2 units in the negative direction of the y-axis
- **b** of 3 units in the negative direction of the x-axis and 5 units in the positive direction of the y-axis
- **c** of 1 unit in the negative direction of the x-axis and 6 units in the negative direction of the y-axis
- **d** defined by the rule $(x, y) \to (x - 3, y + 2)$
- **e** defined by the rule $(x, y) \to (x - 1, y + 1)$.

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Example 1

2 Find the equation for the image of the curve $y = f(x)$, where $f(x) = \frac{1}{x}$, under:

a a translation 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis

b a translation 2 units in the negative direction of the x-axis and 3 units in the positive direction of the y-axis

c a translation $\frac{1}{2}$ unit in the positive direction of the x-axis and 4 units in the positive direction of the y-axis.

3 Sketch the graph of each of the following. Label asymptotes and axis intercepts, and state the domain and range.

a $y = \frac{1}{x} + 3$ **b** $y = \frac{1}{x^2} - 3$ **c** $y = \frac{1}{(x+2)^2}$

d $y = \sqrt{x-2}$ **e** $y = \frac{1}{x-1}$ **f** $y = \frac{1}{x} - 4$

g $y = \frac{1}{x+2}$ **h** $y = \frac{1}{x-3}$ **i** $f(x) = \frac{1}{(x-3)^2}$

j $f(x) = \frac{1}{(x+4)^2}$ **k** $f(x) = \frac{1}{x-1} + 1$ **l** $f(x) = \frac{1}{x-2} + 2$

4 For $y = f(x) = \frac{1}{x}$, sketch the graph of each of the following. Label asymptotes and axis intercepts.

a $y = f(x-1)$ **b** $y = f(x) + 1$ **c** $y = f(x+3)$

d $y = f(x) - 3$ **e** $y = f(x+1)$ **f** $y = f(x) - 1$

5 For each of the following, state a transformation which maps the graph of $y = f(x)$ to the graph of $y = f_1(x)$:

a $f(x) = x^2, \quad f_1(x) = (x+5)^2$ **b** $f(x) = \frac{1}{x}, \quad f_1(x) = \frac{1}{x} + 2$

c $f(x) = \frac{1}{x^2}, \quad f_1(x) = \frac{1}{x^2} + 4$ **d** $f(x) = \frac{1}{x^2} - 3, \quad f_1(x) = \frac{1}{x^2}$

e $f(x) = \frac{1}{x-3}, \quad f_1(x) = \frac{1}{x}$

6 Write down the equation of the image when the graph of each of the functions below is transformed by:

i a translation of 7 units in the positive direction of the x-axis and 1 unit in the positive direction of the y-axis

ii a translation of 2 units in the negative direction of the x-axis and 6 units in the negative direction of the y-axis

iii a translation of 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis

iv a translation of 1 unit in the negative direction of the x-axis and 4 units in the positive direction of the y-axis.

a $y = x^{\frac{1}{4}}$ **b** $y = \sqrt[3]{x}$ **c** $y = \frac{1}{x^3}$ **d** $y = \frac{1}{x^4}$

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7 Find the equation for the image of the graph of each of the following under the stated translation:

a $y = (x-2)^2 + 3$ Translation: $(x, y) \to (x-3, y+2)$
b $y = 2(x+3)^2 + 3$ Translation: $(x, y) \to (x+3, y-3)$
c $y = \dfrac{1}{(x-2)^2} + 3$ Translation: $(x, y) \to (x+4, y-2)$
d $y = (x+2)^3 + 1$ Translation: $(x, y) \to (x-1, y+1)$
e $y = \sqrt[3]{x-3} + 2$ Translation: $(x, y) \to (x-1, y+1)$

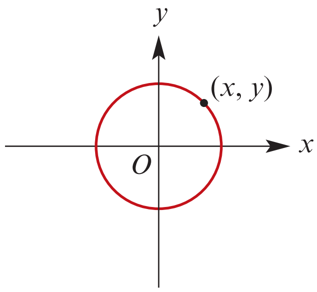
8 For each of the following, state a transformation which maps the graph of $y = f(x)$ to the graph of $y = f_1(x)$:

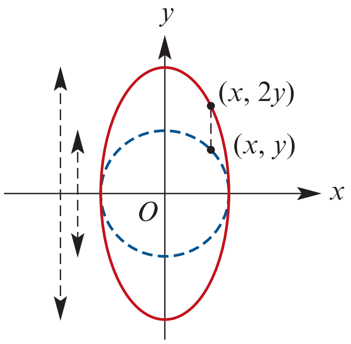
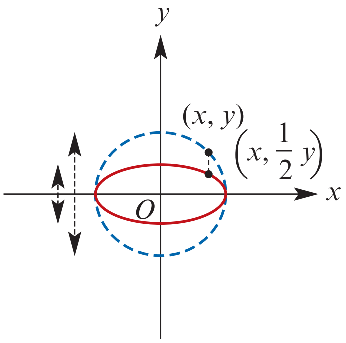
a $f(x) = \dfrac{1}{x^2}$, $f_1(x) = \dfrac{1}{(x-2)^2} + 3$
b $f(x) = \dfrac{1}{x}$, $f_1(x) = \dfrac{1}{x+2} - 3$
c $f(x) = \sqrt{x}$, $f_1(x) = \sqrt{x+4} + 2$

3B Dilations

We start with the example of a circle, as it is easy to visualise the effect of a dilation from an axis.

A dilation of a graph can be thought of as the graph 'stretching away from' or 'shrinking towards' an axis.



Dilation of factor 2 from the x-axis	Dilation of factor $\frac{1}{2}$ from the x-axis
 The graph is 'stretched' to twice the height. The point (x, y) is mapped onto $(x, 2y)$, i.e. $(x, y) \to (x, 2y)$.	 The graph is 'shrunk' to half the height. The point (x, y) is mapped onto $(x, \frac{1}{2}y)$, i.e. $(x, y) \to (x, \frac{1}{2}y)$.

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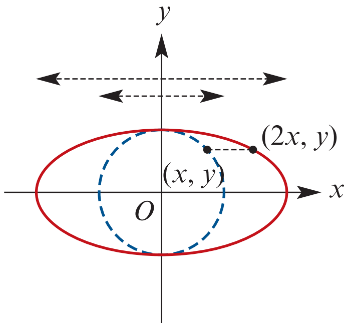
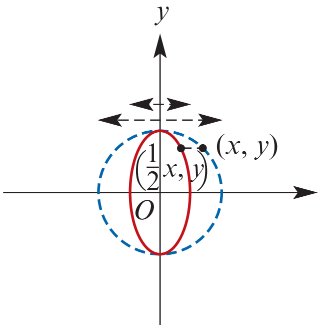
Dilation of factor 2 from the y-axis	Dilation of factor $\frac{1}{2}$ from the y-axis
	 y (x, y) $\left(\frac{1}{2}x, y\right)$ x O
The graph is ‘stretched’ to twice the width. The point (x, y) is mapped onto $(2x, y)$, i.e. $(x, y) \to (2x, y)$.	The graph is ‘shrunk’ to half the width. The point (x, y) is mapped onto $(\frac{1}{2}x, y)$, i.e. $(x, y) \to (\frac{1}{2}x, y)$.

Dilation from the x-axis

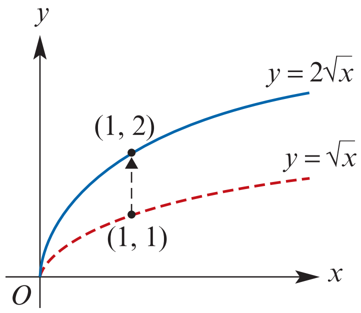
We can determine the equation of the image of a curve under a dilation by following the same approach used for translations.

A dilation of factor 2 from the x-axis is defined by the rule $(x, y) \to (x, 2y)$.

Hence the point with coordinates $(1, 1) \to (1, 2)$.

Consider the curve with equation $y = \sqrt{x}$ and the dilation of factor 2 from the x-axis.

- Let (x', y') be the image of the point with coordinates (x, y) on the curve.
- Hence $x' = x$ and $y' = 2y$, and thus $x = x'$ and $y = \dfrac{y'}{2}$.
- Substituting for x and y, we see that the curve with equation $y = \sqrt{x}$ maps to the curve with equation $\dfrac{y'}{2} = \sqrt{x'}$, i.e. the curve with equation $y = 2\sqrt{x}$.



For b a positive constant, a dilation of factor b from the x-axis is described by the rule

$$(x, y) \to (x, by)$$

or $\quad x' = x \quad$ and $\quad y' = by$

For the graph of $y = f(x)$, the following two processes yield the same result:

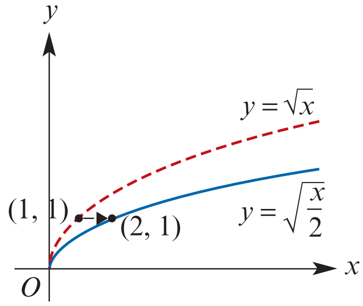
- Applying the dilation from the x-axis $(x, y) \to (x, by)$ to the graph of $y = f(x)$.
- Replacing y with $\dfrac{y}{b}$ in the equation to obtain $y = bf(x)$ and graphing the result.

▸ Dilation from the y-axis

A dilation of factor 2 from the y-axis is defined by the rule $(x, y) \to (2x, y)$. Hence the point with coordinates $(1, 1) \to (2, 1)$.

Again, consider the curve with equation $y = \sqrt{x}$.

- Let (x', y') be the image of the point with coordinates (x, y) on the curve.
- Hence $x' = 2x$ and $y' = y$, and thus $x = \frac{x'}{2}$ and $y = y'$.
- The curve with equation $y = \sqrt{x}$ maps to the curve with equation $y' = \sqrt{\frac{x'}{2}}$.



For a a positive constant, a dilation of factor a from the y-axis is described by the rule

$$(x, y) \to (ax, y)$$

or $\quad x' = ax \quad$ and $\quad y' = y$

For the graph of $y = f(x)$, the following two processes yield the same result:

- Applying the dilation from the y-axis $(x, y) \to (ax, y)$ to the graph of $y = f(x)$.
- Replacing x with $\frac{x}{a}$ in the equation to obtain $y = f\left(\frac{x}{a}\right)$ and graphing the result.

Example 2

Determine the rule of the image when the graph of $y = \frac{1}{x^2}$ is dilated by a factor of 4:

a from the x-axis **b** from the y-axis.

Solution

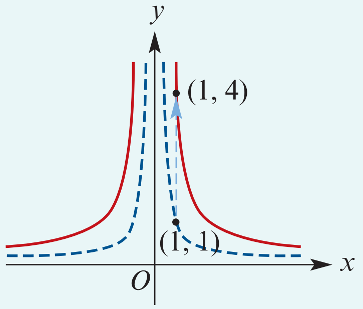
a $(x, y) \to (x, 4y)$

Let (x', y') be the coordinates of the image of (x, y), so $x' = x,\ y' = 4y$.

Rearranging gives $x = x',\ y = \frac{y'}{4}$.

Therefore $y = \frac{1}{x^2}$ becomes $\frac{y'}{4} = \frac{1}{(x')^2}$.

The rule of the transformed function is $y = \frac{4}{x^2}$.



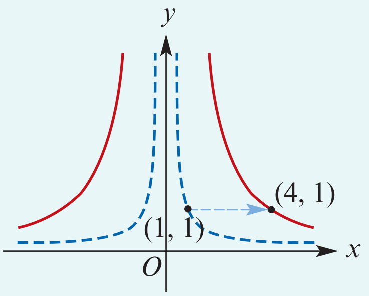
b $(x, y) \to (4x, y)$

Let (x', y') be the coordinates of the image of (x, y), so $x' = 4x,\ y' = y$.

Rearranging gives $x = \frac{x'}{4},\ y = y'$.

Therefore $y = \frac{1}{x^2}$ becomes $y' = \frac{1}{(\frac{x'}{4})^2}$.

The rule of the transformed function is $y = \frac{16}{x^2}$.



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Example 3

Determine the factor of dilation when the graph of $y = \sqrt{3x}$ is obtained by dilating the graph of $y = \sqrt{x}$:

a from the y-axis **b** from the x-axis.

Solution

a Note that a dilation from the y-axis 'changes' the x-values. So write the transformed function as

$$y' = \sqrt{3x'}$$

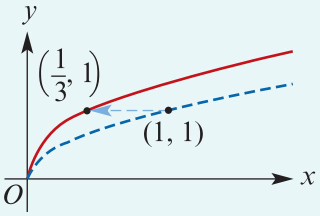
where (x', y') are the coordinates of the image of (x, y).

Therefore $x = 3x'$ and $y = y'$ ('changed' x).

Rearranging gives $x' = \frac{x}{3}$ and $y' = y$.

So the mapping is given by $(x, y) \to \left(\frac{x}{3}, y\right)$.

The graph of $y = \sqrt{x}$ is dilated by a factor of $\frac{1}{3}$ from the y-axis to produce the graph of $y = \sqrt{3x}$.



b Note that a dilation from the x-axis 'changes' the y-values. So write the transformed function as

$$\frac{y'}{\sqrt{3}} = \sqrt{x'}$$

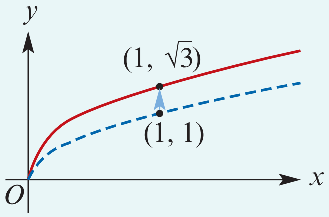
where (x', y') are the coordinates of the image of (x, y).

Therefore $x = x'$ and $y = \frac{y'}{\sqrt{3}}$ ('changed' y).

Rearranging gives $x' = x$ and $y' = \sqrt{3}\,y$.

So the mapping is given by $(x, y) \to (x, \sqrt{3}\,y)$.

The graph of $y = \sqrt{x}$ is dilated by a factor of $\sqrt{3}$ from the x-axis to produce the graph of $y = \sqrt{3x}$.



Section summary

For the graph of $y = f(x)$, we have the following two pairs of equivalent processes:

1. - Applying the **dilation from the x-axis** $(x, y) \to (x, by)$ to the graph of $y = f(x)$.
   - Replacing y with $\frac{y}{b}$ in the equation to obtain $y = bf(x)$ and graphing the result.
2. - Applying the **dilation from the y-axis** $(x, y) \to (ax, y)$ to the graph of $y = f(x)$.
   - Replacing x with $\frac{x}{a}$ in the equation to obtain $y = f\left(\frac{x}{a}\right)$ and graphing the result.

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Exercise 3B

Example 2

1 Determine the rule of the image when the graph of $y = \frac{1}{x}$ is dilated by a factor of 3:

a from the x-axis **b** from the y-axis.

2 Determine the rule of the image when the graph of $y = \frac{1}{x^2}$ is dilated by a factor of 2:

a from the x-axis **b** from the y-axis.

3 Determine the rule of the image when the graph of $y = \sqrt{x}$ is dilated by a factor of 2:

a from the x-axis **b** from the y-axis.

4 Determine the rule of the image when the graph of $y = x^3$ is dilated by a factor of 2:

a from the x-axis **b** from the y-axis.

5 Sketch the graph of each of the following:

a $y = \frac{4}{x}$ **b** $y = \frac{1}{2x}$ **c** $y = \sqrt{3x}$ **d** $y = \frac{2}{x^2}$

6 For $y = f(x) = \frac{1}{x^2}$, sketch the graph of each of the following:

a $y = f(2x)$ **b** $y = 2f(x)$ **c** $y = f\left(\frac{x}{2}\right)$

d $y = 3f(x)$ **e** $y = f(5x)$ **f** $y = f\left(\frac{x}{4}\right)$

7 Sketch the graphs of each of the following on the one set of axes:

a $y = \frac{1}{x}$ **b** $y = \frac{3}{x}$ **c** $y = \frac{3}{2x}$

8 Sketch the graph of the function $f\colon \mathbb{R}^+ \to \mathbb{R}$, $f(x) = 3\sqrt{x}$.

Example 3

9 Determine the factor of dilation when the graph of $y = \sqrt{5x}$ is obtained by dilating the graph of $y = \sqrt{x}$:

a from the y-axis **b** from the x-axis.

10 For each of the following, state a transformation which maps the graph of $y = f(x)$ to the graph of $y = f_1(x)$:

a $f(x) = \frac{1}{x^2}$, $f_1(x) = \frac{5}{x^2}$ **b** $f(x) = \sqrt{x}$, $f_1(x) = 4\sqrt{x}$

c $f(x) = \sqrt{x}$, $f_1(x) = \sqrt{5x}$ **d** $f(x) = \sqrt{\frac{x}{3}}$, $f_1(x) = \sqrt{x}$

e $f(x) = \frac{1}{4x^2}$, $f_1(x) = \frac{1}{x^2}$

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11 Write down the equation of the image when the graph of each of the functions below is transformed by:

- **i** a dilation of factor 4 from the x-axis
- **ii** a dilation of factor $\frac{2}{3}$ from the x-axis
- **iii** a dilation of factor $\frac{1}{2}$ from the y-axis
- **iv** a dilation of factor 5 from the y-axis.

a $y = x^2$ **b** $y = \dfrac{1}{x^2}$ **c** $y = \sqrt[3]{x}$ **d** $y = \dfrac{1}{x^3}$

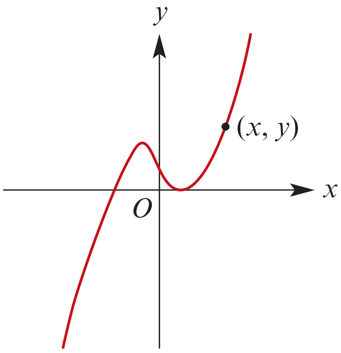
e $y = \dfrac{1}{x^4}$ **f** $y = \sqrt[4]{x}$ **g** $y = x^{\frac{1}{5}}$

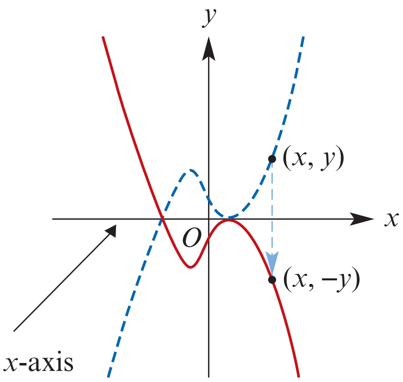
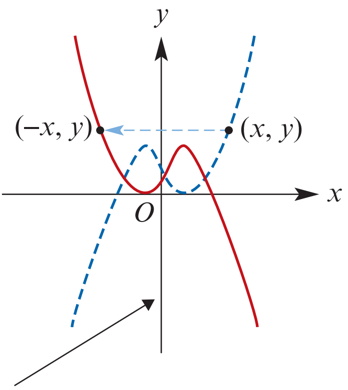
3C Reflections

The special case where the graph of a function is reflected in the line $y = x$ to produce the graph of the inverse relation is discussed separately in Section 1F.

In this chapter we study reflections in the x- or y-axis only.

First consider reflecting the graph of the function shown here in each axis, and observe the effect on a general point (x, y) on the graph.



Reflection in the x-axis	Reflection in the y-axis
 The x-axis acts as a 'mirror' line. The point (x, y) is mapped onto $(x, -y)$, i.e. $(x, y) \rightarrow (x, -y)$	 The y-axis acts as a 'mirror' line. The point (x, y) is mapped onto $(-x, y)$, i.e. $(x, y) \rightarrow (-x, y)$

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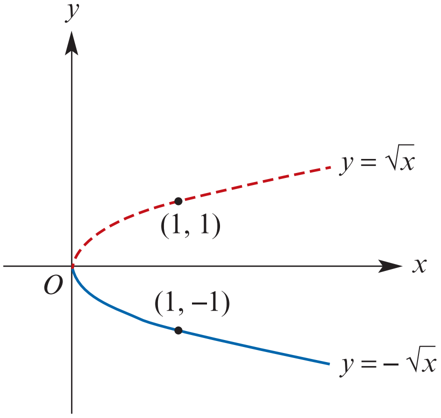
Reflection in the *x*-axis

A reflection in the x-axis can be defined by the rule $(x, y) \to (x, -y)$. Hence the point with coordinates $(1, 1) \to (1, -1)$.

- Let (x', y') be the image of the point (x, y).
- Hence $x' = x$ and $y' = -y$, which gives $x = x'$ and $y = -y'$.
- The curve with equation $y = \sqrt{x}$ maps to the curve with equation $-y' = \sqrt{x'}$, i.e. the curve with equation $y = -\sqrt{x}$.



A reflection in the x-axis is described by the rule

$$(x, y) \to (x, -y)$$

or $\quad x' = x \quad$ and $\quad y' = -y$

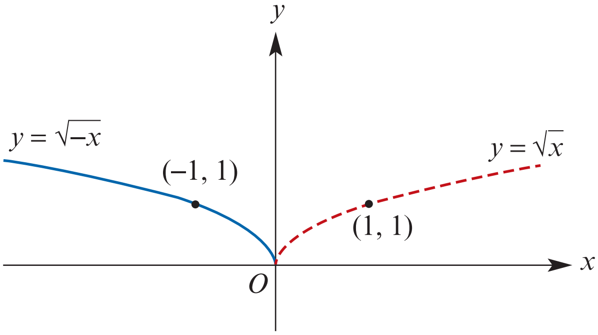
For the graph of $y = f(x)$, the following two processes yield the same result:

- Applying the reflection in the x-axis $(x, y) \to (x, -y)$ to the graph of $y = f(x)$.
- Replacing y with $-y$ in the equation to obtain $y = -f(x)$ and graphing the result.

Reflection in the *y*-axis

A reflection in the y-axis can be defined by the rule $(x, y) \to (-x, y)$. Hence the point with coordinates $(1, 1) \to (-1, 1)$.

- Let (x', y') be the image of the point (x, y).
- Hence $x' = -x$ and $y' = y$, which gives $x = -x'$ and $y = y'$.
- The curve with equation $y = \sqrt{x}$ maps to the curve with equation $y' = \sqrt{-x'}$, i.e. the curve with equation $y = \sqrt{-x}$.



A reflection in the y-axis is described by the rule

$$(x, y) \to (-x, y)$$

or $\quad x' = -x \quad$ and $\quad y' = y$

For the graph of $y = f(x)$, the following two processes yield the same result:

- Applying the reflection in the y-axis $(x, y) \to (-x, y)$ to the graph of $y = f(x)$.
- Replacing x with $-x$ in the equation to obtain $y = f(-x)$ and graphing the result.

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Example 4

Find the equation of the image when the graph of $y = \sqrt{x}$ is reflected:

a in the x-axis **b** in the y-axis.

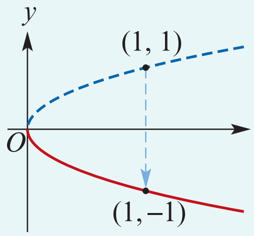
Solution

a Note that a reflection in the x-axis changes the y-values, and so $(x, y) \to (x, -y)$. Let (x', y') be the coordinates of the image of (x, y). Then $x' = x,\ y' = -y$.

Rearranging gives $x = x',\ y = -y'$.

Therefore $y = \sqrt{x}$ becomes $-y' = \sqrt{x'}$.

The rule of the transformed function is $y = -\sqrt{x}$.



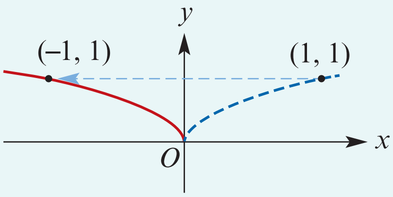
b Note that a reflection in the y-axis changes the x-values, and so $(x, y) \to (-x, y)$. Let (x', y') be the coordinates of the image of (x, y).

Then $x' = -x,\ y' = y$.

Rearranging gives $x = -x',\ y = y'$.

Therefore $y = \sqrt{x}$ becomes $y' = \sqrt{-x'}$.

The rule of the transformed function is $y = \sqrt{-x}$.



Section summary

For the graph of $y = f(x)$, we have the following two pairs of equivalent processes:

1. - Applying the **reflection in the x-axis** $(x, y) \to (x, -y)$ to the graph of $y = f(x)$.
   - Replacing y with $-y$ in the equation to obtain $y = -f(x)$ and graphing the result.
2. - Applying the **reflection in the y-axis** $(x, y) \to (-x, y)$ to the graph of $y = f(x)$.
   - Replacing x with $-x$ in the equation to obtain $y = f(-x)$ and graphing the result.

Exercise 3C

Example 4

1 Find the equation of the image when the graph of $y = (x - 1)^2$ is reflected:

a in the x-axis **b** in the y-axis.

2 Sketch the graph and state the domain of:

a $y = -\left(x^{\frac{1}{3}}\right)$ **b** $y = (-x)^3$

3 State a transformation which maps the graph of $y = \sqrt{x}$ to the graph of $y = \sqrt{-x}$.

4 Find the equation of the image when the graph of each of the functions below is transformed by: **i** a reflection in the x-axis **ii** a reflection in the y-axis.

a $y = x^3$ **b** $y = \sqrt[3]{x}$ **c** $y = \dfrac{1}{x^3}$ **d** $y = \dfrac{1}{x^4}$ **e** $y = x^{\frac{1}{3}}$ **f** $y = x^{\frac{1}{5}}$ **g** $y = x^{\frac{1}{4}}$

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3D Combinations of transformations

In the previous three sections, we considered three types of transformations separately. In the remainder of this chapter we look at situations where a graph may have been transformed by any combination of dilations, reflections and translations.

For example, first consider:

- a dilation of factor 2 from the x-axis
- followed by a reflection in the x-axis.

The rule becomes

$$(x, y) \to (x, 2y) \to (x, -2y)$$

First the dilation is applied and then the reflection. For example, $(1, 1) \to (1, 2) \to (1, -2)$.

Another example is:

- a dilation of factor 2 from the x-axis
- followed by a translation of 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis.

The rule becomes

$$(x, y) \to (x, 2y) \to (x + 2, 2y - 3)$$

First the dilation is applied and then the translation. For example, $(1, 1) \to (1, 2) \to (3, -1)$.

Example 5

Find the equation of the image of $y = \sqrt{x}$ under:

a a dilation of factor 2 from the x-axis followed by a reflection in the x-axis

b a dilation of factor 2 from the x-axis followed by a translation of 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis.

Solution

a From the discussion above, the rule is $(x, y) \to (x, 2y) \to (x, -2y)$.

If (x, y) maps to (x', y'), then $x' = x$ and $y' = -2y$. Thus $x = x'$ and $y = \frac{y'}{-2}$.

So the image of $y = \sqrt{x}$ has equation

$$\frac{y'}{-2} = \sqrt{x'}$$

and hence $y' = -2\sqrt{x'}$. The equation can be written as $y = -2\sqrt{x}$.

b From the discussion above, the rule is $(x, y) \to (x, 2y) \to (x + 2, 2y - 3)$.

If (x, y) maps to (x', y'), then $x' = x + 2$ and $y' = 2y - 3$. Thus $x = x' - 2$ and $y = \frac{y' + 3}{2}$.

So the image of $y = \sqrt{x}$ has equation

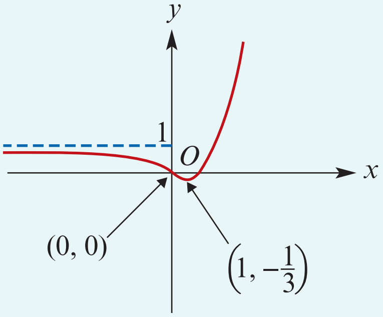
$$\frac{y' + 3}{2} = \sqrt{x' - 2}$$

and hence $y' = 2\sqrt{x' - 2} - 3$. The equation can be written as $y = 2\sqrt{x - 2} - 3$.

Example 6

Sketch the image of the graph shown under the following sequence of transformations:

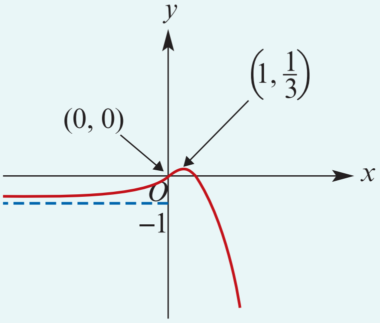
- a reflection in the x-axis
- a dilation of factor 3 from the x-axis
- a translation 2 units in the positive direction of the x-axis and 1 unit in the positive direction of the y-axis.



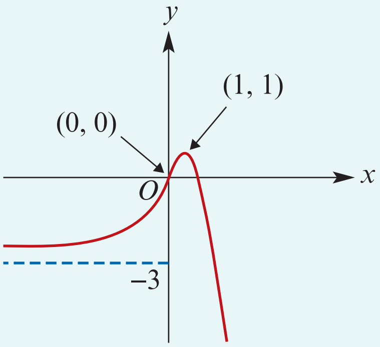
Solution

Consider each transformation in turn and sketch the graph at each stage.

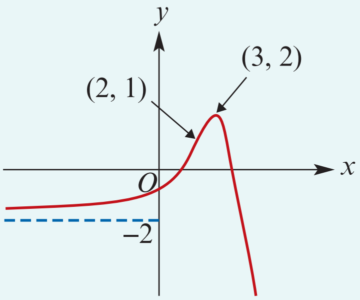
A reflection in the x-axis produces the graph shown on the right.



Next apply the dilation of factor 3 from the x-axis.



Finally, apply the translation 2 units in the positive direction of the x-axis and 1 unit in the positive direction of the y-axis.



Example 7

Find the equation of the image when the graph of $y = \sqrt{x}$ is translated 6 units in the negative direction of the x-axis, reflected in the y-axis and dilated by a factor of 2 from the x-axis.

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Solution

- The translation 6 units in the negative direction of the x-axis maps $(x, y) \to (x - 6, y)$.
- The reflection in the y-axis maps $(x - 6, y) \to (-(x - 6), y)$.
- The dilation of factor 2 from the x-axis maps $(-(x - 6), y) \to (-(x - 6), 2y)$.

In summary: $(x, y) \to (-(x - 6), 2y)$.

Let (x', y') be the coordinates of the image of (x, y). Then $x' = -(x - 6)$ and $y' = 2y$.

Rearranging gives $x = -x' + 6$ and $y = \frac{y'}{2}$.

Therefore $y = \sqrt{x}$ becomes $\frac{y'}{2} = \sqrt{-x' + 6}$.

The rule of the transformed function is $y = 2\sqrt{6 - x}$.

Example 8

For the graph of $y = x^2$:

a Sketch the graph of the image under the sequence of transformations:

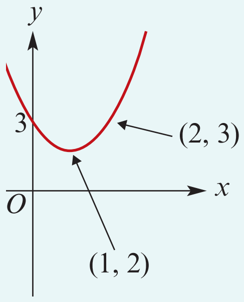
  - a translation of 1 unit in the positive direction of the x-axis and 2 units in the positive direction of the y-axis
  - a dilation of factor 2 from the y-axis
  - a reflection in the x-axis.

b State the rule of the image.

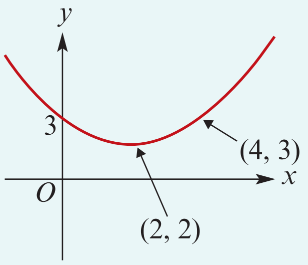
Solution

a Apply each transformation in turn and sketch the graph at each stage.

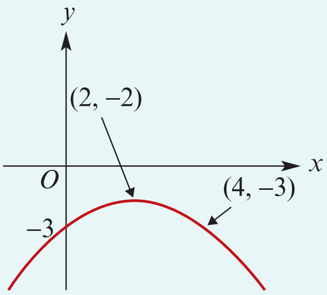
1 The translation:



2 The dilation of factor 2 from the y-axis:



3 The reflection in the x-axis:



b The mapping representing the sequence of transformations is

$$(x, y) \to (x + 1, y + 2) \to (2(x + 1), y + 2) \to (2(x + 1), -(y + 2))$$

Let (x', y') be the image of (x, y). Then $x' = 2(x + 1)$ and $y' = -(y + 2)$.

Rearranging gives $x = \frac{1}{2}(x' - 2)$ and $y = -y' - 2$.

Therefore $y = x^2$ becomes $-y' - 2 = \left(\frac{1}{2}(x' - 2)\right)^2$.

The rule of the image is $y = -\frac{1}{4}(x - 2)^2 - 2$.

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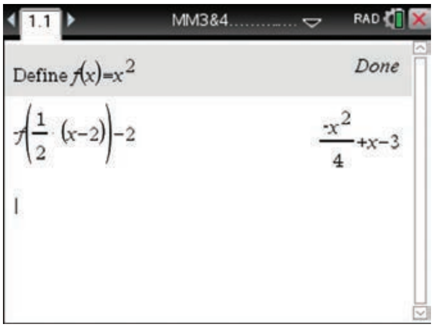
Using the TI-Nspire

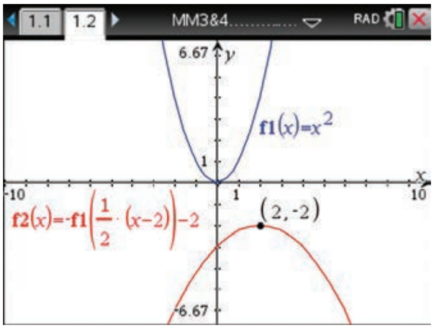
- Define $f(x) = x^2$.
- The rule for the transformed function is $-f\left(\frac{1}{2}(x-2)\right) - 2$.
- The calculator gives the equation of the image of the graph under this sequence of transformations.



Define $f(x)=x^2$ Done

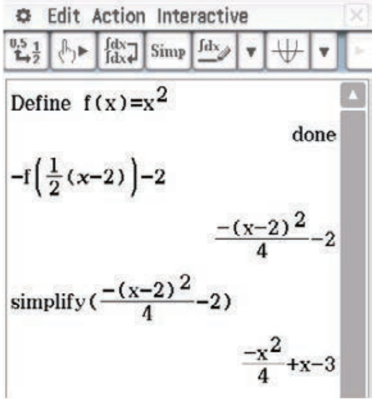
$f\left(\frac{1}{2}\cdot(x-2)\right)-2$ $\quad \frac{-x^2}{4}+x-3$

- The new function can also be entered in the transformation format in a **Graphs** page as shown.


1.1
1.2
MM3&4............
RAD
6.67
y
$f1(x)=x^2$
1
-10
1
10
x
$f2(x)=-f1\left(\frac{1}{2}\cdot(x-2)\right)-2$
(2, -2)
-6.67

Using the Casio ClassPad

- Define $f(x) = x^2$.
- Enter the rule for the transformed function as $-f\left(\frac{1}{2}(x-2)\right) - 2$.
- Highlight the resulting expression and select **Interactive** > **Transformation** > **simplify** to obtain the simplified form.


Edit Action Interactive
Simp

Define f(x)=x^2

done

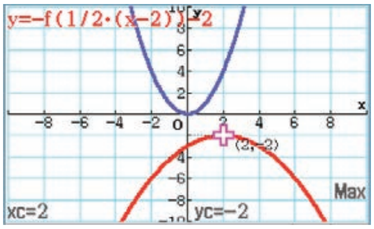
$-f\left(\frac{1}{2}(x-2)\right)-2$

$\frac{-(x-2)^2}{4}-2$

simplify$\left(\frac{-(x-2)^2}{4}-2\right)$

$\frac{-x^2}{4}+x-3$

- To graph both functions, tap on ⊞.
- Highlight each function and drag into the graph window.
- Use ✥ to adjust the window.


y=−f(1/2·(x−2))−2
y
8
6
4
2
−8
−6
−4
−2
0
2
4
6
8
x
−4
−6
−8
−10
(2,−2)
Max
xc=2
yc=−2

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Section summary

A sequence of transformations can be applied, and the rule for transforming points of the plane can be described. For example, the sequence

- a dilation of factor 3 from the x-axis
- followed by a translation of 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis
- followed by a reflection in the x-axis

can be described by the rule $(x, y) \to (x, 3y) \to (x + 2, 3y - 3) \to (x + 2, 3 - 3y)$.

Let $x' = x + 2$ and $y' = 3 - 3y$. Then $x = x' - 2$ and $y = \frac{3 - y'}{3}$.

The graph of $y = f(x)$ maps to $\frac{3 - y'}{3} = f(x' - 2)$. That is, the graph of $y = f(x)$ maps to the graph of $y = 3 - 3f(x - 2)$.

Exercise 3D

Skillsheet
Example 5

1 Find the rule of the image when the graph of each of the functions listed below undergoes each of the following sequences of transformations:

i a dilation of factor 2 from the x-axis, followed by a translation 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis

ii a dilation of factor 3 from the y-axis, followed by a translation 2 units in the negative direction of the x-axis and 4 units in the negative direction of the y-axis

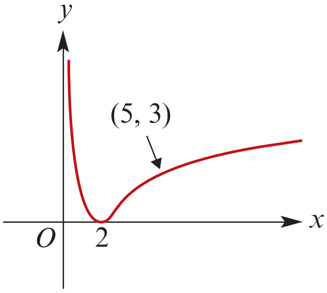
iii a dilation of factor 2 from the x-axis, followed by a reflection in the y-axis.

a $y = x^2$ **b** $y = \sqrt[3]{x}$ **c** $y = \frac{1}{x^2}$

Example 6

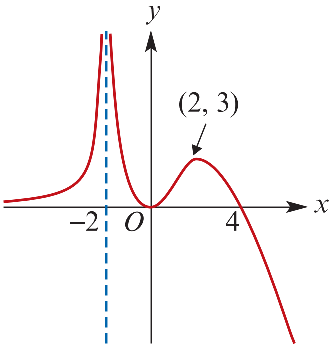
2 Sketch the image of the graph shown under the following sequence of transformations:

- a reflection in the x-axis
- a dilation of factor 2 from the x-axis
- a translation 3 units in the positive direction of the x-axis and 4 units in the positive direction of the y-axis.



3 Sketch the image of the graph shown under the following sequence of transformations:

- a reflection in the y-axis
- a translation 2 units in the negative direction of the x-axis and 3 units in the negative direction of the y-axis
- a dilation of factor 2 from the y-axis.



Example 7

4 Find the rule of the image when the graph of each of the functions listed below undergoes each of the following sequences of transformations:

- **i** a dilation of factor 2 from the x-axis, followed by a reflection in the x-axis, followed by a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis
- **ii** a dilation of factor 2 from the x-axis, followed by a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis, followed by a reflection in the x-axis
- **iii** a reflection in the x-axis, followed by a dilation of factor 2 from the x-axis, followed by a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis
- **iv** a reflection in the x-axis, followed by a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis, followed by a dilation of factor 2 from the x-axis
- **v** a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis, followed by a dilation of factor 2 from the x-axis, followed by a reflection in the x-axis
- **vi** a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis, followed by a reflection in the x-axis, followed by a dilation of factor 2 from the x-axis.

a $y = x^2$ **b** $y = \sqrt[3]{x}$ **c** $y = \dfrac{1}{x}$ **d** $y = x^4$

e $y = \dfrac{1}{x^3}$ **f** $y = \dfrac{1}{x^4}$ **g** $y = x^{-2}$

5 Find the rule of the image when the graph of $y = \sqrt{x}$ is translated 4 units in the negative direction of the x-axis, reflected in the x-axis and dilated by factor 3 from the y-axis.

Example 8

6 For the graph of $y = \dfrac{3}{x^2}$:

a Sketch the graph of the image under the sequence of transformations:

- a dilation of factor 2 from the x-axis
- a translation of 2 units in the negative direction of the x-axis and 1 unit in the negative direction of the y-axis
- a reflection in the x-axis.

b State the rule of the image.

7 For the graph of $y = x^{\frac{1}{3}}$:

a Sketch the graph of the image under the sequence of transformations:

- a reflection in the y-axis
- a translation of 1 unit in the positive direction of the x-axis and 2 units in the negative direction of the y-axis
- a dilation of factor $\frac{1}{2}$ from the y-axis.

b State the rule of the image.

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3E Determining transformations

The method that has been used to find the effect of a transformation on a graph can be used in reverse to find a sequence of transformations that takes a graph to its image.

For example, to find a sequence of transformations which maps $y = \sqrt{x}$ to $y' = -2\sqrt{x'}$, work backwards through the steps in the solution of Example 5a:

- $y = \sqrt{x}$ maps to $\dfrac{y'}{-2} = \sqrt{x'}$.
- Hence $x = x'$ and $y = \dfrac{y'}{-2}$, and therefore $x' = x$ and $y' = -2y$.
- The transformation is a dilation of factor 2 from the x-axis followed by a reflection in the x-axis.

This can also be done by inspection, of course, if you recognise the form of the image. For the combinations of transformations in this course, it is often simpler to do this.

Example 9

a Find a sequence of transformations which takes the graph of $y = x^2$ to the graph of $y = 2(x-2)^2 + 3$.

b Find a sequence of transformations which takes the graph of $y = \sqrt{x}$ to the graph of $y = \sqrt{5x-2}$.

Solution

a The transformation can be found by inspection, but we shall use the method.

The graph of $y = x^2$ maps to $y' = 2(x'-2)^2 + 3$. Rearranging this equation gives

$$\frac{y'-3}{2} = (x'-2)^2$$

We choose to write $y = \dfrac{y'-3}{2}$ and $x = x' - 2$.

Solving for x' and y' gives

$$x' = x + 2 \quad \text{and} \quad y' = 2y + 3$$

So we can write the transformation as

$$(x, y) \to (x, 2y) \to (x+2, 2y+3)$$

This transformation is a dilation of factor 2 from the x-axis followed by a translation of 2 units in the positive direction of the x-axis and 3 units in the positive direction of the y-axis.

b We have $y = \sqrt{x}$ and $y' = \sqrt{5x'-2}$. We choose to write $y = y'$ and $x = 5x' - 2$. Hence

$$x' = \frac{x+2}{5} = \frac{x}{5} + \frac{2}{5} \quad \text{and} \quad y' = y$$

The transformation is a dilation of factor $\frac{1}{5}$ from the y-axis followed by a translation of $\frac{2}{5}$ units in the positive direction of the x-axis.

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Example 10

a Find a sequence of transformations which takes the graph of $y = \dfrac{3}{(x-1)^2} + 6$ to the graph of $y = \dfrac{1}{x^2}$.

b Find a sequence of transformations which takes the graph of $y = (5x-1)^2 + 6$ to the graph of $y = x^2$.

Solution

a Write $\dfrac{y-6}{3} = \dfrac{1}{(x-1)^2}$ and $y' = \dfrac{1}{(x')^2}$. The points (x, y) satisfying $\dfrac{y-6}{3} = \dfrac{1}{(x-1)^2}$ are mapped to the points (x', y') satisfying $y' = \dfrac{1}{(x')^2}$.

Hence we choose to write

$$y' = \frac{y-6}{3} \quad \text{and} \quad x' = x - 1$$

We can write this transformation as

$$(x, y) \to (x-1, y-6) \to \left(x-1, \frac{y-6}{3}\right)$$

This is a translation of 1 unit in the negative direction of the x-axis and 6 units in the negative direction of the y-axis followed by a dilation of factor $\frac{1}{3}$ from the x-axis.

b Write $y - 6 = (5x-1)^2$ and $y' = (x')^2$. The points (x, y) satisfying $y - 6 = (5x-1)^2$ are mapped to the points (x', y') satisfying $y' = (x')^2$.

Hence we choose to write

$$y' = y - 6 \quad \text{and} \quad x' = 5x - 1$$

One transformation is a dilation of factor 5 from the y-axis followed by a translation of 1 unit in the negative direction of the x-axis and 6 units in the negative direction of the y-axis.

We note that the transformations we found are far from being the only possible answers. In fact there are infinitely many choices.

Section summary

The notation developed in this chapter can be used to help find the transformation that takes the graph of a function to its image.

For example, if the graph of $y = f(x)$ is mapped to the graph of $y' = 2f(x'-3)$, we can see that the transformation

$$x' = x + 3 \quad \text{and} \quad y' = 2y$$

is a suitable choice. This is a translation of 3 units to the right followed by a dilation of factor 2 from the x-axis.

There are infinitely many transformations that take the graph of $y = f(x)$ to the graph of $y' = 2f(x'-3)$. The one we chose is conventional.

Exercise 3E

Skillsheet

Example 9

1 For each of the following, find a sequence of transformations that takes:

a the graph of $y = x^3$ to the graph of:

i $y = 2(x - 1)^3 + 3$ **ii** $y = -(x + 1)^3 + 2$ **iii** $y = (2x + 1)^3 - 2$

b the graph of $y = \frac{1}{x^2}$ to the graph of:

i $y = \frac{2}{(x + 3)^2}$ **ii** $y = \frac{1}{(x + 3)^2} + 2$ **iii** $y = \frac{1}{(x - 3)^2} - 2$

c the graph of $y = \sqrt[3]{x}$ to the graph of:

i $y = \sqrt[3]{x + 3} + 2$ **ii** $y = 2\sqrt[3]{3x}$ **iii** $y = -\sqrt[3]{x} + 2$

Example 10

2 **a** Find a sequence of transformations that takes the graph of $y = (x - 1)^2 + 6$ to the graph of $y = x^2$.

b Find a sequence of transformations that takes the graph of $y = 2(x - 1)^2 - 3$ to the graph of $y = x^2$.

c Find a sequence of transformations that takes the graph of $y = \frac{1}{(x - 1)^2} - 6$ to the graph of $y = \frac{1}{x^2}$.

d Find a sequence of transformations that takes the graph of $y = \frac{2}{(x - 1)^2} - 5$ to the graph of $y = \frac{1}{x^2}$.

e Find a sequence of transformations that takes the graph of $y = (2x - 1)^2 + 6$ to the graph of $y = x^2$.

3 **a** Find a sequence of transformations that takes the graph of $y = \frac{5}{(x - 3)^2} - 7$ to the graph of $y = \frac{1}{x^2}$.

b Find a sequence of transformations that takes the graph of $y = (3x + 2)^2 + 5$ to the graph of $y = x^2$.

c Find a sequence of transformations that takes the graph of $y = -3(3x + 1)^2 + 7$ to the graph of $y = x^2$.

d Find a sequence of transformations that takes the graph of $y = 2\sqrt{4 - x}$ to the graph of $y = \sqrt{x}$.

e Find a sequence of transformations that takes the graph of $y = 2\sqrt{4 - x} + 3$ to the graph of $y = -\sqrt{x} + 6$.

4 In each case below, state a sequence of transformations that transforms the graph of the first equation into the graph of the second equation:

a $y = \frac{1}{x}$, $y = \frac{2}{x - 1} + 3$ **b** $y = \frac{1}{x^2}$, $y = \frac{2}{(x + 4)^2} - 7$ **c** $y = \frac{1}{x^3}$, $y = \frac{4}{(1 - x)^3} - 5$

d $y = \sqrt[3]{x}$, $y = 2 - \sqrt[3]{x + 1}$ **e** $y = \frac{1}{\sqrt{x}}$, $y = \frac{2}{\sqrt{-x}} + 3$ **f** $y = \frac{2}{3 - x} + 4$, $y = \frac{1}{x}$

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3F Using transformations to sketch graphs

By considering a rule for a graph as a combination of transformations of a more ‘simple’ rule, we can readily sketch graphs of many apparently ‘complicated’ functions.

Example 11

Identify a sequence of transformations that maps the graph of $y = \dfrac{1}{x}$ onto the graph of $y = \dfrac{4}{x+5} - 3$. Use this to sketch the graph of $y = \dfrac{4}{x+5} - 3$, stating the equations of asymptotes and the coordinates of axis intercepts.

Solution

Rearrange the equation of the transformed graph to have the same ‘shape’ as $y = \dfrac{1}{x}$:

$$\frac{y' + 3}{4} = \frac{1}{x' + 5}$$

where (x', y') are the coordinates of the image of (x, y).

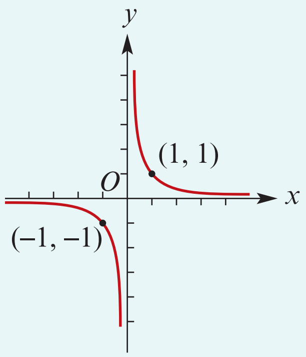
Therefore $x = x' + 5$ and $y = \dfrac{y' + 3}{4}$. Rearranging gives $x' = x - 5$ and $y' = 4y - 3$.

The mapping is $(x, y) \to (x - 5, 4y - 3)$, and so a sequence of transformations is:

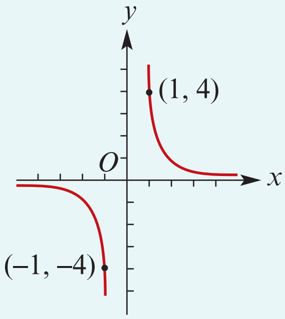
1. a dilation of factor 4 from the x-axis
2. a translation of 5 units in the negative direction of the x-axis
3. a translation of 3 units in the negative direction of the y-axis.

The original graph $y = \dfrac{1}{x}$ is shown on the right.

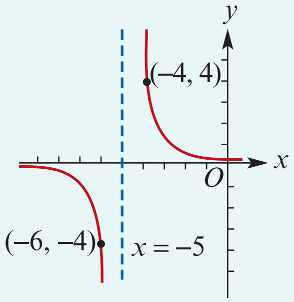
The effect of the transformations is shown below.



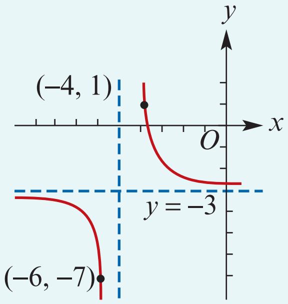
1 Dilation from x-axis:



2 Translation in negative direction of x-axis:



3 Translation in negative direction of y-axis:



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Find the axis intercepts in the usual way, as below. The transformed graph, with asymptotes and intercepts marked, is shown on the right.

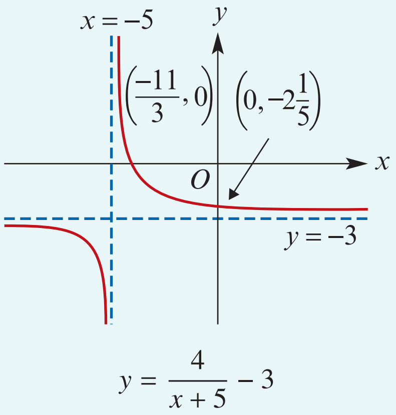
When $x = 0$, $y = \frac{4}{5} - 3 = -2\frac{1}{5}$

When $y = 0$, $\frac{4}{x+5} - 3 = 0$

$$4 = 3x + 15$$

$$3x = -11$$

$$\therefore \quad x = -\frac{11}{3}$$



Once you have done a few of these types of exercises, you can identify the transformations more quickly by carefully observing the rule of the transformed graph and relating it to the 'shape' of the simplest function in its family. Consider the following examples.

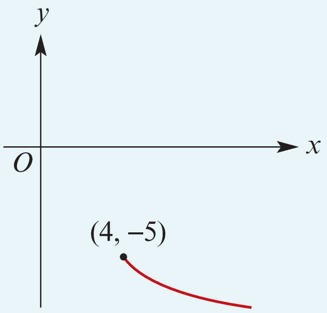
Example 12

Sketch the graph of $y = -\sqrt{x-4} - 5$.

Solution

The graph is obtained from the graph of $y = \sqrt{x}$ by:

- a reflection in the x-axis, followed by a translation of 5 units in the negative direction of the y-axis, and
- a translation of 4 units in the positive direction of the x-axis.



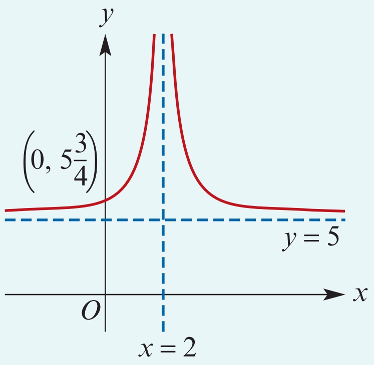
Example 13

Sketch the graph of $y = \frac{3}{(x-2)^2} + 5$.

Solution

This is obtained from the graph of $y = \frac{1}{x^2}$ by:

- a dilation of factor 3 from the x-axis, followed by a translation of 5 units in the positive direction of the y-axis, and
- a translation of 2 units in the positive direction of the x-axis.



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Section summary

In general, the function given by the equation

$$y = Af(n(x+c)) + b$$

where $b, c \in \mathbb{R}^+$ and $A, n \in \mathbb{R}^+$, represents a transformation of the graph of $y = f(x)$ by:

- a dilation of factor A from the x-axis, followed by a translation of b units in the positive direction of the y-axis, and
- a dilation of factor $\frac{1}{n}$ from the y-axis, followed by a translation of c units in the negative direction of the x-axis.

Similar statements can be made for $b, c \in \mathbb{R}^-$. The case where $A \in \mathbb{R}^-$ corresponds to a reflection in the x-axis and a dilation from the x-axis. The case where $n \in \mathbb{R}^-$ corresponds to a reflection in the y-axis and a dilation from the y-axis.

Exercise 3F

Example 11

1 Sketch the graph of each of the following. State the equations of asymptotes and the axis intercepts. State the range of each function.

a $f(x) = \dfrac{3}{x-1}$ **b** $g(x) = \dfrac{2}{x+1} - 1$ **c** $h(x) = \dfrac{3}{(x-2)^2}$

d $f(x) = \dfrac{2}{(x-1)^2} - 1$ **e** $h(x) = \dfrac{-1}{x-3}$ **f** $f(x) = \dfrac{-1}{x+2} + 3$

g $f(x) = \dfrac{2}{(x-3)^2} + 4$

Example 12, 13

2 Sketch the graph of each of the following without using your calculator. State the range of each.

a $y = -\sqrt{x-3}$ **b** $y = -\sqrt{x-3} + 2$ **c** $y = \sqrt{2(x+3)}$

d $y = \dfrac{1}{2x-3}$ **e** $y = 5\sqrt{x+2}$ **f** $y = -5\sqrt{x+2} - 2$

g $y = \dfrac{-3}{x-2}$ **h** $y = \dfrac{-2}{(x+2)^2} - 4$ **i** $y = \dfrac{3}{2x} - 5$

j $y = \dfrac{5}{2x} + 5$ **k** $y = 2(x-3)^2 + 5$

3 Show that $\dfrac{3x+2}{x+1} = 3 - \dfrac{1}{x+1}$ and hence, without using your calculator, sketch the graph of

$$f\colon \mathbb{R} \setminus \{-1\} \to \mathbb{R},\ f(x) = \frac{3x+2}{x+1}$$

4 Show that $\dfrac{4x-5}{2x+1} = 2 - \dfrac{7}{2x+1}$ and hence, without using your calculator, sketch the graph of

$$f : \mathbb{R} \setminus \{-\tfrac{1}{2}\} \to \mathbb{R},\ f(x) = \frac{4x-5}{2x+1}$$

Hint: $f(x) = 2 - \dfrac{7}{2(x+\frac{1}{2})}$

5 Sketch the graph of each of the following without using your calculator. State the range of each.

a $y = \dfrac{2}{x-3} + 4$

b $y = \dfrac{4}{3-x} + 4$

c $y = \dfrac{2}{(x-1)^2} + 1$

d $y = 2\sqrt{x-1} + 2$

e $y = -3\sqrt{x-4} + 1$

f $y = 5\sqrt{2x+4} + 1$

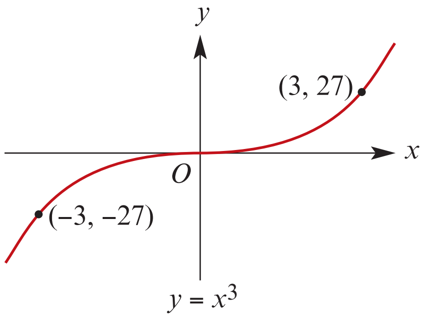
3G Transformations of power functions with positive integer index

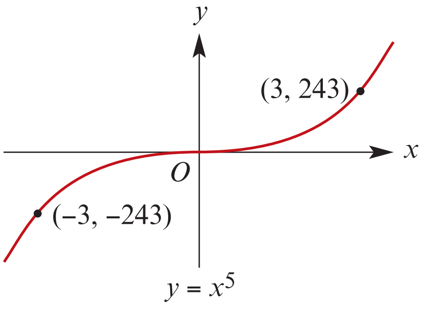
We recall that every quadratic polynomial function can be written in the turning point form $y = a(x-h)^2 + k$. This is not true for polynomials of higher degree. However, there are many polynomials that can be written as $y = a(x-h)^n + k$.

In Chapter 1 we introduced power functions, which include functions with rule $f(x) = x^n$, where n is a positive integer. In this section we continue our study of such functions and, in particular, we look at transformations of these functions.

▶ The function $f(x) = x^n$ where n is an odd positive integer

The diagrams below show the graphs of $y = x^3$ and $y = x^5$.





Assume that n is an odd integer with $n \geq 3$. From Mathematical Methods Units 1 & 2, you will recall that the derivative function of $f(x) = x^n$ has rule

$$f'(x) = nx^{n-1}$$

Hence the gradient is zero when $x = 0$. Since n is odd and therefore $n - 1$ is even, we have $f'(x) = nx^{n-1} > 0$ for all non-zero x. That is, the gradient of the graph of $y = f(x)$ is positive for all non-zero x and is zero when $x = 0$. Recall that, for functions of this form, the stationary point at $(0, 0)$ is called a stationary point of inflection.

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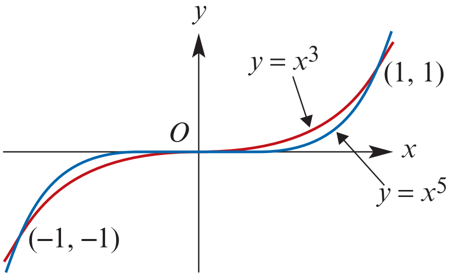
Comparing the graphs of $y = x^n$ and $y = x^m$ for n and m odd

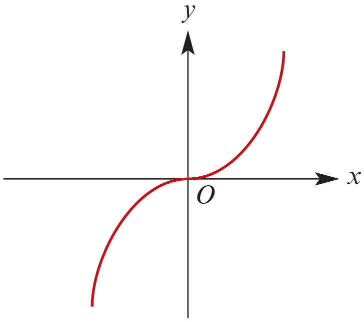
Assume that n and m are odd positive integers with $n > m$. Then:

- $x^n = x^m$ for $x = -1, 0, 1$
- $x^n > x^m$ for $-1 < x < 0$ and for $x > 1$
- $x^n < x^m$ for $0 < x < 1$ and for $x < -1$.



It should be noted that the appearance of graphs is dependent on the scales on the x- and y-axes.

Power functions of odd degree are often depicted as shown.



Transformations of $f(x) = x^n$ where n is an odd positive integer

Transformations of these functions result in graphs with rules of the form $y = a(x - h)^n + k$ where a, h and k are real constants.

Example 14

Sketch the graph of:

a $y = (x - 2)^3 + 1$ **b** $y = -(x - 1)^3 + 2$ **c** $y = 2(x + 1)^3 + 2$

Solution

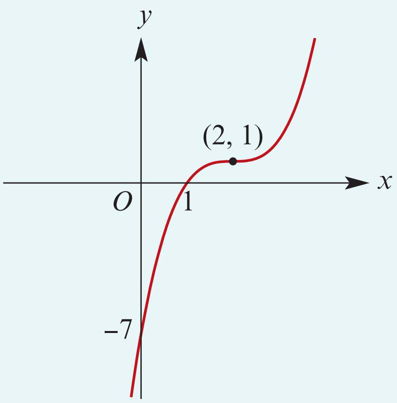
a The translation $(x, y) \to (x + 2, y + 1)$ maps the graph of $y = x^3$ onto the graph of $y = (x - 2)^3 + 1$.

So $(2, 1)$ is a point of zero gradient.

Find the axis intercepts:

- When $x = 0$, $y = (-2)^3 + 1 = -7$
- When $y = 0$,

$$\begin{aligned} 0 &= (x-2)^3 + 1 \\ -1 &= (x-2)^3 \\ -1 &= x - 2 \\ \therefore \quad x &= 1 \end{aligned}$$



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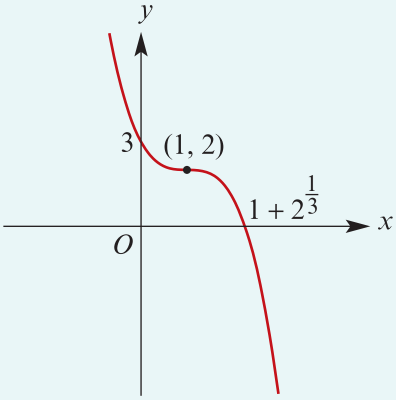
b A reflection in the x-axis followed by the translation $(x, y) \to (x + 1, y + 2)$ maps the graph of $y = x^3$ onto the graph of $y = -(x - 1)^3 + 2$.

So $(1, 2)$ is a point of zero gradient.

Find the axis intercepts:

- When $x = 0$, $y = -(-1)^3 + 2 = 3$
- When $y = 0$,

$$\begin{aligned} 0 &= -(x - 1)^3 + 2 \\ (x - 1)^3 &= 2 \\ x - 1 &= 2^{\frac{1}{3}} \\ \therefore \quad x &= 1 + 2^{\frac{1}{3}} \approx 2.26 \end{aligned}$$



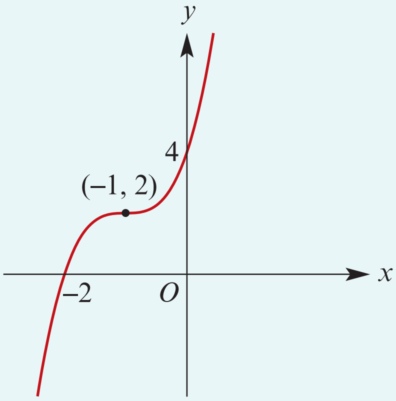
c A dilation of factor 2 from the x-axis followed by the translation $(x, y) \to (x - 1, y + 2)$ maps the graph of $y = x^3$ onto the graph of $y = 2(x + 1)^3 + 2$.

So $(-1, 2)$ is a point of zero gradient.

Find the axis intercepts:

- When $x = 0$, $y = 2 + 2 = 4$
- When $y = 0$,

$$\begin{aligned} 0 &= 2(x + 1)^3 + 2 \\ -1 &= (x + 1)^3 \\ -1 &= x + 1 \\ \therefore \quad x &= -2 \end{aligned}$$



Example 15

The graph of $y = a(x - h)^3 + k$ has a point of zero gradient at $(1, 1)$ and passes through the point $(0, 4)$. Find the values of a, h and k.

Solution

Since $(1, 1)$ is the point of zero gradient,

$$h = 1 \quad \text{and} \quad k = 1$$

So $y = a(x - 1)^3 + 1$ and, since the graph passes through $(0, 4)$,

$$\begin{aligned} 4 &= -a + 1 \\ \therefore \quad a &= -3 \end{aligned}$$

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Example 16

a Find the rule for the image of the graph of $y = x^5$ under the following sequence of transformations:

- reflection in the y-axis
- dilation of factor 2 from the y-axis
- translation 2 units in the positive direction of the x-axis and 3 units in the positive direction of the y-axis.

b Find a sequence of transformations which takes the graph of $y = x^5$ to the graph of $y = 6 - 2(x + 5)^5$.

Solution

a $(x, y) \to (-x, y) \to (-2x, y) \to (-2x + 2, y + 3)$

Let (x', y') be the image of (x, y) under this transformation.

Then $x' = -2x + 2$ and $y' = y + 3$. Hence $x = \dfrac{x' - 2}{-2}$ and $y = y' - 3$.

Therefore the graph of $y = x^5$ maps to the graph of

$$y' - 3 = \left(\frac{x' - 2}{-2}\right)^5$$

i.e. to the graph of

$$y = -\frac{1}{32}(x - 2)^5 + 3$$

b Rearrange $y' = 6 - 2(x' + 5)^5$ to $\dfrac{y' - 6}{-2} = (x' + 5)^5$.

Therefore $y = \dfrac{y' - 6}{-2}$ and $x = x' + 5$, which gives $y' = -2y + 6$ and $x' = x - 5$.

The sequence of transformations is:

- reflection in the x-axis
- dilation of factor 2 from the x-axis
- translation 5 units in the negative direction of the x-axis and 6 units in the positive direction of the y-axis.

The function $f(x) = x^n$ where n is an even positive integer

Assume that n is an even integer with $n \geq 2$. From Mathematical Methods Units 1 & 2, you will recall that the derivative function of $f(x) = x^n$ has rule

$$f'(x) = nx^{n-1}$$

Hence the gradient is zero when $x = 0$. Since n is even and therefore $n - 1$ is odd, we have $f'(x) = nx^{n-1} > 0$ for $x > 0$, and $f'(x) = nx^{n-1} < 0$ for $x < 0$. Thus the graph of $y = f(x)$ has a turning point at $(0, 0)$; this point is a local minimum.

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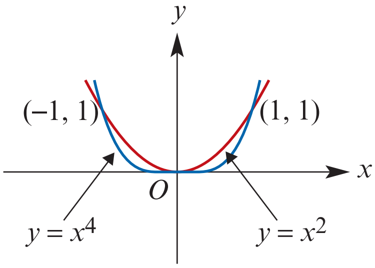
Comparing the graphs of $y = x^n$ and $y = x^m$ for n and m even

Assume that n and m are even positive integers with $n > m$. Then:

- $x^n = x^m$ for $x = -1, 0, 1$
- $x^n > x^m$ for $x < -1$ and for $x > 1$
- $x^n < x^m$ for $-1 < x < 0$ and for $0 < x < 1$.



Example 17

The graph of $y = a(x - h)^4 + k$ has a turning point at $(2, 2)$ and passes through the point $(0, 4)$. Find the values of a, h and k.

Solution

Since $(2, 2)$ is the turning point,

$$h = 2 \quad \text{and} \quad k = 2$$

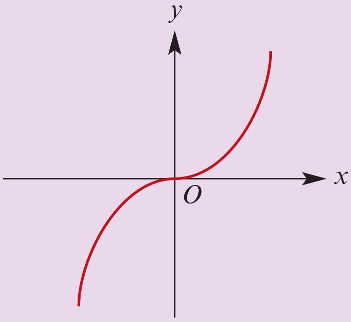
So $y = a(x - 2)^4 + 2$ and, since the graph passes through $(0, 4)$,

$$4 = 16a + 2$$

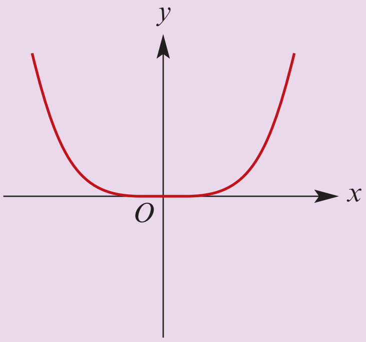
$$\therefore \quad a = \frac{1}{8}$$

Section summary

- A graph with rule of the form $y = a(x - h)^n + k$ can be obtained as a transformation of the graph of $y = x^n$.
- **Odd index** If n is an odd integer with $n \geq 3$, then the graph of $y = x^n$ has a shape similar to the one shown below; there is a point of zero gradient at $(0, 0)$.



- **Even index** If n is an even integer with $n \geq 2$, then the graph of $y = x^n$ has a shape similar to the one shown below; there is a turning point at $(0, 0)$.



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Exercise 3G

Example 14 **1** Sketch the graph of each of the following. State the coordinates of the point of zero gradient and the axis intercepts.

a $f(x) = 2x^3$	**b** $g(x) = -2x^3$
c $h(x) = x^5 + 1$	**d** $f(x) = x^3 - 4$
e $f(x) = (x + 1)^3 - 8$	**f** $f(x) = 2(x - 1)^3 - 2$
g $g(x) = -2(x - 1)^3 + 2$	**h** $h(x) = 3(x - 2)^3 - 4$
i $f(x) = 2(x - 1)^3 + 2$	**j** $h(x) = -2(x - 1)^3 - 4$
k $f(x) = (x + 1)^5 - 32$	**l** $f(x) = 2(x - 1)^5 - 2$

Example 15 **2** The graph of $y = a(x - h)^3 + k$ has a point of zero gradient at $(0, 4)$ and passes through the point $(1, 1)$. Find the values of a, h and k.

3 Find the equation of the image of $y = x^3$ under each of the following transformations:

a a dilation of factor 3 from the x-axis

b a translation with rule $(x, y) \to (x - 1, y + 1)$

c a reflection in the x-axis followed by the translation $(x, y) \to (x + 2, y - 3)$

d a dilation of factor 2 from the x-axis followed by the translation $(x, y) \to (x-1, y-2)$

e a dilation of factor 3 from the y-axis.

Example 16 **4** **a** Find the rule for the image of the graph of $y = x^3$ under the following sequence of transformations:

- reflection in the y-axis
- dilation of factor 3 from the y-axis
- translation 3 units in the positive direction of the x-axis and 1 unit in the positive direction of the y-axis.

b Find a sequence of transformations which takes the graph of $y = x^3$ to the graph of $y = 4 - 3(x + 1)^3$.

5 Find the rule for the image of the graph of $y = x^4$ under the following sequence of transformations:

- reflection in the y-axis
- dilation of factor 2 from the y-axis
- translation 2 units in the negative direction of the x-axis and 1 unit in the negative direction of the y-axis.

6 Find a sequence of transformations which takes the graph of $y = x^4$ to the graph of $y = 5 - 3(x + 1)^4$.

7 By applying suitable transformations to $y = x^4$, sketch the graph of each of the following:

a $y = 3(x - 1)^4 - 2$ **b** $y = -2(x + 2)^4$ **c** $y = (x - 2)^4 - 6$

d $y = 2(x - 3)^4 - 1$ **e** $y = 1 - (x + 4)^4$ **f** $y = -3(x - 2)^4 - 3$

Example 17

8 The graph of $y = a(x - h)^4 + k$ has a turning point at $(-2, 3)$ and passes through the point $(0, -6)$. Find the values of a, h and k.

9 The graph of $y = a(x - h)^4 + k$ has a turning point at $(1, 7)$ and passes through the point $(0, 23)$. Find the values of a, h and k.

3H Determining the rule for a function from its graph

Given sufficient information about a curve, we can determine its rule. For example, if we know the coordinates of two points on a hyperbola of the form

$$y = \frac{a}{x} + b$$

then we can find the rule for the hyperbola, i.e. we can find the values of a and b.

Sometimes the rule has a more specific form. For example, the curve may be a dilation of $y = \sqrt{x}$. Then we know its rule is of the form $y = a\sqrt{x}$, and the coordinates of one point on the curve (with the exception of the origin) will be enough to determine the value of a.

Example 18

The points $(1, 5)$ and $(4, 2)$ lie on a curve with equation $y = \frac{a}{x} + b$. Find the values of a and b.

Solution

When $x = 1$, $y = 5$ and so

$$5 = a + b \qquad (1)$$

When $x = 4$, $y = 2$ and so

$$2 = \frac{a}{4} + b \qquad (2)$$

Subtract (2) from (1):

$$3 = \frac{3a}{4}$$

$$\therefore \quad a = 4$$

Substitute in (1) to find b:

$$5 = 4 + b$$

$$\therefore \quad b = 1$$

The equation of the curve is $y = \frac{4}{x} + 1$.

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Example 19

The points (2, 1) and (10, 6) lie on a curve with equation $y = a\sqrt{x-1} + b$. Find the values of a and b.

Solution

When $x = 2$, $y = 1$ and so

$$1 = a\sqrt{1} + b$$

i.e. $1 = a + b$ (1)

When $x = 10$, $y = 6$ and so

$$6 = a\sqrt{9} + b$$

i.e. $6 = 3a + b$ (2)

Subtract (1) from (2):

$$5 = 2a$$

$$\therefore \quad a = \frac{5}{2}$$

Substitute in (1) to find $b = -\frac{3}{2}$.

The equation of the curve is $y = \frac{5}{2}\sqrt{x-1} - \frac{3}{2}$.

Exercise 3H

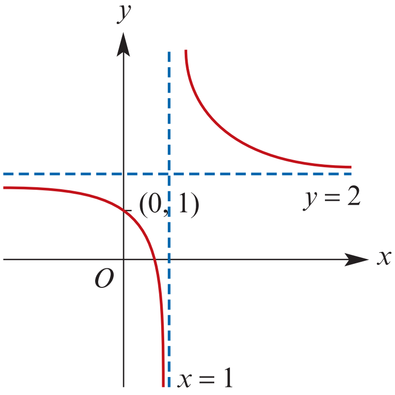
Skillsheet
Example 18

1 The points (1, 4) and (3, 1) lie on a curve with equation $y = \frac{a}{x} + b$. Find the values of a and b.

2 The graph shown has the rule

$$y = \frac{A}{x+b} + B$$

Find the values of A, b and B.



Example 19

3 The points (3, 1) and (11, 6) lie on a curve with equation $y = a\sqrt{x-2} + b$. Find the values of a and b.

4 The points with coordinates (1, 5) and (16, 11) lie on a curve which has a rule of the form $y = A\sqrt{x} + B$. Find A and B.

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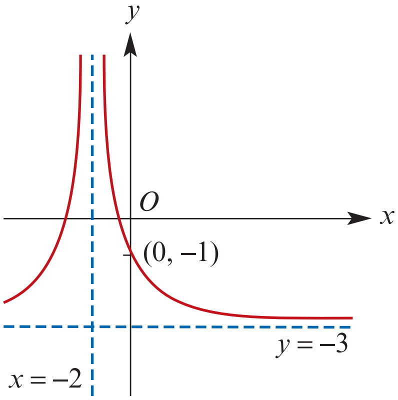
5 The points with coordinates $(1, 1)$ and $(0.5, 7)$ lie on a curve which has a rule of the form $y = \dfrac{A}{x^2} + B$. Find the values of A and B.

6 The graph shown has the rule

$$y = \frac{A}{(x+b)^2} + B$$

Find the values of A, b and B.



7 The points with coordinates $(1, -1)$ and $(2, \frac{3}{4})$ lie on a curve which has a rule of the form $y = \dfrac{a}{x^3} + b$. Find the values of a and b.

8 The points with coordinates $(-1, 4)$ and $(1, -8)$ lie on a curve which has a rule of the form $y = a\sqrt[3]{x} + b$. Find the values of a and b.

3I Using matrices for transformations

The following table gives a summary of some basic transformations of the plane. In each row of the table, the point (x', y') is the image of the point (x, y) under the mapping.

Mapping	Rule	
Reflection in the x-axis	$x' = x$ $y' = -y$	$= x + 0y$ $= 0x + -y$
Reflection in the y-axis	$x' = -x$ $y' = y$	$= -x + 0y$ $= 0x + y$
Dilation of factor k from the y-axis	$x' = kx$ $y' = y$	$= kx + 0y$ $= 0x + y$
Dilation of factor k from the x-axis	$x' = x$ $y' = ky$	$= x + 0y$ $= 0x + ky$
Reflection in the line $y = x$	$x' = y$ $y' = x$	$= 0x + y$ $= x + 0y$
Translation defined by a column matrix $\begin{bmatrix} a \\ b \end{bmatrix}$	$x' = x + a$ $y' = y + b$	

We have discussed most of the transformations from this table already in this chapter. Reflection in the line $y = x$ occurred in our consideration of inverse functions in Chapter 1.

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The first five mappings given in the table are special cases of a general kind of mapping defined by

$$x' = ax + by$$
$$y' = cx + dy$$

where a, b, c, d are real numbers.

This mapping can be defined equivalently using a matrix equation:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

A transformation of the form

$$(x, y) \to (ax + by, cx + dy)$$

is called a **linear transformation**.

These first five mappings can each be defined by a 2 × 2 matrix. This is shown in the following table.

Mapping	Rule	Matrix
Reflection in the x-axis	$x' = 1x + 0y$ $y' = 0x - 1y$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection in the y-axis	$x' = -1x + 0y$ $y' = 0x + 1y$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Dilation of factor k from the y-axis	$x' = kx + 0y$ $y' = 0x + 1y$	$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$
Dilation of factor k from the x-axis	$x' = 1x + 0y$ $y' = 0x + ky$	$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$
Reflection in the line $y = x$	$x' = 0x + 1y$ $y' = 1x + 0y$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Example 20

Find the image of the point (2, 3) under:

a a reflection in the x-axis

b a dilation of factor k from the y-axis.

Solution

a $$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Therefore $(2, 3) \to (2, -3)$.

b $$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2k \\ 3 \end{bmatrix}$$

Therefore $(2, 3) \to (2k, 3)$.

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Example 21

Consider a linear transformation such that $(1, 0) \to (3, -1)$ and $(0, 1) \to (-2, 4)$. Find the image of $(-3, 5)$.

Solution

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

$$\therefore \quad a = 3,\ c = -1 \qquad \text{and} \quad b = -2,\ d = 4$$

The transformation can be defined by the 2×2 matrix $\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix}$.

Let $(-3, 5) \to (x', y')$.

$$\text{Then} \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \times (-3) + (-2) \times 5 \\ (-1) \times (-3) + 4 \times 5 \end{bmatrix}$$

$$= \begin{bmatrix} -19 \\ 23 \end{bmatrix}$$

The image of $(-3, 5)$ is $(-19, 23)$.

Note that a non-linear transformation cannot be represented by a matrix in the way indicated above. For example, the translation

$$x' = x + a$$

$$y' = y + b$$

cannot be represented by a square matrix. However, we can write

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$$

using matrix addition.

Composition of mappings

Consider two linear transformations defined by matrices

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

We can compose the transformation of **A** with the transformation of **B**.

The **composition** consists of the transformation of **A** being applied first and then the transformation of **B**. The matrix of the resulting composition is the product **BA**:

$$\mathbf{BA} = \begin{bmatrix} b_{11}a_{11} + b_{12}a_{21} & b_{11}a_{12} + b_{12}a_{22} \\ b_{21}a_{11} + b_{22}a_{21} & b_{21}a_{12} + b_{22}a_{22} \end{bmatrix}$$

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Example 22

Find the image of the point $(2, -3)$ under a reflection in the x-axis followed by a dilation of factor k from the y-axis.

Solution

Matrix multiplication gives the matrix of the composition of the transformations.

Let **A** be the matrix for reflection in the x-axis, and let **B** be the matrix for dilation of factor k from the y-axis.

The required transformation is defined by the product

$$\mathbf{BA} = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & -1 \end{bmatrix}$$

Since

$$\mathbf{BA}\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 2k \\ 3 \end{bmatrix}$$

the image of $(2, -3)$ is $(2k, 3)$.

Example 23

Express the composition of the transformations dilation of factor k from the y-axis followed by a translation defined by the matrix $\mathbf{C} = \begin{bmatrix} a \\ b \end{bmatrix}$, mapping a point (x, y) to a point (x', y'), as a matrix equation. Hence find x and y in terms of x' and y' respectively.

Solution

Let **A** be the matrix of the dilation transformation, let $\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$ and let $\mathbf{X}' = \begin{bmatrix} x' \\ y' \end{bmatrix}$.

The equation is $\mathbf{AX} + \mathbf{C} = \mathbf{X}'$.

Now $\mathbf{A} = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ and hence $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$.

Thus $\begin{bmatrix} kx + a \\ y + b \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$, which implies $x' = kx + a$ and $y' = y + b$.

Hence $x = \frac{1}{k}(x' - a)$ and $y = y' - b$.

Notes:

- A more direct approach is to write $(x, y) \to (kx, y) \to (kx + a, y + b) = (x', y')$ and then solve for x and y as in the last line of the solution.
- Another approach is to use the inverse of the matrix **A** to solve the matrix equation for **X**. We will look at this briefly in the last section of this chapter.

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▶ Transforming graphs

We now apply matrices to transforming graphs. The notation is consistent with the notation introduced earlier in this chapter.

Example 24

A transformation is defined by the matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Find the equation of the image of the graph of the quadratic equation $y = x^2 + 2x + 3$ under this transformation.

Solution

As before, the transformation maps $(x, y) \to (x', y')$.

Using matrix notation,

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

We can multiply out to find $x' = x$ and $y' = 2y$. Hence $x = x'$ and $y = \dfrac{y'}{2}$.

Substitute in $y = x^2 + 2x + 3$ to find

$$\frac{y'}{2} = (x')^2 + 2x' + 3$$

Hence the image has equation $y = 2x^2 + 4x + 6$.

Example 25

A transformation is described by the matrix equation

$$\mathbf{A}(\mathbf{X} + \mathbf{B}) = \mathbf{X}', \qquad \text{where} \quad \mathbf{A} = \begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Find the image of the straight line with equation $y = 2x + 5$ under this transformation.

Solution

$$\mathbf{A}(\mathbf{X} + \mathbf{B}) = \mathbf{X}'$$

$$\begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x + 1 \\ y + 2 \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$\begin{bmatrix} -3(y + 2) \\ 2(x + 1) \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

Therefore $x' = -3(y + 2)$ and $y' = 2(x + 1)$, and so $x = \dfrac{y'}{2} - 1$ and $y = -\dfrac{x'}{3} - 2$.

The straight line with equation $y = 2x + 5$ is transformed to the straight line with equation

$$-\frac{x'}{3} - 2 = 2\left(\frac{y'}{2} - 1\right) + 5$$

Rearranging gives $y' = -\dfrac{x'}{3} - 5$.

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The notation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is often used to indicate that a transformation is a mapping from the Cartesian plane into the Cartesian plane. The rule can then be defined through the use of matrices as shown in the following example.

Example 26

A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Find the image of the curve with equation $y = \dfrac{1}{x}$ under this transformation.

Solution

Let $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x' \\ y' \end{bmatrix}$. Then $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

$$= \begin{bmatrix} 5x \\ y \end{bmatrix} + \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Hence $x' = 5x + 5$ and $y' = y + 2$, giving $x = \dfrac{x' - 5}{5}$ and $y = y' - 2$.

Substituting in $y = \dfrac{1}{x}$ we obtain

$$y' - 2 = \frac{5}{x' - 5}$$

The image has equation $y = \dfrac{5}{x - 5} + 2$.

Section summary

- Transformation matrices

Mapping	Rule	Matrix
Reflection in the x-axis	$x' = x$ $y' = -y$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection in the y-axis	$x' = -x$ $y' = y$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Dilation of factor k from the y-axis	$x' = kx$ $y' = y$	$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$
Dilation of factor k from the x-axis	$x' = x$ $y' = ky$	$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$
Reflection in the line $y = x$	$x' = y$ $y' = x$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

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- **Composition of transformations** We can consider the composition of two linear transformations defined by matrices

  $$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

  The composition consists of the transformation of **A** being applied first and then the transformation of **B**. The matrix of the resulting composition is the product **BA**.
- **Mapping notation** The notation $T: \mathbb{R}^2 \to \mathbb{R}^2$ is often used to indicate that a transformation is a mapping from the Cartesian plane into the Cartesian plane. A transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ can be defined using matrices; for example, by

  $$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Exercise 3I

Example 20

1 Using matrix methods, find the image of the point $(1, -2)$ under each of the following transformations:

a dilation of factor 3 from the x-axis
b dilation of factor 2 from the y-axis
c reflection in the x-axis
d reflection in the y-axis
e reflection in the line $y = x$.

2 Write down the matrix of each of the following transformations:

a reflection in the line $x = 0$
b reflection in the line $y = x$
c reflection in the line $y = -x$
d dilation of factor 2 from the x-axis
e dilation of factor 3 from the y-axis.

Example 21

3 Consider a linear transformation such that $(1, 0) \to (2, -1)$ and $(0, 1) \to (-2, 5)$. Find the image of $(-2, 1)$ under this transformation.

4 Consider a linear transformation such that $(1, 0) \to (2, -1)$ and $(0, 1) \to (4, -2)$. Find the image of $(-3, 5)$ under this transformation.

Example 22

5 Find the matrix that determines the composition of following transformations, in the given order, and find the image of the point $(-3, 2)$ under this transformation:

- reflection in the x-axis
- dilation of factor 2 from the x-axis.

Example 23

6 Express as a matrix equation the composition of the transformations dilation of factor 3 from the x-axis followed by a translation defined by the matrix $\mathbf{C} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, mapping a point (x, y) to a point (x', y'). Hence find x and y in terms of x' and y' respectively.

Example 24

7 A transformation is defined by the matrix $\begin{bmatrix} -4 & 0 \\ 0 & 2 \end{bmatrix}$. Find the equation of the image of the graph of the quadratic equation $y = x^2 + x + 2$ under this transformation.

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8 A transformation is defined by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$. Find the equation of the image of the graph of the cubic equation $y = x^3 + 2x$ under this transformation.

9 A transformation is defined by the matrix $\begin{bmatrix} 0 & 3 \\ -2 & 0 \end{bmatrix}$. Find the equation of the image of the straight line with equation $y = 2x + 3$ under the transformation.

10 A transformation is defined by the matrix $\begin{bmatrix} 0 & 2 \\ -3 & 0 \end{bmatrix}$. Find the equation of the image of the straight line with equation $y = -2x + 4$ under the transformation.

Example 25 **11** A transformation is described by the equation $\mathbf{A}(\mathbf{X} + \mathbf{B}) = \mathbf{X}'$, where $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -3 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Find the image of the straight line with equation $y = -2x + 6$ under the transformation.

12 A transformation is described by the equation $\mathbf{AX} + \mathbf{B} = \mathbf{X}'$, where $\mathbf{A} = \begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. Find the image of the straight line with equation $y = -2x + 6$ under the transformation.

13 A transformation is described by the equation $\mathbf{AX} + \mathbf{B} = \mathbf{X}'$, where $\mathbf{A} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$. Find the image of the curve with equation $y = -2x^3 + 6x$ under the transformation.

14 A transformation is described by the equation $\mathbf{AX} + \mathbf{B} = \mathbf{X}'$, where $\mathbf{A} = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Find the image of the curve with equation $y = -2x^3 + 6x^2 + 2$ under the transformation.

Example 26 **15** A transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Find the image of the curve with equation $y = x^2$ under this transformation.

16 A transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

Find the image of the curve with equation $y = \dfrac{1}{x^2}$ under this transformation.

17 A transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Find the image of the curve with equation $y = 3(x - 2)^2 - 4$ under this transformation.

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3J Using the inverse of a 2 × 2 matrix for transformations

This section deals with using inverse matrices with transformations. It is not necessary for the study of transformations. Further information on matrices is available in the Interactive Textbook.

- The identity matrix for the family of $n \times n$ matrices is the matrix with ones in the 'top left' to 'bottom right' diagonal and zeroes in all other positions; this matrix is denoted by **I**.
- If **A** and **B** are square matrices of the same dimension such that **AB** = **BA** = **I**, then we say that **A** is the inverse of **B** and that **B** is the inverse of **A**.
- If a square matrix **A** has an inverse matrix, then this inverse is unique and is denoted $\mathbf{A}^{-1}$.
- A square matrix is said to be **regular** if its inverse exists, and said to be **singular** if it does not have an inverse.

Inverse of a 2 × 2 matrix

For $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the inverse of **A** is given by

$$\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

and the **determinant** of **A** is $\det(\mathbf{A}) = ad - bc$.

A 2 × 2 matrix **A** has an inverse if and only if $\det(\mathbf{A}) \neq 0$.

Example 27

For the matrix $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 6 \end{bmatrix}$, find:

a $\det(\mathbf{A})$ **b** $\mathbf{A}^{-1}$ **c** $\mathbf{X}$, if $\mathbf{AX} = \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$ **d** $\mathbf{Y}$, if $\mathbf{YA} = \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$

Solution

a $\det(\mathbf{A}) = 3 \times 6 - 2 = 16$

b $\mathbf{A}^{-1} = \dfrac{1}{16}\begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix}$

c $\mathbf{AX} = \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$

Multiply both sides (from the left) by $\mathbf{A}^{-1}$:

$$\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$$

$$\therefore \quad \mathbf{IX} = \mathbf{X} = \frac{1}{16}\begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$$

$$= \frac{1}{16}\begin{bmatrix} 16 & 32 \\ 16 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

d $\mathbf{YA} = \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$

Multiply both sides (from the right) by $\mathbf{A}^{-1}$:

$$\mathbf{YAA}^{-1} = \frac{1}{16}\begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}\begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix}$$

$$\therefore \quad \mathbf{YI} = \mathbf{Y} = \frac{1}{16}\begin{bmatrix} 24 & 8 \\ 40 & -8 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{5}{2} & -\frac{1}{2} \end{bmatrix}$$

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Note: If a transformation is defined by a matrix of the form $\mathbf{A} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, where $a, b \neq 0$, then $\det(\mathbf{A}) = ab$ and $\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{bmatrix}$.

Example 28

A transformation is defined by the matrix $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$. Find the equation of the image of the graph of the quadratic equation $y = x^2 + 2x + 3$ under this transformation.

Solution

As before, the transformation maps $(x, y) \to (x', y')$.

Using matrix notation,

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

This can be written as the matrix equation

$$\mathbf{AX} = \mathbf{X}'$$

Multiply both sides of the equation (from the left) by $\mathbf{A}^{-1}$:

$$\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\mathbf{X}'$$

$$\therefore \quad \mathbf{X} = \mathbf{A}^{-1}\mathbf{X}'$$

Therefore

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{1}{3}x' \\ \frac{1}{2}y' \end{bmatrix}$$

So $x = \dfrac{x'}{3}$ and $y = \dfrac{y'}{2}$.

The curve with equation $y = x^2 + 2x + 3$ is mapped to the curve with equation

$$\frac{y'}{2} = \left(\frac{x'}{3}\right)^2 + \frac{2x'}{3} + 3$$

Rearranging gives $y' = \dfrac{2(x')^2}{9} + \dfrac{4x'}{3} + 6$.

This makes quite hard work of an easy problem, but it demonstrates a procedure that can be used for any transformation defined by a 2 × 2 non-singular matrix.

Example 29

A transformation is described by the equation

$$\mathbf{A}(\mathbf{X} + \mathbf{B}) = \mathbf{X}', \qquad \text{where} \quad \mathbf{A} = \begin{bmatrix} 0 & -4 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Find the image of the straight line with equation $y = 2x + 5$ under this transformation.

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Solution

First solve the matrix equation for **X**:

$$\mathbf{A}^{-1}\mathbf{A}(\mathbf{X}+\mathbf{B}) = \mathbf{A}^{-1}\mathbf{X}'$$

$$\mathbf{X}+\mathbf{B} = \mathbf{A}^{-1}\mathbf{X}'$$

and so $\mathbf{X} = \mathbf{A}^{-1}\mathbf{X}' - \mathbf{B}$

Therefore

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{4} & 0 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} \dfrac{y'}{2} - 1 \\ -\dfrac{x'}{4} - 3 \end{bmatrix}$$

So $x = \dfrac{y'}{2} - 1$ and $y = -\dfrac{x'}{4} - 3$.

The straight line with equation $y = 2x + 5$ is transformed to the straight line with equation

$$-\frac{x'}{4} - 3 = 2\left(\frac{y'}{2} - 1\right) + 5$$

Rearranging gives $y' = -\dfrac{x'}{4} - 6$.

Section summary

If the inverse of the transformation matrix is known, then this can be used to find the relationship between the image of a point and the original point. For example:

- If $\mathbf{AX} = \mathbf{X}'$, then $\mathbf{X} = \mathbf{A}^{-1}\mathbf{X}'$.
- If $\mathbf{AX} + \mathbf{B} = \mathbf{X}'$, then $\mathbf{X} = \mathbf{A}^{-1}(\mathbf{X}' - \mathbf{B})$.

Exercise 3J

Example 27

1 For the matrices $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -2 & -2 \\ 3 & 2 \end{bmatrix}$, find:

a det(**A**) **b** $\mathbf{A}^{-1}$ **c** det(**B**) **d** $\mathbf{B}^{-1}$

2 Find the inverse of each of the following regular matrices:

a $\begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}$ **b** $\begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix}$ **c** $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$, where k is any non-zero real number

3 Let **A** and **B** be the regular matrices $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$.

a Find $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$.

b Find **AB** and hence find, if possible, $(\mathbf{AB})^{-1}$.

c From $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$, find the products $\mathbf{A}^{-1}\mathbf{B}^{-1}$ and $\mathbf{B}^{-1}\mathbf{A}^{-1}$. What do you notice?

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Example 27 **4** Consider the matrix $\mathbf{A} = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$.

a Find $\mathbf{A}^{-1}$. **b** If $\mathbf{AX} = \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$, find $\mathbf{X}$. **c** If $\mathbf{YA} = \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$, find $\mathbf{Y}$.

5 Let $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 6 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 4 & -1 \\ 2 & 2 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 3 & 4 \\ 2 & 6 \end{bmatrix}$.

a Find $\mathbf{X}$ such that $\mathbf{AX} + \mathbf{B} = \mathbf{C}$. **b** Find $\mathbf{Y}$ such that $\mathbf{YA} + \mathbf{B} = \mathbf{C}$.

Example 28 **6** A transformation is defined by the matrix $\begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}$. Find the equation of the image of the graph of the quadratic equation $y = x^2 + 2x$ under this transformation.

7 A transformation is defined by the matrix $\begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$. Find the equation of the image of the graph of the cubic equation $y = x^3 + 4$ under this transformation.

8 A transformation is defined by the matrix $\begin{bmatrix} 0 & 4 \\ -2 & 0 \end{bmatrix}$. Find the equation of the image of the straight line with equation $y = 5x + 3$ under this transformation.

9 A transformation is defined by the matrix $\begin{bmatrix} 0 & 5 \\ -3 & 0 \end{bmatrix}$. Find the equation of the image of the straight line with equation $y = -2x + 7$ under this transformation.

Example 29 **10** A transformation is described by the equation $\mathbf{A}(\mathbf{X} + \mathbf{B}) = \mathbf{X}'$, where $\mathbf{A} = \begin{bmatrix} 0 & -2 \\ -4 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. Find the image of the straight line with equation $y = -2x + 8$ under the transformation.

11 A transformation is described by the equation $\mathbf{AX} + \mathbf{B} = \mathbf{X}'$, where $\mathbf{A} = \begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. Find the image of the straight line with equation $y = -2x + 6$ under the transformation.

12 A transformation is described by the equation $\mathbf{AX} + \mathbf{B} = \mathbf{X}'$, where $\mathbf{A} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$. Find the image of the curve with equation $y = -2x^3 + 6x$ under the transformation.

13 A transformation is described by the equation $\mathbf{AX} + \mathbf{B} = \mathbf{X}'$, where $\mathbf{A} = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Find the image of the curve with equation $y = -2x^3 + 6x^2 + 2$ under the transformation.

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Chapter summary

Spreadsheet

AS

Nrich

In the following table, the rule for each transformation is given along with the rule for the image of the graph of $y = f(x)$.

Mapping	Rule	Matrix	The graph of $y = f(x)$ maps to
Reflection in the x-axis	$x' = x$ $y' = -y$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$y = -f(x)$
Reflection in the y-axis	$x' = -x$ $y' = y$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	$y = f(-x)$
Dilation of factor a from the y-axis	$x' = ax$ $y' = y$	$\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$	$y = f\left(\dfrac{x}{a}\right)$
Dilation of factor b from the x-axis	$x' = x$ $y' = by$	$\begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix}$	$y = bf(x)$
Reflection in the line $y = x$	$x' = y$ $y' = x$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$x = f(y)$
Translation defined by $\begin{bmatrix} h \\ k \end{bmatrix}$	$x' = x + h$ $y' = y + k$	Not linear	$y - k = f(x - h)$

Technology-free questions

1 Sketch the graph of each of the following. Label any asymptotes and axis intercepts. State the range of each function.

a $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R},\ f(x) = \dfrac{1}{x} - 3$

b $f\colon (2, \infty) \to \mathbb{R},\ f(x) = \dfrac{1}{x - 2}$

c $f\colon \mathbb{R} \setminus \{1\} \to \mathbb{R},\ f(x) = \dfrac{2}{x - 1} - 3$

d $f\colon (2, \infty) \to \mathbb{R},\ f(x) = \dfrac{-3}{2 - x} + 4$

e $f\colon \mathbb{R} \setminus \{1\} \to \mathbb{R},\ f(x) = 1 - \dfrac{1}{x - 1}$

2 Sketch the graph of each of the following:

a $f(x) = 2\sqrt{x - 3} + 1$

b $g(x) = \dfrac{3}{(x - 2)^2} - 1$

c $h(x) = \dfrac{-3}{(x - 2)^2} - 1$

3 Sketch the graph of each of the following. State the coordinates of the point of zero gradient and the axis intercepts.

a $f(x) = -2(x + 1)^3$

b $g(x) = -2(x - 1)^5 + 8$

c $h(x) = 2(x - 2)^5 + 1$

d $f(x) = 4(x - 1)^3 - 4$

4 The points with coordinates $(1, 6)$ and $(16, 12)$ lie on a curve which has a rule of the form $y = a\sqrt{x} + b$. Find a and b.

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5 A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -4 \\ -1 \end{bmatrix}$$

Find the image of the curve with equation $y = \sqrt{x}$ under this transformation.

6 A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -4 \\ -\frac{1}{2} \end{bmatrix}$$

Find the image of the curve with equation $y = 2\sqrt{x-4} + 3$ under this transformation.

7 The points with coordinates $(1, 3)$ and $(3, 7)$ lie on a curve with equation of the form $y = \dfrac{a}{x} + b$. Find the values of a and b.

8 **a** Find the rule for the image of the graph of $y = -x^2$ under the following sequence of transformations:

- reflection in the y-axis
- dilation of factor 2 from the y-axis
- translation 4 units in the positive direction of the x-axis and 6 units in the positive direction of the y-axis.

b Find a sequence of transformations which takes the graph of $y = x^4$ to the graph of $y = 6 - 4(x + 1)^4$.

9 Identify a sequence of transformations that maps the graph of $y = \dfrac{1}{x^2}$ onto the graph of $y = \dfrac{3}{(x-5)^2} + 3$. Use this to sketch the graph of $y = \dfrac{3}{(x-5)^2} + 3$, stating the equations of asymptotes and the coordinates of axis intercepts.

10 Find a sequence of transformations that takes the graph of $y = 2x^2 - 3$ to the graph of $y = x^2$.

11 Find a sequence of transformations that takes the graph of $y = 2(x - 3)^3 + 4$ to the graph of $y = x^3$.

Multiple-choice questions

1 The point $P(3, -4)$ lies on the graph of a function f. The graph of f is translated 3 units up (parallel to the y-axis) and reflected in the x-axis. The coordinates of the final image of P are

A $(6, 4)$ **B** $(3, 1)$ **C** $(3, -1)$ **D** $(-3, 1)$ **E** $(3, 7)$

2 The graph of $y = x^3 + 4$ is translated 3 units ‘down’ and 2 units ‘to the right’. The resulting graph has equation

A $y = (x - 2)^3 + 2$ **B** $y = (x - 2)^3 + 1$ **C** $y = (x - 2)^3 + 5$

D $y = (x + 2)^3 + 1$ **E** $y = (x + 2)^3 + 6$

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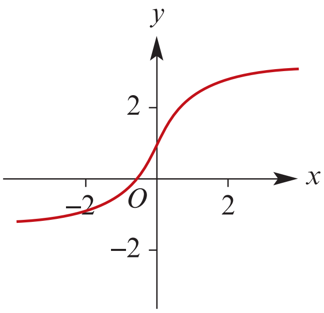
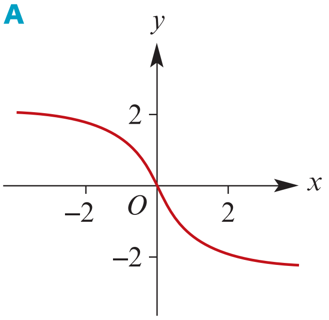
3 The graph of $y = f(x)$ is shown on the right.

Which one of the following could be the graph of $y = f(-x)$?



A



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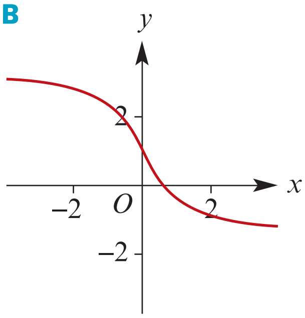

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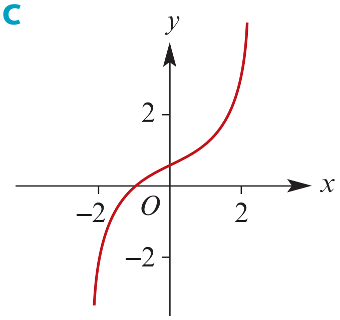

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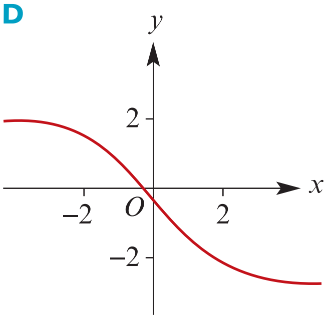
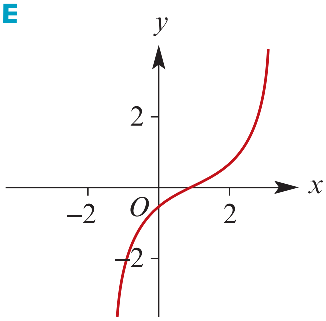


E



4 The graph of the function with rule $y = x^2$ is reflected in the x-axis and then translated 4 units in the negative direction of the x-axis and 3 units in the negative direction of the y-axis. The rule for the new function is

A $y = (-x + 4)^2 - 3$ **B** $y = -(x - 4)^2 + 3$ **C** $y = -(x - 3)^2 + 4$

D $y = (-x - 4)^2 + 3$ **E** $y = -(x + 4)^2 - 3$

5 The graph of $y = \dfrac{a}{x+b} + c$ is shown on the right. A possible set of values for a, b and c respectively is

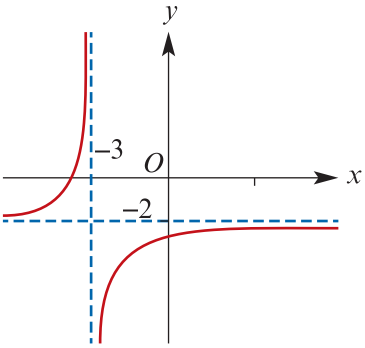
A −1, 3, 2

B 1, 2, −3

C −1, −3, −2

D −1, 3, −2

E 1, 2, −3



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6 The graph of the function f is obtained from the graph of $y = x^{\frac{1}{3}}$ by a reflection in the y-axis followed by a dilation of factor 5 from the x-axis. The rule for f is

A $f(x) = -5x^{\frac{1}{3}}$ **B** $f(x) = \frac{1}{5}(-x)^{\frac{1}{3}}$ **C** $f(x) = (-5x)^{\frac{1}{3}}$

D $f(x) = -\frac{1}{5}x^{\frac{1}{3}}$ **E** $f(x) = -5(-x)^{\frac{1}{3}}$

7 The transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ with rule

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

maps the curve with equation $y = \sqrt[3]{x}$ to the curve with equation

A $y = 1 + \sqrt[3]{\dfrac{x-2}{3}}$ **B** $y = \sqrt[3]{\dfrac{x+2}{3}} - 1$ **C** $y = 2\sqrt[3]{\dfrac{x-3}{3}} - 1$

D $y = -1 - \sqrt[3]{\dfrac{x+2}{3}}$ **E** $y = -1 + \sqrt[3]{\dfrac{3x+2}{3}}$

8 A transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ that maps the graph of $y = \dfrac{1}{x}$ to the graph of $y = \dfrac{3}{2x+1} - 4$ is given by

A $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} \\ -4 \end{bmatrix}$ **B** $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ -1 \end{bmatrix}$

C $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ -4 \end{bmatrix}$ **D** $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ -4 \end{bmatrix}$

E $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} \\ -2 \end{bmatrix}$

9 A transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ that maps the graph of $y = -\dfrac{5}{2x-1} + 3$ to the graph of $y = \dfrac{1}{x}$ is given by

A $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & -\frac{1}{5} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ \frac{3}{5} \end{bmatrix}$ **B** $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{5} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ \frac{3}{5} \end{bmatrix}$

C $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{5} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ -\frac{3}{5} \end{bmatrix}$ **D** $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ -\frac{3}{5} \end{bmatrix}$

E $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} \\ -\frac{3}{5} \end{bmatrix}$

10 Let $f(x) = 3x - 2$ and $g(x) = x^2 - 4x + 2$. A sequence of transformations that takes the graph of $y = g(x)$ to the graph of $y = g(f(x))$ is

A a dilation of factor $\frac{1}{3}$ from the y-axis followed by a translation $\frac{2}{3}$ units in the positive direction of the x-axis

B a dilation of factor 3 from the y-axis followed by a translation 2 units in the negative direction of the x-axis

C a dilation of factor $\frac{1}{3}$ from the y-axis followed by a translation $\frac{1}{2}$ unit in the positive direction of the x-axis

D a dilation of factor 3 from the y-axis followed by a translation 2 units in the positive direction of the x-axis

E a dilation of factor $\frac{1}{3}$ from the y-axis followed by a translation 2 units in the positive direction of the x-axis

Extended-response questions

1 Consider the function $f\colon D \to \mathbb{R}$ with rule $f(x) = \dfrac{24}{x+2} - 6$, where D is the maximal domain for this rule.

a Find D.

b Describe a sequence of transformations which, when applied to the graph of $y = \dfrac{1}{x}$, produces the graph of $y = f(x)$. Specify the order in which these transformations are to be applied.

c Find the coordinates of the points where the graph of f cuts the axes.

Let $g\colon (-2, \infty) \to \mathbb{R}$, $g(x) = f(x)$.

d Find the rule for g^{-1}, the inverse of g.

e Write down the domain of g^{-1}.

f Sketch the graphs of $y = g(x)$ and $y = g^{-1}(x)$ on the one set of axes.

g Find the value(s) of x for which $g(x) = x$ and hence the value(s) of x for which $g(x) = g^{-1}(x)$.

2 Consider the function $f\colon D \to \mathbb{R}$ with rule $f(x) = 4 - 2\sqrt{2x+6}$, where D is the maximal domain for this rule.

a Find D.

b Describe a sequence of transformations which, when applied to the graph of $y = \sqrt{x}$, produces the graph of $y = f(x)$. Specify the order in which these transformations are to be applied.

c Find the coordinates of the points where the graph of f cuts the axes.

d Find the rule for f^{-1}, the inverse of f.

e Find the domain of f^{-1}.

f Sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the one set of axes.

g Find the value(s) of x for which $f(x) = x$ and hence the value(s) of x for which $f(x) = f^{-1}(x)$.

3 **a** **i** Find the dilation from the x-axis which takes $y = x^2$ to the parabola with its vertex at the origin that passes through the point $(25, 15)$.

ii State the rule which reflects this dilated parabola in the x-axis.

iii State the rule which takes the reflected parabola of part **ii** to a parabola with x-axis intercepts $(0, 0)$ and $(50, 0)$ and vertex $(25, 15)$.

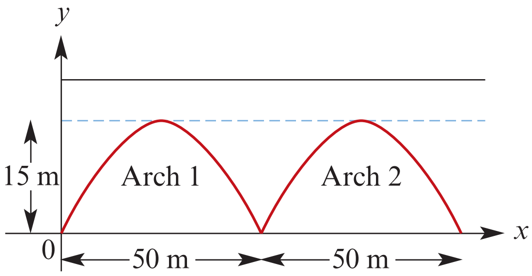
iv State the rule which takes the curve $y = x^2$ to the parabola defined in part **iii**.

b The plans for the entrance of a new building involve twin parabolic arches as shown in the diagram.

i From the results of part **a**, give the equation for the curve of arch 1.

ii Find the translation which maps the curve of arch 1 to the curve of arch 2.

iii Find the equation of the curve of arch 2.



c The architect wishes to have flexibility in her planning and so wants to develop an algorithm for determining the equations of the curves when each arch has width m metres and height n metres.

i Find the rule for the transformation which takes the graph of $y = x^2$ to the current arch 1 with these new dimensions.

ii Find the equation for the curve of arch 1.

iii Find the equation for the curve of arch 2.

4 Consider the function $g\colon D \to \mathbb{R}$ with rule $g(x) = \dfrac{3}{(3x-4)^2} + 6$, where D is the maximal domain for this rule.

a Find D.

b Find the smallest value of a such that $f\colon (a, \infty) \to \mathbb{R}$, $f(x) = g(x)$ is a one-to-one function.

c Find the inverse function of f.

d Find the value of x for which $f(x) = f^{-1}(x)$.

e On the one set of axes, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$.

5 **a** Sketch the curve with equation $f(x) = \dfrac{50}{20-x}$, for $x \neq 20$.

b For $g(x) = \dfrac{50x}{20-x}$:

i Show that $g(x) = \dfrac{1000}{20-x} - 50$.

ii Sketch the graph of $y = g(x)$.

iii Show that $g(x) = 20f(x) - 50$.

c Find the rule for the function g^{-1}.

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6 When the transformation with rule $(x, y) \rightarrow (y, x)$ (a reflection in the line $y = x$) is applied to the graph of a one-to-one function f, the resulting image has rule $y = f^{-1}(x)$, i.e. the graph of the inverse function is obtained.

a For the graph of $y = f(x)$, find the rule for the image of f, in terms of $f^{-1}(x)$, for each of the following sequences of transformations:

i
- a translation of 3 units in the positive direction of the x-axis
- a translation of 5 units in the positive direction of the y-axis
- a reflection in the line $y = x$

ii
- a reflection in the line $y = x$
- a translation of 3 units in the positive direction of the x-axis
- a translation of 5 units in the positive direction of the y-axis

iii
- a dilation of factor 3 from the x-axis
- a dilation of factor 5 from the y-axis
- a reflection in the line $y = x$

iv
- a reflection in the line $y = x$
- a dilation of factor 5 from the y-axis
- a dilation of factor 3 from the x-axis.

b Find the image of the graph of $y = f(x)$, in terms of $f^{-1}(x)$, under the transformation with rule $(x, y) \rightarrow (ay + b, cx + d)$, where a, b, c and d are positive constants, and describe this transformation in words.

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Chapter 4

4

Polynomial functions

Objectives

- To revise the properties of **quadratic functions**.
- To add, subtract and multiply polynomials.
- To be able to use the technique of **equating coefficients**.
- To **divide polynomials**.
- To use the **remainder theorem**, the **factor theorem** and the **rational-root theorem** to identify the linear factors of cubic and quartic polynomials.
- To draw and use **sign diagrams**.
- To find the rules for given polynomial graphs.
- To apply **polynomial functions** to problem solving.

A polynomial function of degree 2 is called a **quadratic function**. The general rule for such a function is

$$f(x) = ax^2 + bx + c, \quad a \neq 0$$

A polynomial function of degree 3 is called a **cubic function**. The general rule for such a function is

$$f(x) = ax^3 + bx^2 + cx + d, \quad a \neq 0$$

A polynomial function of degree 4 is called a **quartic function**. The general rule for such a function is

$$f(x) = ax^4 + bx^3 + cx^2 + dx + e, \quad a \neq 0$$

In this chapter we revise quadratic functions, and build on our previous study of cubic and quartic functions.

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4A Quadratic functions

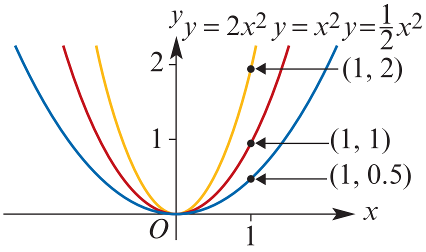
In this section, we revise material on quadratic functions covered in Mathematical Methods Units 1 & 2.

Transformations of parabolas

Dilation from the x-axis

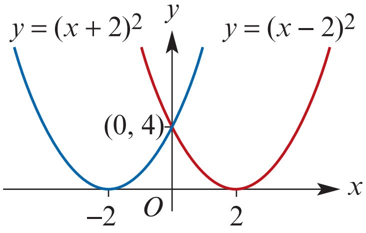
For $a > 0$, the graph of the function $y = ax^2$ is obtained from the graph of $y = x^2$ by a dilation of factor a from the x-axis.

The graphs on the right are those of $y = x^2$, $y = 2x^2$ and $y = \frac{1}{2}x^2$, i.e. $a = 1, 2$ and $\frac{1}{2}$.



Translation parallel to the x-axis

The graphs of $y = (x + 2)^2$ and $y = (x - 2)^2$ are shown.

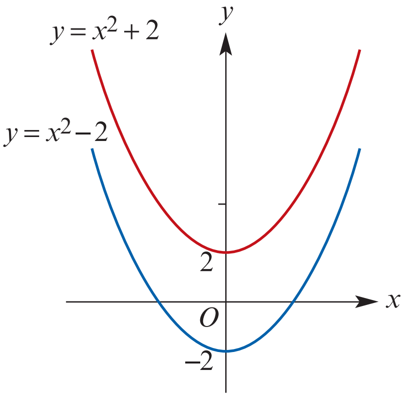


For $h > 0$, the graph of $y = (x + h)^2$ is obtained from the graph of $y = x^2$ by a translation of h units in the negative direction of the x-axis.

For $h < 0$, the graph of $y = (x + h)^2$ is obtained from the graph of $y = x^2$ by a translation of $-h$ units in the positive direction of the x-axis.

Translation parallel to the y-axis

The graphs of $y = x^2 + 2$ and $y = x^2 - 2$ are shown.



For $k > 0$, the graph of $y = x^2 + k$ is obtained from the graph of $y = x^2$ by a translation of k units in the positive direction of the y-axis.

For $k < 0$, the translation is in the negative direction of the y-axis.

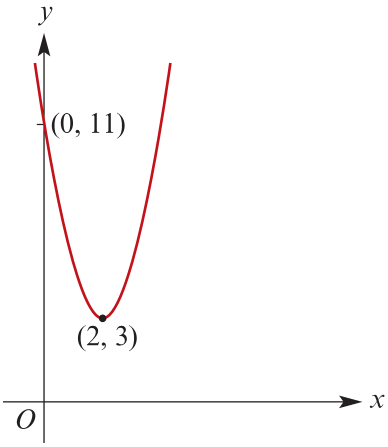
Combinations of transformations

The graph of the function

$$f(x) = 2(x - 2)^2 + 3$$

is obtained by transforming the graph of the function $f(x) = x^2$ by:

- dilation of factor 2 from the x-axis
- translation of 2 units in the positive direction of the x-axis
- translation of 3 units in the positive direction of the y-axis.



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Graphing quadratics in turning point form

By applying dilations, reflections and translations to the basic parabola $y = x^2$, we can sketch the graph of any quadratic expressed in **turning point form** $y = a(x - h)^2 + k$:

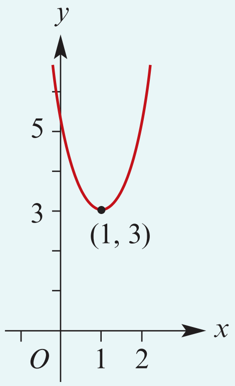
- If $a > 0$, the graph has a minimum point.
- If $a < 0$, the graph has a maximum point.
- The vertex is the point (h, k).
- The axis of symmetry is $x = h$.
- If h and k are positive, then the graph of $y = a(x - h)^2 + k$ is obtained from the graph of $y = ax^2$ by translating h units in the positive direction of the x-axis and k units in the positive direction of the y-axis.
- Similar results hold for different combinations of h and k positive and negative.

Example 1

Sketch the graph of $y = 2(x - 1)^2 + 3$.

Solution

The graph of $y = 2x^2$ is translated 1 unit in the positive direction of the x-axis and 3 units in the positive direction of the y-axis.



The vertex has coordinates $(1, 3)$.

The axis of symmetry is the line $x = 1$.

The graph will be narrower than $y = x^2$.

The range is $[3, \infty)$.

To add further detail to our graph, we can find the axis intercepts:

y-axis intercept

When $x = 0$, $y = 2(0 - 1)^2 + 3 = 5$.

x-axis intercepts

In this example, the minimum value of y is 3, and so y cannot be 0. Therefore this graph has no x-axis intercepts.

Note: Another way to see this is to let $y = 0$ and try to solve for x:

$$\begin{aligned} 0 &= 2(x-1)^2 + 3 \\ -3 &= 2(x-1)^2 \\ -\tfrac{3}{2} &= (x-1)^2 \end{aligned}$$

As the square root of a negative number is not a real number, this equation has no real solutions.

Example 2

Sketch the graph of $y = -(x+1)^2 + 4$.

Solution

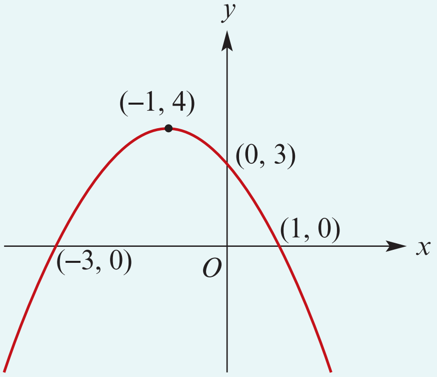
The vertex has coordinates $(-1, 4)$ and so the axis of symmetry is the line $x = -1$.

When $x = 0$, $y = -(0+1)^2 + 4 = 3$.
$\therefore$ the y-axis intercept is 3.

When $y = 0$,

$$\begin{aligned} -(x+1)^2 + 4 &= 0 \\ (x+1)^2 &= 4 \\ x + 1 &= \pm 2 \\ x &= \pm 2 - 1 \end{aligned}$$

$\therefore$ the x-axis intercepts are 1 and -3.



The axis of symmetry

For a quadratic function written in polynomial form $y = ax^2 + bx + c$, the axis of symmetry of its graph has the equation $x = -\frac{b}{2a}$.

Therefore the x-coordinate of the turning point is $-\frac{b}{2a}$. Substitute this value into the quadratic polynomial to find the y-coordinate of the turning point.

Example 3

For each of the following quadratic functions, use the axis of symmetry to find the turning point of the graph, express the function in the form $y = a(x-h)^2 + k$, and hence find the maximum or minimum value and the range:

a $y = x^2 - 4x + 3$

b $y = -2x^2 + 12x - 7$

Solution	*Explanation*
a The x-coordinate of the turning point is 2.	Here $a = 1$ and $b = -4$, so the axis of symmetry is $x = -\left(\frac{-4}{2}\right) = 2$.
When $x = 2$, $y = 4 - 8 + 3 = -1$.	
The coordinates of the turning point are $(2, -1)$. Hence the equation is $y = (x-2)^2 - 1$.	For the turning point form $y = a(x-h)^2 + k$, we have found that $a = 1$, $h = 2$ and $k = -1$.
The minimum value is -1 and the range is $[-1, \infty)$.	Since $a > 0$, the parabola has a minimum.

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b The x-coordinate of the turning point is 3.

When $x = 3,\ y = -2\times(3)^2 + 12\times 3 - 7 = 11$.

The coordinates of the turning point are $(3, 11)$. Hence the equation is $y = -2(x-3)^2 + 11$.

The maximum value is 11 and the range is $(-\infty, 11]$.

Here $a = -2$ and $b = 12$, so the axis of symmetry is $x = -\left(\dfrac{12}{-4}\right) = 3$.

For the turning point form $y = a(x-h)^2 + k$, we have found that $a = -2$, $h = 3$ and $k = 11$.

Since $a < 0$, the parabola has a maximum.

Graphing quadratics in polynomial form

It is not essential to convert a quadratic to turning point form in order to sketch its graph.

For a quadratic in polynomial form, we can find the x- and y-axis intercepts and the axis of symmetry by other methods and use these details to sketch the graph.

Step 1 Find the y-axis intercept.
Step 2 Find the x-axis intercepts.
Step 3 Find the equation of the axis of symmetry.
Step 4 Find the coordinates of the turning point.

Example 4

Find the x- and y-axis intercepts and the turning point, and hence sketch the graph of $y = x^2 + x - 12$.

Solution

Step 1 $c = -12$. Therefore the y-axis intercept is -12.

Step 2 Let $y = 0$. Then

$$0 = x^2 + x - 12$$
$$0 = (x+4)(x-3)$$
$$\therefore\quad x = -4 \text{ or } x = 3$$

The x-axis intercepts are -4 and 3.

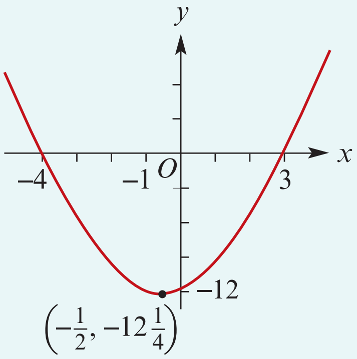
Step 3 The axis of symmetry is the line with equation

$$x = \frac{-4+3}{2} = -\frac{1}{2}$$

Step 4 When $x = -\frac{1}{2}$, $y = (-\frac{1}{2})^2 + (-\frac{1}{2}) - 12$

$$= -12\tfrac{1}{4}$$

The turning point has coordinates $(-\frac{1}{2}, -12\frac{1}{4})$.



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▸ Completing the square

By completing the square, all quadratics in polynomial form, $y = ax^2 + bx + c$, may be transposed into turning point form, $y = a(x - h)^2 + k$. We have seen that this can be used to sketch the graphs of quadratic polynomials.

To complete the square of $x^2 + bx + c$:

- Take half the coefficient of x (that is, $\frac{b}{2}$) and add and subtract its square $\frac{b^2}{4}$.

To complete the square of $ax^2 + bx + c$:

- First take out a as a factor and then complete the square inside the bracket.

Example 5

By completing the square, write the quadratic $f(x) = 2x^2 - 4x - 5$ in turning point form, and hence sketch the graph of $y = f(x)$.

Solution

$$
\begin{aligned}
f(x) &= 2x^2 - 4x - 5 \\
&= 2\left(x^2 - 2x - \frac{5}{2}\right) \\
&= 2\left(x^2 - 2x + 1 - 1 - \frac{5}{2}\right) \qquad \text{add and subtract } \left(\frac{b}{2}\right)^2 \text{ to ‘complete the square’} \\
&= 2\left[(x^2 - 2x + 1) - \frac{7}{2}\right] \\
&= 2\left[(x - 1)^2 - \frac{7}{2}\right] \\
&= 2(x - 1)^2 - 7
\end{aligned}
$$

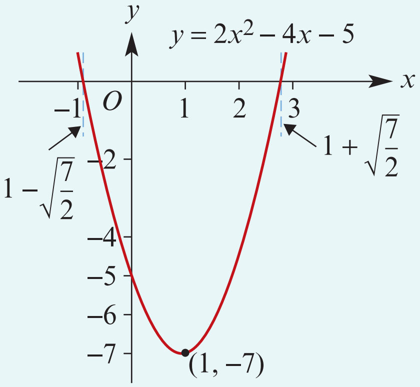
The x-axis intercepts can be determined after completing the square:

$$
\begin{aligned}
2x^2 - 4x - 5 &= 0 \\
2(x - 1)^2 - 7 &= 0 \\
(x - 1)^2 &= \frac{7}{2} \\
x - 1 &= \pm\sqrt{\frac{7}{2}}
\end{aligned}
$$

$$\therefore \quad x = 1 + \sqrt{\frac{7}{2}} \quad \text{or} \quad x = 1 - \sqrt{\frac{7}{2}}$$

This information can now be used to sketch the graph:

- The y-axis intercept is $c = -5$.
- The turning point is $(1, -7)$.
- The x-axis intercepts are $1 + \sqrt{\frac{7}{2}}$ and $1 - \sqrt{\frac{7}{2}}$.



▶ The quadratic formula

The solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are given by the **quadratic formula**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

It should be noted that the equation of the axis of symmetry can be derived from this general formula: the axis of symmetry is the line with equation

$$x = -\frac{b}{2a}$$

Example 6

Sketch the graph of $f(x) = -3x^2 - 12x - 7$ by:

- finding the equation of the axis of symmetry
- finding the coordinates of the turning point
- using the general quadratic formula to find the x-axis intercepts.

Solution

Since $c = -7$, the y-axis intercept is -7.

Axis of symmetry
$$\begin{aligned} x &= -\frac{b}{2a} \\ &= -\left(\frac{-12}{2 \times (-3)}\right) \\ &= -2 \end{aligned}$$

Turning point

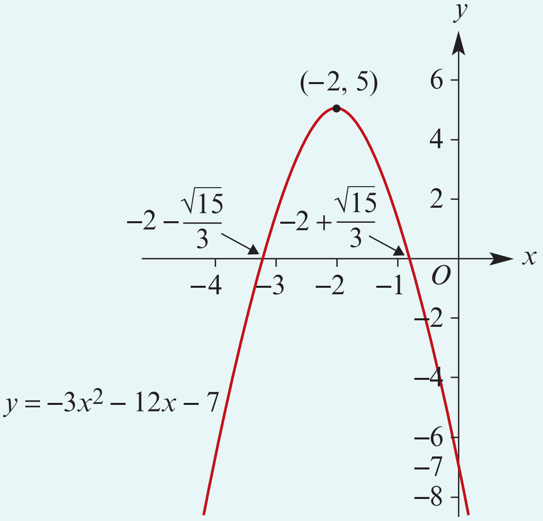
When $x = -2$, $y = -3(-2)^2 - 12(-2) - 7 = 5$.

The turning point coordinates are $(-2, 5)$.

x-axis intercepts

$$-3x^2 - 12x - 7 = 0$$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-12) \pm \sqrt{(-12)^2 - 4(-3)(-7)}}{2(-3)} \\ &= \frac{12 \pm \sqrt{60}}{-6} \\ &= \frac{12 \pm 2\sqrt{15}}{-6} \\ &= -2 \pm \tfrac{1}{3}\sqrt{15} \end{aligned}$$



The discriminant

The **discriminant** Δ of a quadratic polynomial $ax^2 + bx + c$ is

$$\Delta = b^2 - 4ac$$

For the equation $ax^2 + bx + c = 0$:

- If $\Delta > 0$, there are two solutions.
- If $\Delta = 0$, there is one solution.
- If $\Delta < 0$, there are no real solutions.

For the equation $ax^2 + bx + c = 0$ where a, b and c rational numbers:

- If Δ is a perfect square and $\Delta \neq 0$, then the equation has two rational solutions.
- If $\Delta = 0$, then the equation has one rational solution.
- If Δ is not a perfect square and $\Delta > 0$, then the equation has two irrational solutions.

Example 7

Without sketching graphs, determine whether the graph of each of the following functions crosses, touches or does not intersect the x-axis:

a $f(x) = 2x^2 - 4x - 6$

b $f(x) = -4x^2 + 12x - 9$

c $f(x) = 3x^2 - 2x + 8$

Solution	*Explanation*
a $\Delta = b^2 - 4ac$	
$= (-4)^2 - 4 \times 2 \times (-6)$	Here $a = 2$, $b = -4$, $c = -6$.
$= 16 + 48$	
$= 64 > 0$	
The graph crosses the x-axis twice.	As $\Delta > 0$, there are two x-axis intercepts.
b $\Delta = b^2 - 4ac$	
$= (12)^2 - 4 \times (-4) \times (-9)$	Here $a = -4$, $b = 12$, $c = -9$.
$= 144 - 144$	
$= 0$	
The graph touches the x-axis once.	As $\Delta = 0$, there is only one x-axis intercept.
c $\Delta = b^2 - 4ac$	
$= (-2)^2 - 4 \times 3 \times 8$	Here $a = 3$, $b = -2$, $c = 8$.
$= 4 - 96$	
$= -92 < 0$	
The graph does not intersect the x-axis.	As $\Delta < 0$, there are no x-axis intercepts.

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Example 8

Find the values of m for which the equation $3x^2 - 2mx + 3 = 0$ has:

a one solution **b** no solution **c** two distinct solutions.

Solution

For the quadratic $3x^2 - 2mx + 3$, the discriminant is $\Delta = 4m^2 - 36$.

a For one solution:

$$\Delta = 0$$

i.e. $4m^2 - 36 = 0$

$$m^2 = 9$$

$\therefore \quad m = \pm 3$

b For no solution:

$$\Delta < 0$$

i.e. $4m^2 - 36 < 0$

From the graph, this is equivalent to

$$-3 < m < 3$$

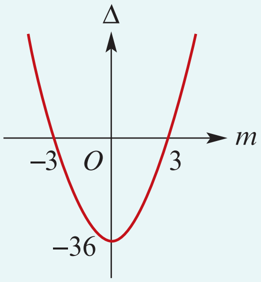
c For two distinct solutions:

$$\Delta > 0$$

i.e. $4m^2 - 36 > 0$

From the graph it can be seen that

$$m > 3 \quad \text{or} \quad m < -3$$



Section summary

- The graph of $y = a(x - h)^2 + k$ is a parabola congruent to the graph of $y = ax^2$. The vertex (or turning point) is the point (h, k). The axis of symmetry is $x = h$.
- The axis of symmetry of the graph of $y = ax^2 + bx + c$ has equation $x = -\dfrac{b}{2a}$.
- By completing the square, all quadratic functions in polynomial form $y = ax^2 + bx + c$ may be transposed into the turning point form $y = a(x - h)^2 + k$.
- To complete the square of $x^2 + bx + c$:
  - Take half the coefficient of x (that is, $\frac{b}{2}$) and add and subtract its square $\frac{b^2}{4}$.
- To complete the square of $ax^2 + bx + c$:
  - First take out a as a factor and then complete the square inside the bracket.
- The solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are given by the **quadratic formula**

  $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

  From the formula it can be seen that:
  - If $b^2 - 4ac > 0$, there are two solutions.
  - If $b^2 - 4ac = 0$, there is one solution.
  - If $b^2 - 4ac < 0$, there are no real solutions.

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Exercise 4A

Example 1, 2

1 Sketch the graphs of the following functions:

a $f(x) = 2(x - 1)^2$

b $f(x) = 2(x - 1)^2 - 2$

c $f(x) = -2(x - 1)^2$

d $f(x) = 4 - 2(x + 1)^2$

e $f(x) = 4 + 2(x + \frac{1}{2})^2$

f $f(x) = 2(x + 1)^2 - 1$

g $f(x) = 3(x - 2)^2 - 4$

h $f(x) = (x + 1)^2 - 1$

i $f(x) = 5x^2 - 1$

j $f(x) = 2(x + 1)^2 - 4$

Example 3

2 For each of the following quadratic functions, use the axis of symmetry to find the turning point of the graph, express the function in the form $y = a(x - h)^2 + k$, and hence find the maximum or minimum value and the range:

a $f(x) = x^2 + 3x - 2$

b $f(x) = x^2 - 6x + 8$

c $f(x) = 2x^2 + 8x - 6$

d $f(x) = 4x^2 + 8x - 7$

e $f(x) = 2x^2 - 5x$

f $f(x) = 7 - 2x - 3x^2$

g $f(x) = -2x^2 + 9x + 11$

Example 4

3 Find the x- and y-axis intercepts and the turning point, and hence sketch the graph of each of the following:

a $y = -x^2 + 2x$

b $y = x^2 - 6x + 8$

c $y = -x^2 - 5x - 6$

d $y = -2x^2 + 8x - 6$

e $y = 4x^2 - 12x + 9$

f $y = 6x^2 + 3x - 18$

Example 5

4 Sketch the graph of each of the following by first completing the square:

a $y = x^2 + 2x - 6$

b $y = x^2 - 4x - 10$

c $y = -x^2 - 5x - 3$

d $y = -2x^2 + 8x - 10$

e $y = x^2 - 7x + 3$

Example 6

5 Sketch the graph of $f(x) = 3x^2 - 2x - 1$ by first finding the equation of the axis of symmetry, then finding the coordinates of the vertex, and finally using the quadratic formula to calculate the x-axis intercepts.

6 Sketch the graph of $f(x) = -3x^2 - 2x + 2$ by first finding the equation of the axis of symmetry, then finding the coordinates of the vertex, and finally using the quadratic formula to calculate the x-axis intercepts.

7 Sketch the graphs of the following functions, clearly labelling the axis intercepts and turning points:

a $f(x) = x^2 + 3x - 2$

b $f(x) = 2x^2 + 4x - 7$

c $f(x) = 5x^2 - 10x - 1$

d $f(x) = -2x^2 + 4x - 1$

e $y = 2.5x^2 + 3x + 0.3$

f $y = -0.6x^2 - 1.3x - 0.1$

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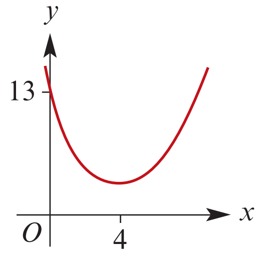
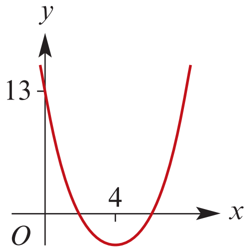
8 **a** Which of the graphs shown could represent $y = (x - 4)^2 - 3$?

b Which graph could represent $y = 3 - (x - 4)^2$?

A



B



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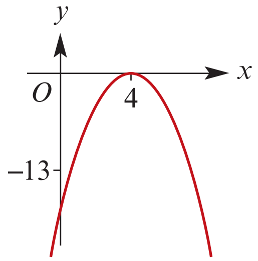

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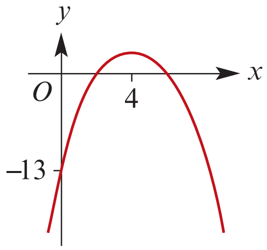


9 Match each of the following functions with the appropriate graph below:

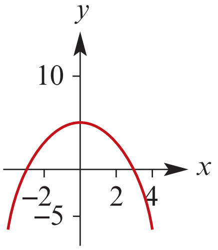
a $y = \frac{1}{3}(x + 4)(8 - x)$

b $y = x^2 - \frac{x}{2} + 1$

c $y = -10 + 2(x - 1)^2$

d $y = \frac{1}{2}(9 - x^2)$

A



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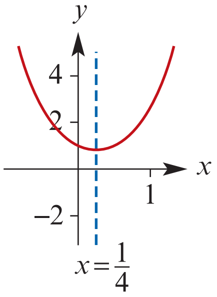

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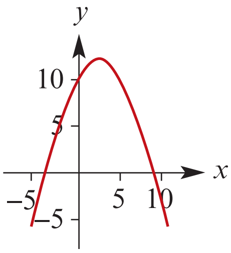
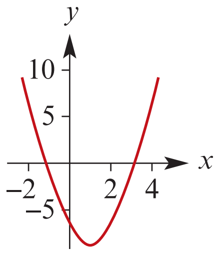


D



Example 7 **10** Without sketching the graphs of the following functions, determine whether they cross, touch or do not intersect the x-axis:

a $f(x) = x^2 - 5x + 2$

b $f(x) = -4x^2 + 2x - 1$

c $f(x) = x^2 - 6x + 9$

d $f(x) = 8 - 3x - 2x^2$

e $f(x) = 3x^2 + 2x + 5$

f $f(x) = -x^2 - x - 1$

Example 8 **11** For which values of m does the equation $mx^2 - 2mx + 3 = 0$ have:

a two solutions for x

b one solution for x?

12 Find the value of m for which $(4m + 1)x^2 - 6mx + 4$ is a perfect square.

13 Find the values of a for which the equation $(a - 3)x^2 + 2ax + (a + 2) = 0$ has no solutions for x.

14 Prove that the equation $x^2 + (a + 1)x + (a - 2) = 0$ always has two distinct solutions.

15 Show that the equation $(k + 1)x^2 - 2x - k = 0$ has a solution for all values of k.

16 For which values of k does the equation $kx^2 - 2kx = 5$ have:

a two solutions for x **b** one solution for x?

17 For which values of k does the equation $(k - 3)x^2 + 2kx + (k + 2) = 0$ have:

a two solutions for x **b** one solution for x?

18 Show that the equation $ax^2 - (a + b)x + b = 0$ has a solution for all values of a and b.

4B Determining the rule for a parabola

In this section we revise methods for finding the rule of a quadratic function from information about its graph. The following three forms are useful. You will see others in the worked examples.

1 $y = a(x - e)(x - f)$ This can be used if two x-axis intercepts and the coordinates of one other point are known.

2 $y = a(x - h)^2 + k$ This can be used if the coordinates of the turning point and one other point are known.

3 $y = ax^2 + bx + c$ This can be used if the coordinates of three points on the parabola are known.

Example 9

A parabola has x-axis intercepts -3 and 4 and it passes through the point $(1, 24)$. Find the rule for this parabola.

Solution	*Explanation*
$y = a(x + 3)(x - 4)$ When $x = 1$, $y = 24$. Thus $24 = a(1 + 3)(1 - 4)$ $24 = -12a$ $\therefore\ a = -2$ The rule is $y = -2(x + 3)(x - 4)$.	Two x-axis intercepts are given. Therefore use the form $y = a(x - e)(x - f)$.

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Example 10

The coordinates of the turning point of a parabola are $(2, 6)$ and the parabola passes through the point $(3, 3)$. Find the rule for this parabola.

Solution

$y = a(x - 2)^2 + 6$

When $x = 3$, $y = 3$. Thus

$$\begin{aligned} 3 &= a(3 - 2)^2 + 6 \\ 3 &= a + 6 \\ \therefore \quad a &= -3 \end{aligned}$$

The rule is $y = -3(x - 2)^2 + 6$.

Explanation

The coordinates of the turning point and one other point on the parabola are given. Therefore use $y = a(x - h)^2 + k$.

Example 11

A parabola passes through the points $(1, 4)$, $(0, 5)$ and $(-1, 10)$. Find the rule for this parabola.

Solution

$y = ax^2 + bx + c$

When $x = 1$, $y = 4$.
When $x = 0$, $y = 5$.
When $x = -1$, $y = 10$.

Therefore

$$\begin{aligned} 4 &= a + b + c && (1) \\ 5 &= c && (2) \\ 10 &= a - b + c && (3) \end{aligned}$$

Substitute from equation (2) into equations (1) and (3):

$$\begin{aligned} -1 &= a + b && (1') \\ 5 &= a - b && (3') \end{aligned}$$

Add (1′) and (3′):

$$\begin{aligned} 4 &= 2a \\ \therefore \quad a &= 2 \end{aligned}$$

Substitute into equation (1′):

$$\begin{aligned} -1 &= 2 + b \\ \therefore \quad b &= -3 \end{aligned}$$

The rule is $y = 2x^2 - 3x + 5$.

Explanation

The coordinates of three points on the parabola are given. Therefore we substitute values into the general polynomial form $y = ax^2 + bx + c$ to obtain three equations in three unknowns.

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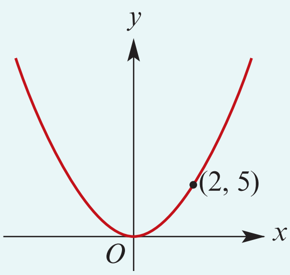
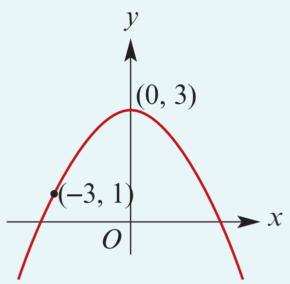
Example 12

Find the equation of each of the following parabolas:

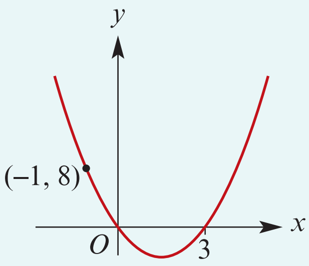
a



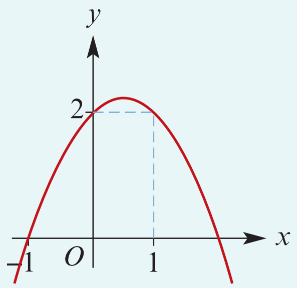
b



c



d



Solution

a This is of the form $y = ax^2$ (since the graph has its vertex at the origin).

As the point $(2, 5)$ is on the parabola,

$$5 = a(2)^2$$

$$\therefore \quad a = \frac{5}{4}$$

The rule is $y = \frac{5}{4}x^2$.

b This is of the form $y = ax^2 + c$ (since the graph is symmetric about the y-axis).

For $(0, 3)$: $\quad 3 = a(0)^2 + c$

$$\therefore \quad c = 3$$

For $(-3, 1)$: $\quad 1 = a(-3)^2 + 3$

$$1 = 9a + 3$$

$$\therefore \quad a = -\frac{2}{9}$$

The rule is $y = -\frac{2}{9}x^2 + 3$.

c This is of the form $y = ax(x - 3)$.

As the point $(-1, 8)$ is on the parabola,

$$8 = -a(-1 - 3)$$

$$8 = 4a$$

$$\therefore \quad a = 2$$

The rule is $y = 2x(x - 3)$.

d This is of the form $y = ax^2 + bx + c$.

The y-axis intercept is 2 and so $c = 2$.

As $(-1, 0)$ and $(1, 2)$ are on the parabola,

$$0 = a - b + 2 \qquad (1)$$

$$2 = a + b + 2 \qquad (2)$$

Add equations (1) and (2):

$$2 = 2a + 4$$

$$2a = -2$$

$$\therefore \quad a = -1$$

Substitute $a = -1$ in (1) to obtain $b = 1$.

The rule is $y = -x^2 + x + 2$.

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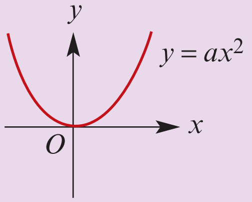
Section summary

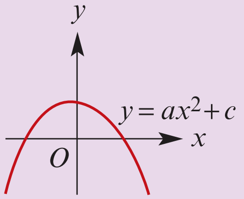
To find a quadratic rule to fit given points, first choose the best form of quadratic expression to work with. Then substitute in the coordinates of the known points to determine the unknown parameters. Some possible forms are given here:

i



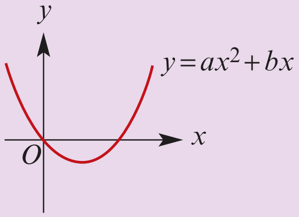
One point is needed to determine a.

ii



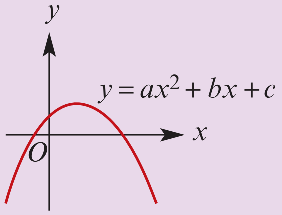
Two points are needed to determine a and c.

iii



Two points are needed to determine a and b.

iv



Three points are needed to determine a, b and c.

Exercise 4B

Example 9
Skillsheet

1 A parabola has x-axis intercepts -3 and -2 and it passes through the point $(1, -24)$. Find the rule for this parabola.

2 A parabola has x-axis intercepts -3 and $-\frac{3}{2}$ and it passes through the point $(1, 20)$. Find the rule for this parabola.

Example 10

3 The coordinates of the turning point of a parabola are $(-2, 4)$ and the parabola passes through the point $(4, 58)$. Find the rule for this parabola.

4 The coordinates of the turning point of a parabola are $(-2, -3)$ and the parabola passes through the point $(-3, -5)$. Find the rule for this parabola.

Example 11

5 A parabola passes through the points $(1, 19)$, $(0, 18)$ and $(-1, 7)$. Find the rule for this parabola.

6 A parabola passes through the points $(2, -14)$, $(0, 10)$ and $(-4, 10)$. Find the rule for this parabola.

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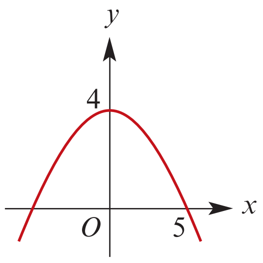
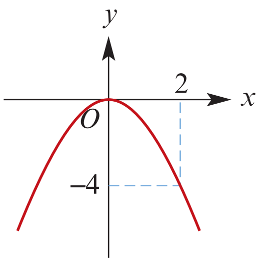
Example 12

7 Determine the equation of each of the following parabolas:

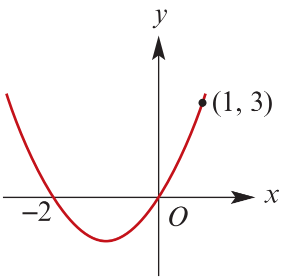
a



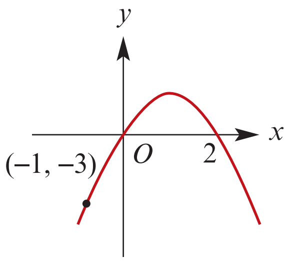
b



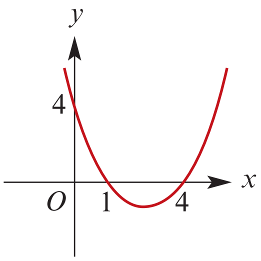
c



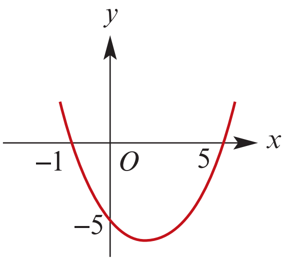
d



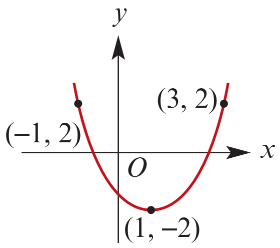
e



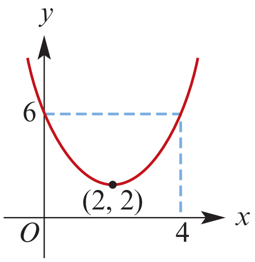
f



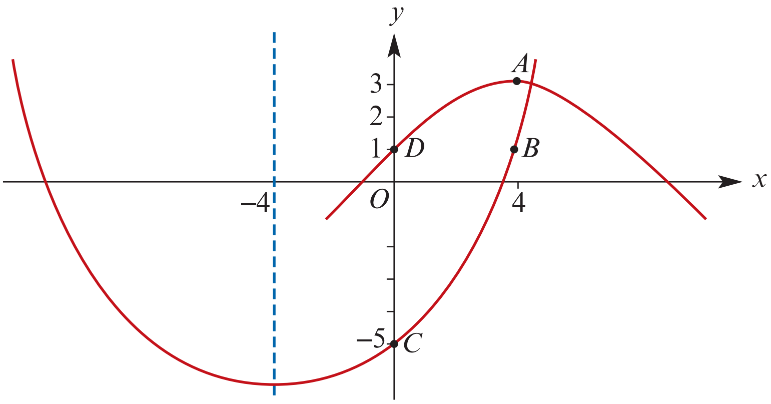
g



h



8 Find quadratic expressions for the two curves in the diagram, given that the coefficient of x in each case is 1. The marked points are $A(4, 3)$, $B(4, 1)$, $C(0, -5)$ and $D(0, 1)$.


y
A
3
2
1
D
B
x
−4
O
4
−5
C

9 The graph of the quadratic function $f(x) = A(x + b)^2 + B$ has a vertex at $(-2, 4)$ and passes through the point $(0, 8)$. Find the values of A, b and B.

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4C The language of polynomials

- A **polynomial function** is a function that can be written in the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

  where $n \in \mathbb{N} \cup \{0\}$ and the coefficients $a_0, \dots, a_n$ are real numbers with $a_n \neq 0$.
- The number 0 is called the **zero polynomial**.
- The **leading term**, $a_n x^n$, of a polynomial is the term of highest index among those terms with a non-zero coefficient.
- The **degree of a polynomial** is the index n of the leading term.
- A **monic polynomial** is a polynomial whose leading term has coefficient 1.
- The **constant term** is the term of index 0. (This is the term not involving x.)

Example 13

Let $P(x) = x^4 - 3x^3 - 2$. Find:

a $P(1)$ **b** $P(-1)$ **c** $P(2)$ **d** $P(-2)$

Solution

a $P(1) = 1^4 - 3 \times 1^3 - 2$
$= 1 - 3 - 2$
$= -4$

b $P(-1) = (-1)^4 - 3 \times (-1)^3 - 2$
$= 1 + 3 - 2$
$= 2$

c $P(2) = 2^4 - 3 \times 2^3 - 2$
$= 16 - 24 - 2$
$= -10$

d $P(-2) = (-2)^4 - 3 \times (-2)^3 - 2$
$= 16 + 24 - 2$
$= 38$

Example 14

a Let $P(x) = 2x^4 - x^3 + 2cx + 6$. If $P(1) = 21$, find the value of c.

b Let $Q(x) = 2x^6 - x^3 + ax^2 + bx + 20$. If $Q(-1) = Q(2) = 0$, find the values of a and b.

Solution

a $P(x) = 2x^4 - x^3 + 2cx + 6$ and $P(1) = 21$.

$$\begin{aligned} P(1) &= 2(1)^4 - (1)^3 + 2c + 6 \\ &= 2 - 1 + 2c + 6 \\ &= 7 + 2c \end{aligned}$$

Since $P(1) = 21$,

$$7 + 2c = 21$$
$$\therefore \quad c = 7$$

Explanation

We will substitute $x = 1$ into $P(x)$ to form an equation and solve.

b $Q(x) = 2x^6 - x^3 + ax^2 + bx + 20$ and $Q(-1) = Q(2) = 0$.

First find $Q(-1)$ and $Q(2)$ in terms of a and b.

$$\begin{aligned} Q(-1) &= 2(-1)^6 - (-1)^3 + a(-1)^2 - b + 20 \\ &= 2 + 1 + a - b + 20 \\ &= 23 + a - b \end{aligned}$$

$$\begin{aligned} Q(2) &= 2(2)^6 - (2)^3 + a(2)^2 + 2b + 20 \\ &= 128 - 8 + 4a + 2b + 20 \\ &= 140 + 4a + 2b \end{aligned}$$

Since $Q(-1) = Q(2) = 0$, this gives

Form simultaneous equations in a and b by putting $Q(-1) = 0$ and $Q(2) = 0$.

$$\begin{aligned} 23 + a - b &= 0 \qquad (1) \\ 140 + 4a + 2b &= 0 \qquad (2) \end{aligned}$$

Divide (2) by 2:

$$70 + 2a + b = 0 \qquad (3)$$

Add (1) and (3):

$$\begin{aligned} 93 + 3a &= 0 \\ \therefore \quad a &= -31 \end{aligned}$$

Substitute in (1) to obtain $b = -8$.

▶ The arithmetic of polynomials

The operations of addition, subtraction and multiplication for polynomials are naturally defined, as shown in the following examples.

Let $P(x) = x^3 + 3x^2 + 2$ and $Q(x) = 2x^2 + 4$. Then

$$\begin{aligned} P(x) + Q(x) &= (x^3 + 3x^2 + 2) + (2x^2 + 4) \\ &= x^3 + 5x^2 + 6 \end{aligned}$$

$$\begin{aligned} P(x) - Q(x) &= (x^3 + 3x^2 + 2) - (2x^2 + 4) \\ &= x^3 + x^2 - 2 \end{aligned}$$

$$\begin{aligned} P(x)Q(x) &= (x^3 + 3x^2 + 2)(2x^2 + 4) \\ &= (x^3 + 3x^2 + 2) \times 2x^2 + (x^3 + 3x^2 + 2) \times 4 \\ &= 2x^5 + 6x^4 + 4x^2 + 4x^3 + 12x^2 + 8 \\ &= 2x^5 + 6x^4 + 4x^3 + 16x^2 + 8 \end{aligned}$$

The sum, difference and product of two polynomials is a polynomial.

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Example 15

Let $P(x) = x^3 - 6x + 3$ and $Q(x) = x^2 - 3x + 1$. Find:

a $P(x) + Q(x)$ **b** $P(x) - Q(x)$ **c** $P(x)Q(x)$

Solution

a $P(x) + Q(x)$

$$\begin{aligned} &= x^3 - 6x + 3 + x^2 - 3x + 1 \\ &= x^3 + x^2 - 6x - 3x + 3 + 1 \\ &= x^3 + x^2 - 9x + 4 \end{aligned}$$

b $P(x) - Q(x)$

$$\begin{aligned} &= x^3 - 6x + 3 - (x^2 - 3x + 1) \\ &= x^3 - 6x + 3 - x^2 + 3x - 1 \\ &= x^3 - x^2 - 6x + 3x + 3 - 1 \\ &= x^3 - x^2 - 3x + 2 \end{aligned}$$

c

$$\begin{aligned} P(x)Q(x) &= (x^3 - 6x + 3)(x^2 - 3x + 1) \\ &= x^3(x^2 - 3x + 1) - 6x(x^2 - 3x + 1) + 3(x^2 - 3x + 1) \\ &= x^5 - 3x^4 + x^3 - 6x^3 + 18x^2 - 6x + 3x^2 - 9x + 3 \\ &= x^5 - 3x^4 + (x^3 - 6x^3) + (18x^2 + 3x^2) - (6x + 9x) + 3 \\ &= x^5 - 3x^4 - 5x^3 + 21x^2 - 15x + 3 \end{aligned}$$

We use the notation $\deg(f)$ to denote the degree of a polynomial f. For $f, g \neq 0$, we have

$$\deg(f + g) \leq \max\{\deg(f), \deg(g)\}$$

$$\deg(f \times g) = \deg(f) + \deg(g)$$

Equating coefficients

Two polynomials P and Q are equal only if their corresponding coefficients are equal. For two cubic polynomials, $P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ and $Q(x) = b_3x^3 + b_2x^2 + b_1x + b_0$, they are equal if and only if $a_3 = b_3$, $a_2 = b_2$, $a_1 = b_1$ and $a_0 = b_0$.

For example, if

$$P(x) = 4x^3 + 5x^2 - x + 3 \quad \text{and} \quad Q(x) = b_3x^3 + b_2x^2 + b_1x + b_0$$

then $P(x) = Q(x)$ if and only if $b_3 = 4$, $b_2 = 5$, $b_1 = -1$ and $b_0 = 3$.

Example 16

The polynomial $P(x) = x^3 + 3x^2 + 2x + 1$ can be written in the form $(x - 2)(x^2 + bx + c) + r$ where b, c and r are real numbers. Find the values of b, c and r.

Solution

Expand the required form:

$$\begin{aligned} (x - 2)(x^2 + bx + c) + r &= x(x^2 + bx + c) - 2(x^2 + bx + c) + r \\ &= x^3 + bx^2 + cx - 2x^2 - 2bx - 2c + r \\ &= x^3 + (b - 2)x^2 + (c - 2b)x - 2c + r \end{aligned}$$

If $x^3 + 3x^2 + 2x + 1 = x^3 + (b-2)x^2 + (c-2b)x - 2c + r$ for all real numbers x, then by equating coefficients:

coefficient of x^2 $\quad 3 = b - 2 \quad \therefore\ b = 5$

coefficient of x $\quad 2 = c - 2b \quad \therefore\ c = 2b + 2 = 12$

constant term $\quad 1 = -2c + r \quad \therefore\ r = 2c + 1 = 25$

Hence $b = 5$, $c = 12$ and $r = 25$.

This means $P(x) = (x-2)(x^2 + 5x + 12) + 25$.

Example 17

a If $x^3 + 3x^2 + 3x + 8 = a(x+1)^3 + b$ for all $x \in \mathbb{R}$, find the values of a and b.

b Show that $x^3 + 6x^2 + 6x + 8$ cannot be written in the form $a(x+c)^3 + b$ for real numbers a, b and c.

Solution

a Expand the right-hand side of the equation:

$$\begin{aligned} a(x+1)^3 + b &= a(x^3 + 3x^2 + 3x + 1) + b \\ &= ax^3 + 3ax^2 + 3ax + a + b \end{aligned}$$

If $x^3 + 3x^2 + 3x + 8 = ax^3 + 3ax^2 + 3ax + a + b$ for all $x \in \mathbb{R}$, then by equating coefficients:

coefficient of x^3 $\quad 1 = a$

coefficient of x^2 $\quad 3 = 3a$

coefficient of x $\quad 3 = 3a$

constant term $\quad 8 = a + b$

Hence $a = 1$ and $b = 7$.

b Expand the proposed form:

$$\begin{aligned} a(x+c)^3 + b &= a(x^3 + 3cx^2 + 3c^2x + c^3) + b \\ &= ax^3 + 3cax^2 + 3c^2ax + c^3a + b \end{aligned}$$

Suppose $x^3 + 6x^2 + 6x + 8 = ax^3 + 3cax^2 + 3c^2ax + c^3a + b$ for all $x \in \mathbb{R}$. Then

coefficient of x^3 $\quad 1 = a \quad (1)$

coefficient of x^2 $\quad 6 = 3ca \quad (2)$

coefficient of x $\quad 6 = 3c^2a \quad (3)$

constant term $\quad 8 = c^3a + b \quad (4)$

From (1), we have $a = 1$. So from (2), we have $c = 2$.
But substituting $a = 1$ and $c = 2$ into (3) gives $6 = 12$, which is a contradiction.

Section summary

- A **polynomial function** is a function that can be written in the form
  $$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$
  where $n \in \mathbb{N} \cup \{0\}$ and the coefficients $a_0, \dots, a_n$ are real numbers with $a_n \neq 0$.
  The **leading term** is $a_n x^n$ (the term of highest index) and the **constant term** is a_0 (the term not involving x).
- The **degree of a polynomial** is the index n of the leading term.
- The sum, difference and product of two polynomials is a polynomial. Division does not always lead to another polynomial.
- Two polynomials P and Q are equal only if their corresponding coefficients are equal. Two cubic polynomials, $P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ and $Q(x) = b_3x^3 + b_2x^2 + b_1x + b_0$, are equal if and only if $a_3 = b_3$, $a_2 = b_2$, $a_1 = b_1$ and $a_0 = b_0$.

Exercise 4C

Example 13

1 Let $P(x) = x^3 - 2x^2 + 3x + 1$. Find:

a $P(1)$ **b** $P(-1)$ **c** $P(2)$ **d** $P(-2)$ **e** $P(\frac{1}{2})$ **f** $P(-\frac{1}{2})$

2 Let $P(x) = x^3 + 3x^2 - 4x + 6$. Find:

a $P(0)$ **b** $P(1)$ **c** $P(2)$ **d** $P(-1)$ **e** $P(a)$ **f** $P(2a)$

Example 14

3 **a** Let $P(x) = x^3 + 3x^2 - ax - 30$. If $P(2) = 0$, find the value of a.

b Let $P(x) = x^3 + ax^2 + 5x - 14$. If $P(3) = 68$, find the value of a.

c Let $P(x) = x^4 - x^3 - 2x + c$. If $P(1) = 6$, find the value of c.

d Let $P(x) = 2x^6 - 5x^3 + ax^2 + bx + 12$. If $P(-1) = P(2) = 0$, find a and b.

e Let $P(x) = x^5 - 2x^4 + ax^3 + bx^2 + 12x - 36$. If $P(3) = P(1) = 0$, find a and b.

Example 15

4 Let $f(x) = 2x^3 - x^2 + 3x$, $g(x) = 2 - x$ and $h(x) = x^2 + 2x$. Simplify each of the following:

a $f(x) + g(x)$ **b** $f(x) + h(x)$ **c** $f(x) - g(x)$

d $3f(x)$ **e** $f(x)\,g(x)$ **f** $g(x)\,h(x)$

g $f(x) + g(x) + h(x)$ **h** $f(x)\,h(x)$

5 Expand each of the following products and collect like terms:

a $(x - 2)(x^2 - 3x + 4)$ **b** $(x - 5)(x^2 - 2x + 3)$ **c** $(x + 1)(2x^2 - 3x - 4)$

d $(x + 2)(x^2 + bx + c)$ **e** $(2x - 1)(x^2 - 4x - 3)$

Example 16

6 It is known that $x^3 - x^2 - 6x - 4 = (x + 1)(x^2 + bx + c)$ for all values of x, for suitable values of b and c.

a Expand $(x + 1)(x^2 + bx + c)$ and collect like terms.

b Find b and c by equating coefficients.

c Hence write $x^3 - x^2 - 6x - 4$ as a product of three linear factors.

Example 17

7 **a** If $2x^3 - 18x^2 + 54x - 49 = a(x-3)^3 + b$ for all $x \in \mathbb{R}$, find the values of a and b.

b If $-2x^3 + 18x^2 - 54x + 52 = a(x+c)^3 + b$ for all $x \in \mathbb{R}$, find the values of a, b and c.

c Show that $x^3 - 5x^2 - 2x + 24$ cannot be written in the form $a(x+c)^3 + b$ for real numbers a, b and c.

8 Find the values of A and B such that $A(x+3) + B(x+2) = 4x + 9$ for all real numbers x.

9 Find the values of A, B and C in each of the following:

a $x^2 - 4x + 10 = A(x+B)^2 + C$ for all $x \in \mathbb{R}$

b $4x^2 - 12x + 14 = A(x+B)^2 + C$ for all $x \in \mathbb{R}$

c $x^3 - 9x^2 + 27x - 22 = A(x+B)^3 + C$ for all $x \in \mathbb{R}$.

4D Division and factorisation of polynomials

The division of polynomials was introduced in Mathematical Methods Units 1 & 2.

When we divide the polynomial $P(x)$ by the polynomial $D(x)$ we obtain two polynomials, $Q(x)$ the **quotient** and $R(x)$ the **remainder**, such that

$$P(x) = D(x)Q(x) + R(x)$$

and either $R(x) = 0$ or $R(x)$ has degree less than $D(x)$.

Here $P(x)$ is the **dividend** and $D(x)$ is the **divisor**.

The following example illustrates the process of dividing.

Example 18

Divide $x^3 + x^2 - 14x - 24$ by $x + 2$.

Solution

$$
\begin{array}{r@{}r}
 & x^2 - x - 12 \\
x+2 & \big)\overline{\,x^3 + x^2 - 14x - 24} \\
 & \underline{x^3 + 2x^2 \phantom{{}- 14x - 24}} \\
 & -x^2 - 14x - 24 \\
 & \underline{-x^2 - 2x \phantom{{}- 24}} \\
 & -12x - 24 \\
 & \underline{-12x - 24} \\
 & 0
\end{array}
$$

Explanation

- Divide x, from $x + 2$, into the leading term x^3 to get x^2.
- Multiply x^2 by $x + 2$ to give $x^3 + 2x^2$.
- Subtract from $x^3 + x^2 - 14x - 24$, leaving $-x^2 - 14x - 24$.
- Now divide x, from $x + 2$, into $-x^2$ to get $-x$.
- Multiply $-x$ by $x + 2$ to give $-x^2 - 2x$.
- Subtract from $-x^2 - 14x - 24$, leaving $-12x - 24$.
- Divide x into $-12x$ to get -12.
- Multiply -12 by $x + 2$ to give $-12x - 24$.
- Subtract from $-12x - 24$, leaving remainder of 0.

In this example we see that $x + 2$ is a factor of $x^3 + x^2 - 14x - 24$, as the remainder is zero. Thus $(x^3 + x^2 - 14x - 24) \div (x + 2) = x^2 - x - 12$ with zero remainder.

$$\therefore \quad \frac{x^3 + x^2 - 14x - 24}{x + 2} = x^2 - x - 12$$

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Example 19

Divide $3x^4 - 9x^2 + 27x - 8$ by $x - 2$.

Solution

$$
\begin{array}{r@{\,}l}
 & 3x^3 + 6x^2 + 3x + 33 \\ \cline{2-2}
x - 2 \,\big) & 3x^4 + 0x^3 - 9x^2 + 27x - 8 \\
 & \underline{3x^4 - 6x^3 \qquad\qquad\qquad\quad} \\
 & \qquad\quad 6x^3 - 9x^2 + 27x - 8 \\
 & \qquad\quad \underline{6x^3 - 12x^2 \qquad\qquad} \\
 & \qquad\qquad\qquad 3x^2 + 27x - 8 \\
 & \qquad\qquad\qquad \underline{3x^2 - 6x \qquad\;} \\
 & \qquad\qquad\qquad\qquad\quad 33x - 8 \\
 & \qquad\qquad\qquad\qquad\quad \underline{33x - 66} \\
 & \qquad\qquad\qquad\qquad\qquad\quad 58
\end{array}
$$

Therefore

$$3x^4 - 9x^2 + 27x - 8 = (x - 2)(3x^3 + 6x^2 + 3x + 33) + 58$$

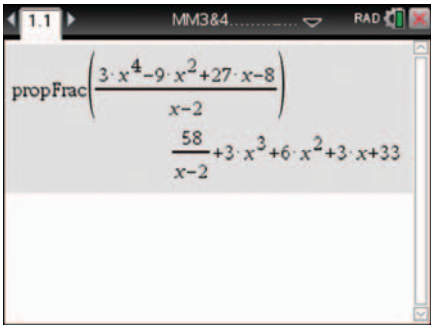
or, equivalently,

$$\frac{3x^4 - 9x^2 + 27x - 8}{x - 2} = 3x^3 + 6x^2 + 3x + 33 + \frac{58}{x - 2}$$

In this example, the dividend is $3x^4 - 9x^2 + 27x - 8$, the divisor is $x - 2$, and the remainder is 58.

Using the TI-Nspire

Use **propFrac** from (menu) > **Algebra** > **Fraction Tools** > **Proper Fraction** as shown.



propFrac$\left(\dfrac{3\cdot x^4-9\cdot x^2+27\cdot x-8}{x-2}\right)$

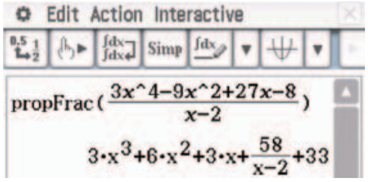
$\dfrac{58}{x-2}+3\cdot x^3+6\cdot x^2+3\cdot x+33$

Using the Casio ClassPad

- Enter and highlight

$$\frac{3x^4 - 9x^2 + 27x - 8}{x - 2}$$

- Select **Interactive** > **Transformation** > **propFrac**.



propFrac$\left(\dfrac{3x\text{^}4-9x\text{^}2+27x-8}{x-2}\right)$

$3\cdot x^3+6\cdot x^2+3\cdot x+\dfrac{58}{x-2}+33$

A second method for division, called **equating coefficients**, can be seen in the explanation column of the next example.

Example 20

Divide $3x^3 + 2x^2 - x - 2$ by $2x + 1$.

Solution

$$
\begin{array}{r l}
 & \frac{3}{2}x^2 + \frac{1}{4}x - \frac{5}{8} \\
2x + 1 \big) & \overline{3x^3 + 2x^2 - x - 2} \\
 & \underline{3x^3 + \frac{3}{2}x^2} \\
 & \qquad \frac{1}{2}x^2 - x - 2 \\
 & \qquad \underline{\frac{1}{2}x^2 + \frac{1}{4}x} \\
 & \qquad \qquad -\frac{5}{4}x - 2 \\
 & \qquad \qquad \underline{-\frac{5}{4}x - \frac{5}{8}} \\
 & \qquad \qquad \qquad -1\frac{3}{8}
\end{array}
$$

Explanation

We show the alternative method here.

First write the identity

$3x^3 + 2x^2 - x - 2 = (2x + 1)(ax^2 + bx + c) + r$

Equate coefficients of x^3:

$3 = 2a$. Therefore $a = \frac{3}{2}$.

Equate coefficients of x^2:

$2 = a + 2b$. Therefore $b = \frac{1}{2}(2 - \frac{3}{2}) = \frac{1}{4}$.

Equate coefficients of x:

$-1 = 2c + b$. Therefore $c = \frac{1}{2}(-1 - \frac{1}{4}) = -\frac{5}{8}$.

Equate constant terms:

$-2 = c + r$. Therefore $r = -2 + \frac{5}{8} = -\frac{11}{8}$.

A third method, called **synthetic division**, is described in the Interactive Textbook.

Dividing by a non-linear polynomial

We give one example of dividing by a non-linear polynomial. The technique is exactly the same as when dividing by a linear polynomial.

Example 21

Divide $3x^3 - 2x^2 + 3x - 4$ by $x^2 - 1$.

Solution

$$
\begin{array}{r l}
 & 3x - 2 \\
x^2 + 0x - 1 \big) & \overline{3x^3 - 2x^2 + 3x - 4} \\
 & \underline{3x^3 + 0x^2 - 3x} \\
 & \qquad -2x^2 + 6x - 4 \\
 & \qquad \underline{-2x^2 + 0x + 2} \\
 & \qquad \qquad \qquad 6x - 6
\end{array}
$$

Therefore

$$3x^3 - 2x^2 + 3x - 4 = (x^2 - 1)(3x - 2) + 6x - 6$$

or, equivalently,

$$\frac{3x^3 - 2x^2 + 3x - 4}{x^2 - 1} = 3x - 2 + \frac{6x - 6}{x^2 - 1}$$

Explanation

We write $x^2 - 1$ as $x^2 + 0x - 1$.

The remainder theorem and the factor theorem

The following two results are recalled from Mathematical Methods Units 1 & 2.

The remainder theorem

Suppose that, when the polynomial $P(x)$ is divided by $x - \alpha$, the quotient is $Q(x)$ and the remainder is R. Then

$$P(x) = (x - \alpha)Q(x) + R$$

Now, as the two expressions are equal for all values of x, they are equal for $x = \alpha$.

$$\therefore \quad P(\alpha) = (\alpha - \alpha)Q(\alpha) + R \qquad \therefore \quad R = P(\alpha)$$

i.e. the remainder when $P(x)$ is divided by $x - \alpha$ is equal to $P(\alpha)$. We therefore have

$$P(x) = (x - \alpha)Q(x) + P(\alpha)$$

More generally:

Remainder theorem

When $P(x)$ is divided by $\beta x + \alpha$, the remainder is $P\left(-\dfrac{\alpha}{\beta}\right)$.

Example 22

Find the remainder when $P(x) = 3x^3 + 2x^2 + x + 1$ is divided by $2x + 1$.

Solution

By the remainder theorem, the remainder is

$$P\left(-\frac{1}{2}\right) = 3\left(-\frac{1}{2}\right)^3 + 2\left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right) + 1$$

$$= -\frac{3}{8} + \frac{2}{4} - \frac{1}{2} + 1 = \frac{5}{8}$$

The factor theorem

Now, in order for $x - \alpha$ to be a factor of the polynomial $P(x)$, the remainder must be zero. We state this result as the **factor theorem**.

Factor theorem

For a polynomial $P(x)$:

- If $P(\alpha) = 0$, then $x - \alpha$ is a factor of $P(x)$.
- Conversely, if $x - \alpha$ is a factor of $P(x)$, then $P(\alpha) = 0$.

More generally:

- If $\beta x + \alpha$ is a factor of $P(x)$, then $P\left(-\dfrac{\alpha}{\beta}\right) = 0$.
- Conversely, if $P\left(-\dfrac{\alpha}{\beta}\right) = 0$, then $\beta x + \alpha$ is a factor of $P(x)$.

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Example 23

Given that $x + 1$ and $x - 2$ are factors of $6x^4 - x^3 + ax^2 - 6x + b$, find the values of a and b.

Solution

Let $P(x) = 6x^4 - x^3 + ax^2 - 6x + b$.

By the factor theorem, we have $P(-1) = 0$ and $P(2) = 0$. Hence

$$6 + 1 + a + 6 + b = 0 \qquad (1)$$

$$96 - 8 + 4a - 12 + b = 0 \qquad (2)$$

Rearranging gives:

$$a + b = -13 \qquad (1')$$

$$4a + b = -76 \qquad (2')$$

Subtract $(1')$ from $(2')$:

$$3a = -63$$

Therefore $a = -21$ and, from $(1')$, $b = 8$.

Example 24

Show that $x + 1$ is a factor of $x^3 - 4x^2 + x + 6$ and hence find the other linear factors.

Solution

Let $\quad P(x) = x^3 - 4x^2 + x + 6$

Then $\quad P(-1) = (-1)^3 - 4(-1)^2 + (-1) + 6$

$\qquad\qquad = 0$

Thus $x + 1$ is a factor (by the factor theorem).

Divide by $x + 1$ to find the other factor:

$$
\begin{array}{r}
x^2 - 5x + 6 \\
x + 1\,\big)\overline{x^3 - 4x^2 + x + 6} \\
\underline{x^3 + x^2\phantom{{}+x+6}} \\
-5x^2 + x + 6 \\
\underline{-5x^2 - 5x\phantom{{}+6}} \\
6x + 6 \\
\underline{6x + 6} \\
0
\end{array}
$$

$$\therefore x^3 - 4x^2 + x + 6 = (x + 1)(x^2 - 5x + 6)$$

$$= (x + 1)(x - 3)(x - 2)$$

The linear factors of $x^3 - 4x^2 + x + 6$ are $(x + 1)$, $(x - 3)$ and $(x - 2)$.

Explanation

We can use the factor theorem to find one factor, and then divide to find the other two linear factors.

Here is an alternative method:

Once we have found that $x + 1$ is a factor, we know that we can write

$$x^3 - 4x^2 + x + 6 = (x + 1)(x^2 + bx + c)$$

By equating constant terms, we have $6 = 1 \times c$. Hence $c = 6$.

By equating coefficients of x^2, we have $-4 = 1 + b$. Hence $b = -5$.

$$\therefore \quad x^3 - 4x^2 + x + 6 = (x + 1)(x^2 - 5x + 6)$$

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Sums and differences of cubes

If $P(x) = x^3 - a^3$, then $x - a$ is a factor and so by division:

$$x^3 - a^3 = (x - a)(x^2 + ax + a^2)$$

If a is replaced by $-a$, then

$$x^3 - (-a)^3 = (x - (-a))(x^2 + (-a)x + (-a)^2)$$

This gives:

$$x^3 + a^3 = (x + a)(x^2 - ax + a^2)$$

Example 25

Factorise:

a $8x^3 + 64$

b $125a^3 - b^3$

Solution

a $$\begin{aligned} 8x^3 + 64 &= (2x)^3 + (4)^3 \\ &= (2x + 4)(4x^2 - 8x + 16) \end{aligned}$$

b $$\begin{aligned} 125a^3 - b^3 &= (5a)^3 - b^3 \\ &= (5a - b)(25a^2 + 5ab + b^2) \end{aligned}$$

The rational-root theorem

Consider the cubic polynomial

$$P(x) = 2x^3 - x^2 - x - 3$$

If the equation $P(x) = 0$ has a solution α that is an integer, then α divides the constant term -3. We can easily show that $P(1) \neq 0$, $P(-1) \neq 0$, $P(3) \neq 0$ and $P(-3) \neq 0$. Hence the equation $P(x) = 0$ has no solution that is an integer.

Does it have a rational solution, that is, a fraction for a solution?

The **rational-root theorem** helps us with this. It says that if α and β have highest common factor 1 (i.e. α and β are relatively prime) and $\beta x + \alpha$ is a factor of $2x^3 - x^2 - x - 3$, then β divides 2 and α divides -3.

Therefore, if $-\dfrac{\alpha}{\beta}$ is a solution of the equation $P(x) = 0$ (where α and β are relatively prime), then β must divide 2 and α must divide -3. So the only value of β that needs to be considered is 2, and $\alpha = \pm 3$ or $\alpha = \pm 1$.

We can test these through the factor theorem. That is, check $P\left(\pm\dfrac{1}{2}\right)$ and $P\left(\pm\dfrac{3}{2}\right)$. We find

$$\begin{aligned} P\left(\frac{3}{2}\right) &= 2\left(\frac{3}{2}\right)^3 - \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right) - 3 \\ &= 2 \times \frac{27}{8} - \frac{9}{4} - \frac{3}{2} - 3 \\ &= 0 \end{aligned}$$

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We have found that $2x - 3$ is a factor of $P(x) = 2x^3 - x^2 - x - 3$.

Dividing through we find that

$$2x^3 - x^2 - x - 3 = (2x - 3)(x^2 + x + 1)$$

We can show that $x^2 + x + 1$ has no linear factors by showing that the discriminant of this quadratic is negative.

Example 26

Use the rational-root theorem to help factorise $P(x) = 3x^3 + 8x^2 + 2x - 5$.

Solution

$$P(1) = 8 \neq 0, \qquad P(-1) = -2 \neq 0,$$
$$P(5) = 580 \neq 0, \qquad P(-5) = -190 \neq 0,$$
$$P\left(-\frac{5}{3}\right) = 0$$

Therefore $3x + 5$ is a factor.

Dividing gives

$$3x^3 + 8x^2 + 2x - 5 = (3x + 5)(x^2 + x - 1)$$

We complete the square for $x^2 + x - 1$ to factorise:

$$\begin{aligned} x^2 + x - 1 &= x^2 + x + \frac{1}{4} - \frac{1}{4} - 1 \\ &= \left(x + \frac{1}{2}\right)^2 - \frac{5}{4} \\ &= \left(x + \frac{1}{2} + \frac{\sqrt{5}}{2}\right)\left(x + \frac{1}{2} - \frac{\sqrt{5}}{2}\right) \end{aligned}$$

Hence

$$P(x) = (3x + 5)\left(x + \frac{1}{2} + \frac{\sqrt{5}}{2}\right)\left(x + \frac{1}{2} - \frac{\sqrt{5}}{2}\right)$$

Explanation

The only possible integer solutions are ±5 or ±1. So there are no integer solutions. We now use the rational-root theorem.

If $-\frac{\alpha}{\beta}$ is a solution, the only value of β that needs to be considered is 3 and $\alpha = \pm 5$ or $\alpha = \pm 1$.

Here is the complete statement of the theorem:

Rational-root theorem

Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of degree n with all the coefficients a_i integers. Let α and β be integers such that the highest common factor of α and β is 1 (i.e. α and β are relatively prime).

If $\beta x + \alpha$ is a factor of $P(x)$, then β divides a_n and α divides a_0.

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▶ Solving polynomial equations

The factor theorem may be used in the solution of equations.

Example 27

Factorise $P(x) = x^3 - 4x^2 - 11x + 30$ and hence solve the equation $x^3 - 4x^2 - 11x + 30 = 0$.

Solution

$$P(1) = 1 - 4 - 11 + 30 \neq 0$$
$$P(-1) = -1 - 4 + 11 + 30 \neq 0$$
$$P(2) = 8 - 16 - 22 + 30 = 0$$

Therefore $x - 2$ is a factor.

Dividing $x^3 - 4x^2 - 11x + 30$ by $x - 2$ gives

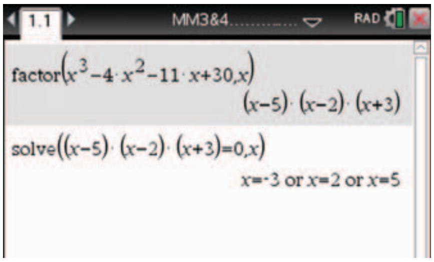
$$\begin{aligned} P(x) &= (x-2)(x^2 - 2x - 15) \\ &= (x-2)(x-5)(x+3) \end{aligned}$$

Now we see that $P(x) = 0$ if and only if

$$x - 2 = 0 \quad \text{or} \quad x - 5 = 0 \quad \text{or} \quad x + 3 = 0$$
$$\therefore \quad x = 2 \quad \text{or} \quad x = 5 \quad \text{or} \quad x = -3$$

Using the TI-Nspire

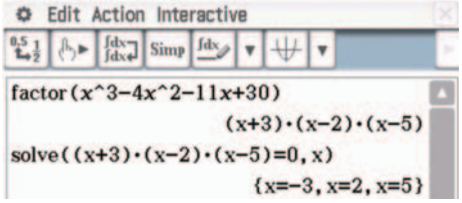
Use **factor** (menu > **Algebra** > **Factor**) and **solve** (menu > **Algebra** > **Solve**) as shown.



```
factor(x^3-4·x^2-11·x+30,x)
                    (x-5)·(x-2)·(x+3)
solve((x-5)·(x-2)·(x+3)=0,x)
                    x=-3 or x=2 or x=5
```

Using the Casio ClassPad

- Enter and highlight $x^3 - 4x^2 - 11x + 30$.
- Select **Interactive** > **Transformation** > **factor**.
- Copy and paste the answer to the next entry line.
- Select **Interactive** > **Equation/Inequality** > **solve** and ensure the variable is x.



```
factor(x^3-4x^2-11x+30)
                    (x+3)·(x-2)·(x-5)
solve((x+3)·(x-2)·(x-5)=0,x)
                    {x=-3,x=2,x=5}
```

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Section summary

- **Division of polynomials** When we divide the polynomial $P(x)$ by the polynomial $D(x)$ we obtain two polynomials, $Q(x)$ the **quotient** and $R(x)$ the **remainder**, such that

  $$P(x) = D(x)Q(x) + R(x)$$

  and either $R(x) = 0$ or $R(x)$ has degree less than $D(x)$.
- Two methods for dividing polynomials are long division and equating coefficients.
- **Remainder theorem** When $P(x)$ is divided by $\beta x + \alpha$, the remainder is $P\left(-\frac{\alpha}{\beta}\right)$.
- **Factor theorem**
  - If $\beta x + \alpha$ is a factor of $P(x)$, then $P\left(-\frac{\alpha}{\beta}\right) = 0$.
  - Conversely, if $P\left(-\frac{\alpha}{\beta}\right) = 0$, then $\beta x + \alpha$ is a factor of $P(x)$.
- A cubic polynomial can be factorised by using the factor theorem to find the first linear factor and then using polynomial division or the method of equating coefficients to complete the factorisation.
- **Rational-root theorem** Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of degree n with all the coefficients a_i integers. Let α and β be integers such that the highest common factor of α and β is 1 (i.e. α and β are relatively prime). If $\beta x + \alpha$ is a factor of $P(x)$, then β divides a_n and α divides a_0.
- Difference of two cubes: $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$
- Sum of two cubes: $x^3 + a^3 = (x + a)(x^2 - ax + a^2)$

Exercise 4D

Skillsheet
Example 18

1 For each of the following, divide the first polynomial by the second:

a $x^3 - x^2 - 14x + 24, \quad x + 4$
b $2x^3 + x^2 - 25x + 12, \quad x - 3$

Example 19

2 For each of the following, divide the first polynomial by the second:

a $x^3 - x^2 - 15x + 25, \quad x + 3$
b $2x^3 - 4x + 12, \quad x - 3$

Example 20

3 For each of the following, divide the first polynomial by the second:

a $2x^3 - 2x^2 - 15x + 25, \quad 2x + 3$
b $4x^3 + 6x^2 - 4x + 12, \quad 2x - 3$

4 For each of the following, divide the first expression by the second:

a $2x^3 - 7x^2 + 15x - 3, \quad x - 3$
b $5x^5 + 13x^4 - 2x^2 - 6, \quad x + 1$

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Example 21 **5** For each of the following, divide the first expression by the second:

a $x^4 - 9x^3 + 25x^2 - 8x - 2,\ x^2 - 2$

b $x^4 + x^3 + x^2 - x - 2,\ x^2 - 1$

Example 22 **6** **a** Find the remainder when $x^3 + 3x - 2$ is divided by $x + 2$.

b Find the value of a for which $(1 - 2a)x^2 + 5ax + (a - 1)(a - 8)$ is divisible by $x - 2$ but not by $x - 1$.

7 Given that $f(x) = 6x^3 + 5x^2 - 17x - 6$:

a Find the remainder when $f(x)$ is divided by $x - 2$.

b Find the remainder when $f(x)$ is divided by $x + 2$.

c Factorise $f(x)$ completely.

8 **a** Prove that the expression $x^3 + (k - 1)x^2 + (k - 9)x - 7$ is divisible by $x + 1$ for all values of k.

b Find the value of k for which the expression has a remainder of 12 when divided by $x - 2$.

Example 23 **9** The polynomial $f(x) = 2x^3 + ax^2 - bx + 3$ has a factor $x + 3$. When $f(x)$ is divided by $x - 2$, the remainder is 15.

a Calculate the values of a and b.

b Find the other two linear factors of $f(x)$.

10 The expression $4x^3 + ax^2 - 5x + b$ leaves remainders of -8 and 10 when divided by $2x - 3$ and $x - 3$ respectively. Calculate the values of a and b.

11 Find the remainder when $(x + 1)^4$ is divided by $x - 2$.

12 Let $P(x) = x^5 - 3x^4 + 2x^3 - 2x^2 + 3x + 1$.

a Show that neither $x - 1$ nor $x + 1$ is a factor of $P(x)$.

b Given that $P(x)$ can be written in the form $(x^2 - 1)Q(x) + ax + b$, where $Q(x)$ is a polynomial and a and b are constants, hence or otherwise, find the remainder when $P(x)$ is divided by $x^2 - 1$.

Example 24 **13** Show that $x + 1$ is a factor of $2x^3 - 5x^2 - 4x + 3$ and find the other linear factors.

14 **a** Show that both $x - \sqrt{3}$ and $x + \sqrt{3}$ are factors of $x^4 + x^3 - x^2 - 3x - 6$.

b Hence write down one quadratic factor of $x^4 + x^3 - x^2 - 3x - 6$, and find a second quadratic factor.

Example 25 **15** Factorise each of the following:

a $8a^3 + 27b^3$

b $64 - a^3$

c $125x^3 + 64y^3$

d $(a - b)^3 + (a + b)^3$

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Example 26 **16** Use the rational-root theorem to help factorise each of the following:

a $12x^3 + 20x^2 - x - 6$

b $4x^3 - 2x^2 + 6x - 3$

17 Use the rational-root theorem to help factorise each of the following:

a $4x^3 + 3x - 18$

b $8x^3 - 12x^2 - 2x + 3$

Example 27 **18** Solve each of the following equations for x:

a $(2 - x)(x + 4)(x - 2)(x - 3) = 0$
b $x^3(2 - x) = 0$
c $(2x - 1)^3(2 - x) = 0$
d $(x + 2)^3(x - 2)^2 = 0$
e $x^4 - 4x^2 = 0$
f $x^4 - 9x^2 = 0$
g $12x^4 + 11x^3 - 26x^2 + x + 2 = 0$
h $x^4 + 2x^3 - 3x^2 - 4x + 4 = 0$
i $6x^4 - 5x^3 - 20x^2 + 25x - 6 = 0$

19 Find the x-axis intercepts and y-axis intercept of the graph of each of the following:

a $y = x^3 - x^2 - 2x$
b $y = x^3 - 2x^2 - 5x + 6$
c $y = x^3 - 4x^2 + x + 6$
d $y = 2x^3 - 5x^2 + x + 2$
e $y = x^3 + 2x^2 - x - 2$
f $y = 3x^3 - 4x^2 - 13x - 6$
g $y = 5x^3 + 12x^2 - 36x - 16$
h $y = 6x^3 - 5x^2 - 2x + 1$
i $y = 2x^3 - 3x^2 - 29x - 30$

20 The expressions $px^4 - 5x + q$ and $x^4 - 2x^3 - px^2 - qx - 8$ have a common factor $x - 2$. Find the values of p and q.

21 Find the remainder when $f(x) = x^4 - x^3 + 5x^2 + 4x - 36$ is divided by $x + 1$.

22 Factorise each of the following polynomials, using a calculator to help find at least one linear factor:

a $x^3 - 11x^2 - 125x + 1287$
b $x^3 - 9x^2 - 121x + 1089$
c $2x^3 - 9x^2 - 242x + 1089$
d $4x^3 - 367x + 1287$

23 Factorise each of the following:

a $x^4 - x^3 - 43x^2 + x + 42$

b $x^4 + 4x^3 - 27x - 108$

24 Factorise each of the following polynomials, using a calculator to help find at least one linear factor:

a $2x^4 - 25x^3 + 57x^2 + 9x + 405$
b $x^4 + 13x^3 + 40x^2 + 81x + 405$
c $x^4 + 3x^3 - 4x^2 + 3x - 135$
d $x^4 + 4x^3 - 35x^2 - 78x + 360$

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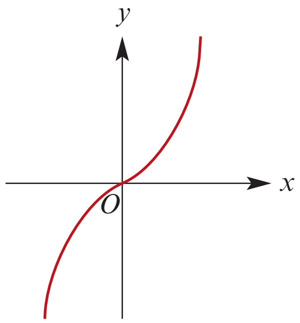
4E The general cubic function

Not all cubic functions can be written in the form $f(x) = a(x - h)^3 + k$. In this section we consider the general cubic function. The form of a general cubic function is

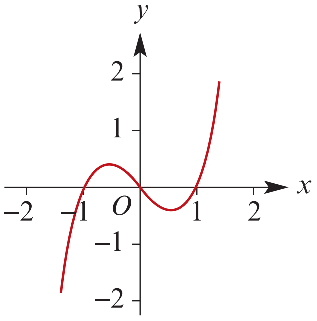
$$f(x) = ax^3 + bx^2 + cx + d, \quad \text{where } a \neq 0$$

It is impossible to fully investigate cubic functions without the use of calculus. Cubic functions will be revisited in Chapter 10.

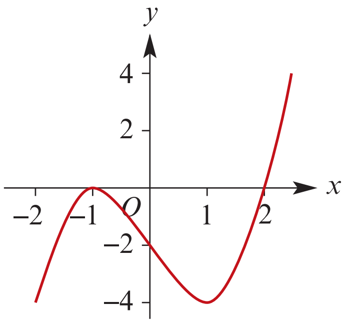
The 'shapes' of cubic graphs vary. Below is a gallery of cubic graphs, demonstrating the variety of 'shapes' that are possible.



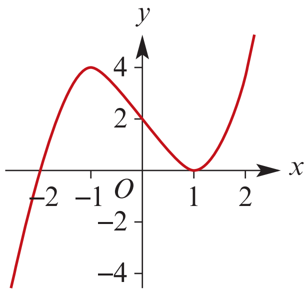
$f(x) = x^3 + x$



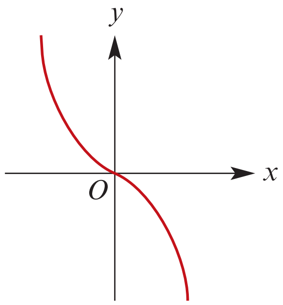
$f(x) = x^3 - x$



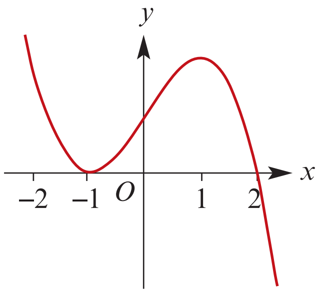
$f(x) = x^3 - 3x - 2$



$f(x) = x^3 - 3x + 2$



$f(x) = -x^3 - x$



$f(x) = -x^3 + 3x + 2$

Notes:

- A cubic graph can have one, two or three x-axis intercepts.
- Not all cubic graphs have a stationary point. For example, the graph of $f(x) = x^3 + x$ shown above has no points of zero gradient.
- The turning points do not occur symmetrically between consecutive x-axis intercepts as they do for quadratics. Differential calculus must be used to determine them.
- If a cubic graph has a turning point on the x-axis, this corresponds to a **repeated factor**. For example, the graph of $f(x) = x^3 - 3x - 2$ shown above has a turning point at $(-1, 0)$. The factorisation is $f(x) = (x + 1)^2(x - 2)$.

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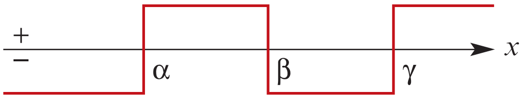
Sign diagrams

A **sign diagram** is a number-line diagram that shows when an expression is positive or negative. For a cubic function with rule $f(x) = (x - \alpha)(x - \beta)(x - \gamma)$, where $\alpha < \beta < \gamma$, the sign diagram is as shown.



Example 28

Draw a sign diagram for the cubic function $f(x) = x^3 - 4x^2 - 11x + 30$.

Solution

From Example 27, we have

$$f(x) = (x + 3)(x - 2)(x - 5)$$

Therefore $f(-3) = f(2) = f(5) = 0$. We note that

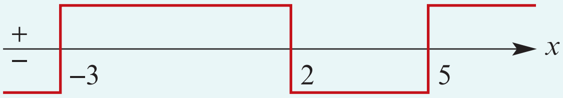
$f(x) > 0$ for $x > 5$

$f(x) < 0$ for $2 < x < 5$

$f(x) > 0$ for $-3 < x < 2$

$f(x) < 0$ for $x < -3$

Hence the sign diagram may be drawn as shown.



Example 29

For the cubic function with rule $f(x) = -x^3 + 19x - 30$:

a Sketch the graph of $y = f(x)$ using a calculator to find the coordinates of the turning points, correct to two decimal places.

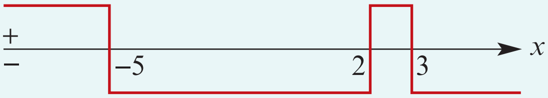
b Sketch the graph of $y = \frac{1}{2}f(x - 1)$.

Solution

a $f(x) = -x^3 + 19x - 30$

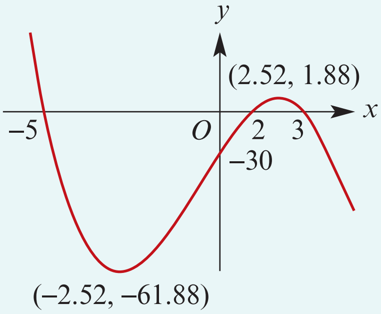
$= (3 - x)(x - 2)(x + 5)$

$= -(x + 5)(x - 2)(x - 3)$



The x-axis intercepts are at $x = -5$, $x = 2$ and $x = 3$ and the y-axis intercept is at $y = -30$.

The turning points can be found using a CAS calculator. The method is described following this example.



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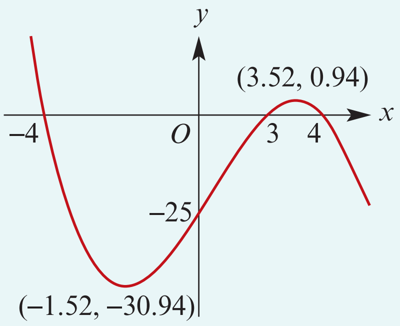
b The rule for the transformation is

$$(x, y) \to \left(x + 1, \frac{1}{2}y\right)$$

This is a dilation of factor $\frac{1}{2}$ from the x-axis followed by a translation 1 unit to the right. Transformations of the turning points:

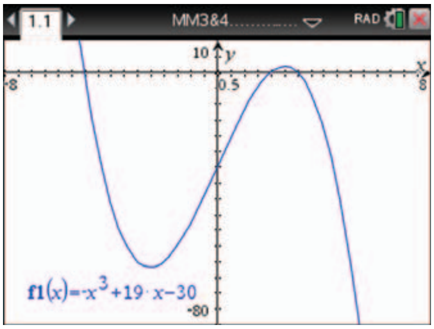
$$(2.52, 1.88) \to (3.52, 0.94)$$

$$(-2.52, -61.88) \to (-1.52, -30.94)$$

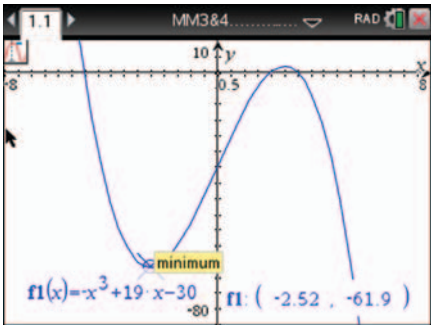


Using the TI-Nspire

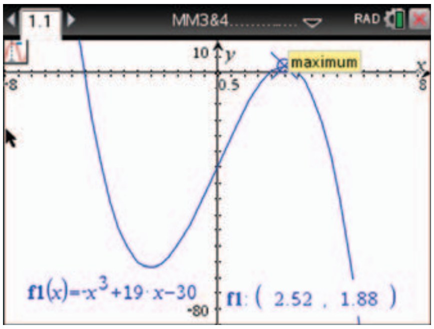
- Enter the function in a **Graphs** page.
- Use menu > **Window/Zoom** > **Window Settings** to set an appropriate window.
- Use either menu > **Trace** > **Graph Trace** or menu > **Analyze Graph** > **Maximum or Minimum** to display the approximate (decimal) coordinates of key points on the graph.


1.1
MM3&4
RAD
10
y
0.5
-8
8
x
f1(x)=-x^3+19·x-30
-80

- In **Graph Trace**, the tracing point (×) can be moved either by using the arrow keys (◄ ►) or by typing a specific x-value then enter. When the tracing point reaches a local minimum, it displays 'minimum'.
- Pressing enter will paste the coordinates to the point on the graph.
- Press esc to exit the command.


1.1
MM3&4
RAD
10
y
0.5
-8
8
x
minimum
f1(x)=-x^3+19·x-30
-80
f1: (-2.52 , -61.9)

- Here **Graph Trace** has been used to find the turning points of the cubic function.
- If you use **Analyze Graph** instead, select the lower bound by moving to the left of the key point and clicking ( ) and then select the upper bound by moving to the right (►) of the key point and clicking.


1.1
MM3&4
RAD
10
y
0.5
maximum
-8
8
x
f1(x)=-x^3+19·x-30
-80
f1: (2.52 , 1.88)

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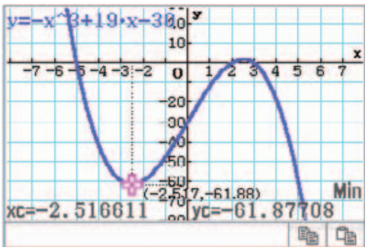
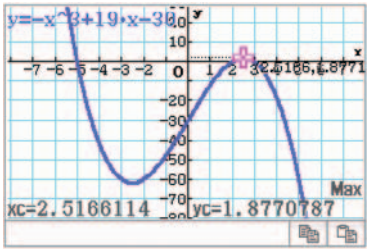
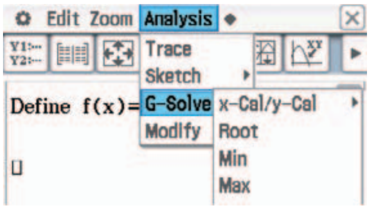
Using the Casio ClassPad

a
- In the Main $\sqrt{\alpha}$ screen, define the function f.
- Tap ⊞ to open the graph window.
- Highlight the function and drag into the graph window.
- To find the local minimum, select **Analysis** > **G-Solve** > **Min**.
- To find the local maximum, select **Analysis** > **G-Solve** > **Max**.

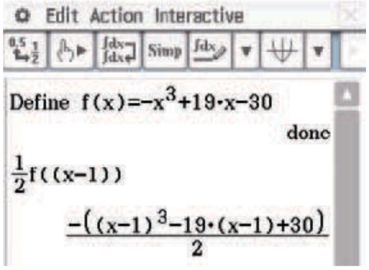

y=−x^3+19·x−30
(−2.517,−61.88)
Min
xc=−2.516611
yc=−61.87708


y=−x^3+19·x−30
(2.5166,1.8771)
Max
xc=2.5166114
yc=1.8770787


Edit Zoom Analysis
Trace
Sketch
G-Solve
Modify
x-Cal/y-Cal
Root
Min
Max

Define f(x)=

b
- Enter the transformed function as $\frac{1}{2}f(x - 1)$.
- Highlight the transformed function and drag into the graph window.
- The coordinates of the turning points can be found as above.

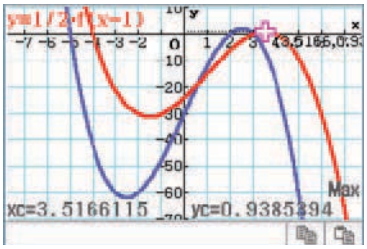

Edit Action Interactive

Define $f(x)=-x^3+19\cdot x-30$

done

$\frac{1}{2}f((x-1))$

$-\frac{((x-1)^3-19\cdot(x-1)+30)}{2}$


y=1/2·f(x−1)
(3.5166,0.93
Max
xc=3.5166115
yc=0.9385394

Section summary

- The graph of a cubic function can have one, two or three x-axis intercepts.
- The graph of a cubic function can have zero, one or two stationary points.
- To sketch a cubic in factorised form $y = a(x - \alpha)(x - \beta)(x - \gamma)$:
  - Find the y-axis intercept.
  - Find the x-axis intercepts.
  - Prepare a sign diagram.
  - Consider the y-values as x increases to the right of all x-axis intercepts.
  - Consider the y-values as x decreases to the left of all x-axis intercepts.
- If there is a repeated factor to the power 2, then the y-values have the same sign immediately to the left and right of the corresponding x-axis intercept.

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Exercise 4E

Example 28

1 Draw a sign diagram for each of the following expressions:

a $(3 - x)(x - 1)(x - 6)$
b $(3 + x)(x - 1)(x + 6)$
c $(x - 5)(x + 1)(2x - 6)$
d $(4 - x)(5 - x)(1 - 2x)$
e $(x - 5)^2(x - 4)$
f $(x - 5)^2(4 - x)$

2 First factorise and then draw a sign diagram for each of the following expressions:

a $x^3 - 4x^2 + x + 6$
b $4x^3 + 3x^2 - 16x - 12$
c $x^3 - 7x^2 + 4x + 12$
d $2x^3 + 3x^2 - 11x - 6$

Example 29

3 **a** Use a calculator to plot the graph of $y = f(x)$ where $f(x) = x^3 - 2x^2 + 1$.

b On the same screen, plot the graphs of:

i $y = f(x - 2)$ **ii** $y = f(x + 2)$ **iii** $y = 3f(x)$

4 **a** Use a calculator to plot the graph of $y = f(x)$ where $f(x) = x^3 + x^2 - 4x + 2$.

b On the same screen, plot the graphs of:

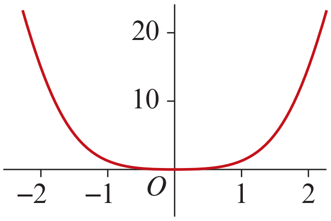
i $y = f(2x)$ **ii** $y = f\left(\frac{x}{2}\right)$ **iii** $y = 2f(x)$

4F Polynomials of higher degree

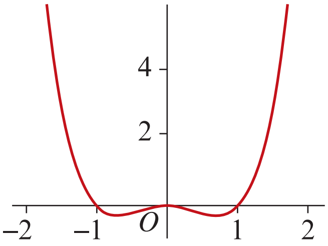
The general form for a quartic function is

$$f(x) = ax^4 + bx^3 + cx^2 + dx + e, \quad \text{where } a \neq 0$$

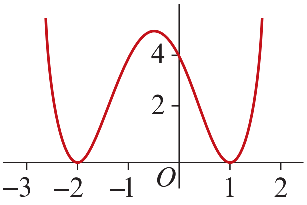
A gallery of quartic functions is shown below.



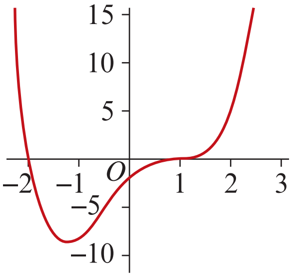
$f(x) = x^4$



$f(x) = x^4 - x^2$



$f(x) = (x - 1)^2(x + 2)^2$



$f(x) = (x - 1)^3(x + 2)$

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The techniques that have been developed for cubic functions may now be applied to quartic functions and to polynomial functions of higher degree in general.

For a polynomial $P(x)$ of degree n, there are at most n solutions to the equation $P(x) = 0$. Therefore the graph of $y = P(x)$ has at most n x-axis intercepts.

The graph of a polynomial of even degree may have no x-axis intercepts: for example, $P(x) = x^2 + 1$. But the graph of a polynomial of odd degree must have at least one x-axis intercept.

Example 30

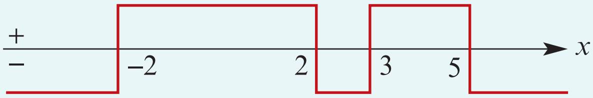
Draw a sign diagram for each quartic expression:

a $(2 - x)(x + 2)(x - 3)(x - 5)$

b $x^4 + x^2 - 2$

Solution

a



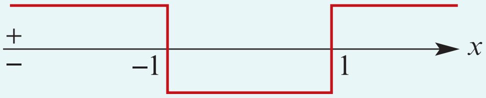
b Let $P(x) = x^4 + x^2 - 2$.

Then $P(1) = 1 + 1 - 2 = 0$.

Thus $x - 1$ is a factor.

$$
\begin{array}{rr}
 & x^3 + x^2 + 2x + 2 \\
\cline{2-2}
x - 1 \,\big) & x^4 + 0x^3 + x^2 + 0x - 2 \\
 & x^4 - x^3 \qquad\qquad\qquad \\
\cline{2-2}
 & x^3 + x^2 + 0x - 2 \\
 & x^3 - x^2 \qquad\quad \\
\cline{2-2}
 & 2x^2 + 0x - 2 \\
 & 2x^2 - 2x \qquad \\
\cline{2-2}
 & 2x - 2 \\
 & 2x - 2 \\
\cline{2-2}
 & 0
\end{array}
$$

$$
\begin{aligned}
\therefore \quad P(x) &= (x - 1)(x^3 + x^2 + 2x + 2) \\
&= (x - 1)[x^2(x + 1) + 2(x + 1)] \\
&= (x - 1)(x + 1)(x^2 + 2)
\end{aligned}
$$



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Example 31

For $p(x) = x^4 - 2x^2 + 1$, find the coordinates of the points where the graph of $y = p(x)$ intersects the x- and y-axes, and hence sketch the graph.

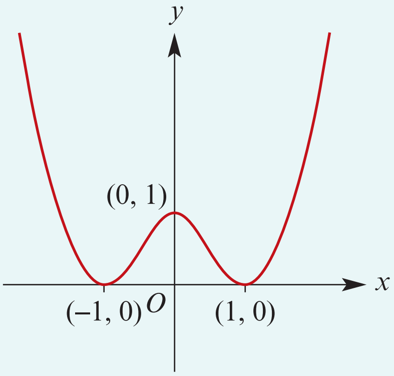
Solution

Note that

$$\begin{aligned} p(x) &= (x^2)^2 - 2(x^2) + 1 \\ &= (x^2 - 1)^2 \\ &= [(x - 1)(x + 1)]^2 \\ &= (x - 1)^2(x + 1)^2 \end{aligned}$$

Therefore the x-axis intercepts are 1 and -1.

When $x = 0$, $y = 1$. So the y-axis intercept is 1.



Explanation

Alternatively, we can factorise $p(x)$ by using the factor theorem and division.

Note that

$$p(1) = 1 - 2 + 1 = 0$$

Therefore $x - 1$ is a factor.

$$\begin{aligned} p(x) &= (x - 1)(x^3 + x^2 - x - 1) \\ &= (x - 1)[x^2(x + 1) - (x + 1)] \\ &= (x - 1)(x + 1)(x^2 - 1) \\ &= (x - 1)^2(x + 1)^2 \end{aligned}$$

Section summary

- The graph of a quartic function can have zero, one, two, three or four x-axis intercepts.
- The graph of a quartic function can have one, two or three stationary points.
- To sketch a quartic in factorised form $y = a(x - \alpha)(x - \beta)(x - \gamma)(x - \delta)$:
  - Find the y-axis intercept.
  - Find the x-axis intercepts.
  - Prepare a sign diagram.
  - Consider the y-values as x increases to the right of all x-axis intercepts.
  - Consider the y-values as x decreases to the left of all x-axis intercepts.
- If there is a repeated factor to an even power, then the y-values have the same sign immediately to the left and right of the corresponding x-axis intercept.

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Exercise 4F

Example 30

1 Draw a sign diagram for each quartic expression:

a $(3 - x)(x + 4)(x - 5)(x - 1)$ **b** $x^4 - 2x^3 - 3x^2 + 4x + 4$

Example 31

2 For $h(x) = 81x^4 - 72x^2 + 16$, find the coordinates of the points where the graph of $y = h(x)$ intersects the x- and y-axes, and hence sketch the graph.

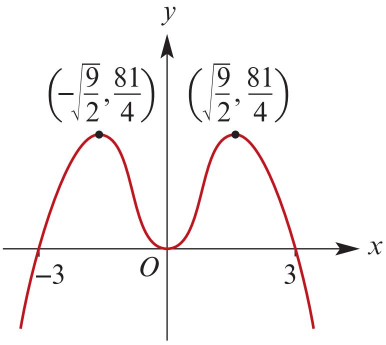
Hint: First express $h(x)$ as the square of a quadratic expression.

3 **a** Use a calculator to plot the graph of $y = f(x)$, where $f(x) = x^4 - 2x^3 + x + 1$.

b On the same screen, plot the graphs of:

i $y = f(x - 2)$ **ii** $y = f(2x)$ **iii** $y = f\left(\frac{x}{2}\right)$

4 The graph of $y = 9x^2 - x^4$ is as shown. Sketch the graph of each of the following by applying suitable transformations:



a $y = 9(x - 1)^2 - (x - 1)^4$

b $y = 18x^2 - 2x^4$

c $y = 18(x + 1)^2 - 2(x + 1)^4$

d $y = 9x^2 - x^4 - \frac{81}{4}$

e $y = 9x^2 - x^4 + 1$

(Do not find the x-axis intercepts for part **e**.)

5 Sketch the graph of $f(x) = x^6 - x^2$. (Use a calculator to find the stationary points.)

6 Sketch the graph of $f(x) = x^5 - x^3$. (Use a calculator to find the stationary points.)

4G Determining the rule for the graph of a polynomial

A straight line is determined by any two points on the line. More generally, the graph of a polynomial function of degree n is completely determined by any $n + 1$ points on the curve.

For example, for a cubic function with rule $y = f(x)$, if it is known that $f(a_1) = b_1$, $f(a_2) = b_2$, $f(a_3) = b_3$ and $f(a_4) = b_4$, then the rule can be determined.

Finding the rule for a parabola has been discussed in Section 4B.

The method for finding the rule from a graph of a cubic function will depend on what information is given in the graph.

If the cubic function has rule of the form $f(x) = a(x - h)^3 + k$ and the point of inflection (h, k) is given, then one other point needs to be known in order to find the value of a.

For those that are not of this form, the information given may be some or all of the x-axis intercepts as well as the coordinates of other points including possibly the y-axis intercept.

Example 32

a A cubic function has rule of the form $y = a(x - 2)^3 + 2$. The point $(3, 10)$ is on the graph of the function. Find the value of a.

b A cubic function has rule of the form $y = a(x - 1)(x + 2)(x - 4)$. The point $(5, 16)$ is on the graph of the function. Find the value of a.

c A cubic function has rule of the form $f(x) = ax^3 + bx$. The points $(1, 16)$ and $(2, 30)$ are on the graph of the function. Find the values of a and b.

Solution

a $y = a(x - 2)^3 + 2$

When $x = 3$, $y = 10$. Solve for a:

$$\begin{aligned} 10 &= a(3-2)^3 + 2 \\ 8 &= a \times 1^3 \\ \therefore \quad a &= 8 \end{aligned}$$

b $y = a(x - 1)(x + 2)(x - 4)$

When $x = 5$, $y = 16$ and so

$$\begin{aligned} 16 &= a(5-1)(5+2)(5-4) \\ 16 &= 28a \\ \therefore \quad a &= \frac{4}{7} \end{aligned}$$

c $f(x) = ax^3 + bx$

We know $f(1) = 16$ and $f(2) = 30$:

$$16 = a + b \qquad (1)$$

$$30 = a(2)^3 + 2b \qquad (2)$$

Multiply (1) by 2 and subtract from (2):

$$\begin{aligned} -2 &= 6a \\ \therefore \quad a &= -\frac{1}{3} \end{aligned}$$

Substitute in (1):

$$\begin{aligned} 16 &= -\frac{1}{3} + b \\ \therefore \quad b &= \frac{49}{3} \end{aligned}$$

Explanation

In each of these problems, we substitute the given values to find the unknowns.

The coordinates of the point of inflection of a graph which is a translation of $y = ax^3$ are known and the coordinates of one further point are known.

Three x-axis intercepts are known and the coordinates of a fourth point are known.

Form simultaneous equations in a and b.

Example 33

For the cubic function with rule $f(x) = ax^3 + bx^2 + cx + d$, it is known that the points with coordinates $(-1, -18)$, $(0, -5)$, $(1, -4)$ and $(2, -9)$ lie on the graph. Find the values of a, b, c and d.

Solution

The following equations can be formed:

$$
\begin{aligned}
-a + b - c + d &= -18 \qquad (1)\\
d &= -5 \qquad (2)\\
a + b + c + d &= -4 \qquad (3)\\
8a + 4b + 2c + d &= -9 \qquad (4)
\end{aligned}
$$

Adding (1) and (3) gives

$$2b + 2d = -22$$

Since $d = -5$, we obtain $b = -6$.

There are now only two unknowns.

Equations (3) and (4) become:

$$
\begin{aligned}
a + c &= 7 \qquad (3')\\
8a + 2c &= 20 \qquad (4')
\end{aligned}
$$

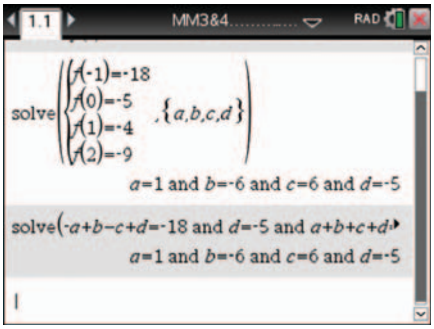
Multiply (3′) by 2 and subtract from (4′) to obtain

$$6a = 6$$

Thus $a = 1$ and $c = 6$.

Using the TI-Nspire

- Define $f(x) = ax^3 + bx^2 + cx + d$.
- Use the simultaneous equations template (menu > **Algebra** > **Solve System of Equations** > **Solve System of Equations**) to solve for a, b, c, d given that $f(-1) = -18$, $f(0) = -5$, $f(1) = -4$ and $f(2) = -9$.



$$\text{solve}\left(\begin{cases} f(-1)=-18\\ f(0)=-5\\ f(1)=-4\\ f(2)=-9\end{cases}, \{a,b,c,d\}\right)$$

```
                    a=1 and b=-6 and c=6 and d=-5
solve(-a+b-c+d=-18 and d=-5 and a+b+c+d▸
                    a=1 and b=-6 and c=6 and d=-5
```

- Alternatively, enter: solve($-a + b - c + d = -18$ and $d = -5$ and $a + b + c + d = -4$ and $8a + 4b + 2c + d = -9$, $\{a, b, c, d\}$). The word ‘and’ can be typed directly or found in the catalog (📖 1 A).

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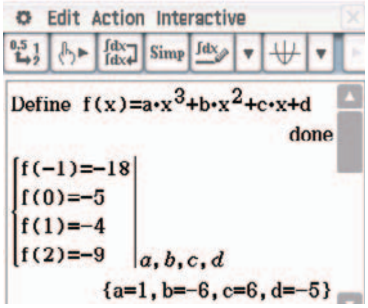
Using the Casio ClassPad

- Open the main screen and define the function $f(x) = ax^3 + bx^2 + cx + d$ using the Var keyboard.
- Tap the simultaneous equations icon {■ three times.
- Enter $f(-1) = -18$, $f(0) = -5$, $f(1) = -4$ and $f(2) = -9$ as the simultaneous equations to be solved, with variables a, b, c, d. Tap EXE.

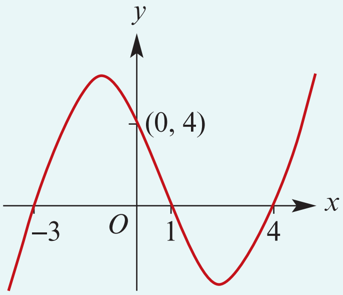
Note: The function name f must be selected from the abc keyboard.



```
Define f(x)=a·x³+b·x²+c·x+d
                                done
⎧f(-1)=-18
⎪f(0)=-5
⎨f(1)=-4
⎩f(2)=-9       |a,b,c,d
          {a=1,b=-6,c=6,d=-5}
```

Example 34

The graph shown is that of a cubic function. Find the rule for this cubic function.



Solution

From the graph, the function is of the form

$$y = a(x-4)(x-1)(x+3)$$

The point $(0, 4)$ is on the graph. Hence

$$4 = a(-4)(-1)3$$

$$\therefore \quad a = \frac{1}{3}$$

The rule is $y = \frac{1}{3}(x-4)(x-1)(x+3)$.

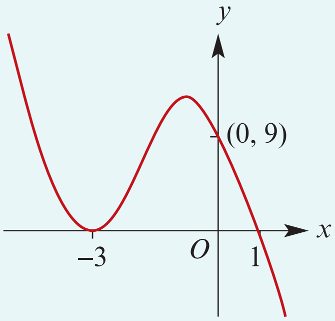
Explanation

The x-axis intercepts are -3, 1 and 4.

So $x+3$, $x-1$ and $x-4$ are linear factors.

Example 35

The graph shown is that of a cubic function. Find the rule for this cubic function.



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Solution

From the graph, the function is of the form

$$y = k(x-1)(x+3)^2$$

The point $(0, 9)$ is on the graph. Hence

$$9 = k(-1)(9)$$

$$\therefore \quad k = -1$$

The rule is $y = -(x-1)(x+3)^2$.

Explanation

The graph touches the x-axis at $x = -3$. Therefore $x + 3$ is a repeated factor.

Example 36

The graph of a cubic function passes through the points $(0, 1)$, $(1, 4)$, $(2, 17)$ and $(-1, 2)$. Find the rule for this cubic function.

Solution

The cubic function will have a rule of the form

$$y = ax^3 + bx^2 + cx + d$$

The values of a, b, c and d have to be determined.

As the point $(0, 1)$ is on the graph, we have $d = 1$.

By using the points $(1, 4)$, $(2, 17)$ and $(-1, 2)$, three simultaneous equations are produced:

$$\begin{aligned} 4 &= a + b + c + 1 \\ 17 &= 8a + 4b + 2c + 1 \\ 2 &= -a + b - c + 1 \end{aligned}$$

These become:

$$\begin{aligned} 3 &= a + b + c & (1) \\ 16 &= 8a + 4b + 2c & (2) \\ 1 &= -a + b - c & (3) \end{aligned}$$

Add (1) and (3):

$$\begin{aligned} 4 &= 2b \\ \therefore \quad b &= 2 \end{aligned}$$

Substitute in (1) and (2):

$$\begin{aligned} 1 &= a + c & (4) \\ 8 &= 8a + 2c & (5) \end{aligned}$$

Multiply (4) by 2 and subtract from (5):

$$\begin{aligned} 6 &= 6a \\ \therefore \quad a &= 1 \end{aligned}$$

From (4), we now have $c = 0$. Hence the rule is $y = x^3 + 2x^2 + 1$.

Exercise 4G

Skillsheet

Example 32

1 **a** A cubic function has rule of the form $y = a(x-5)^3 - 2$. The point $(4, 0)$ is on the graph of the function. Find the value of a.

b A cubic function has rule of the form $y = a(x-1)(x+1)(x+2)$. The point $(3, 120)$ is on the graph of the function. Find the value of a.

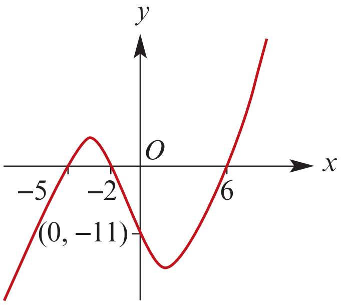
c A cubic function has rule of the form $f(x) = ax^3 + bx$. The points $(2, -20)$ and $(-1, 20)$ are on the graph of the function. Find the values of a and b.

Example 33

2 For the cubic function with rule $f(x) = ax^3 + bx^2 + cx + d$, it is known that the points with coordinates $(-1, 14)$, $(0, 5)$, $(1, 0)$ and $(2, -19)$ lie on the graph of the cubic. Find the values of a, b, c and d.

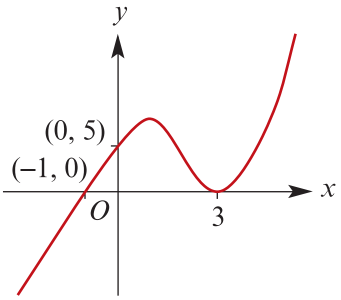
Example 34

3 Determine the rule for the cubic function with the graph shown below.



Example 35

4 Determine the rule for the cubic function with the graph shown below.



Example 36

5 Find the rule for the cubic function that passes through the following points:

a $(0, 1)$, $(1, 3)$, $(-1, -1)$ and $(2, 11)$

b $(0, 1)$, $(1, 1)$, $(-1, 1)$ and $(2, 7)$

c $(0, -2)$, $(1, 0)$, $(-1, -6)$ and $(2, 12)$

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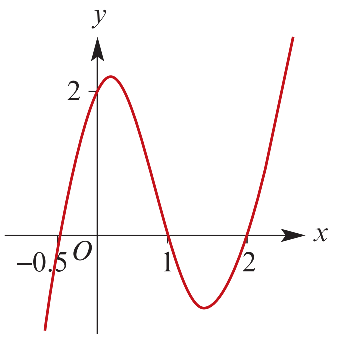
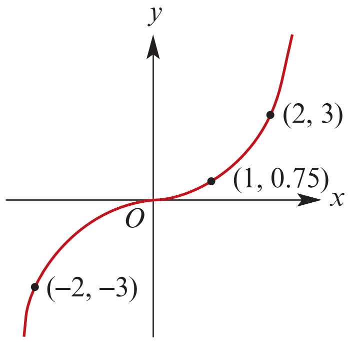
6 Find expressions which define the following cubic curves:

a

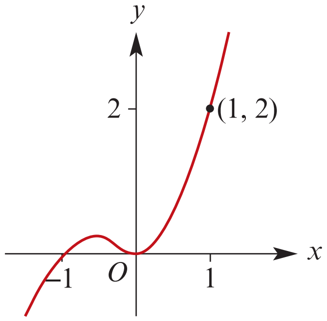


b

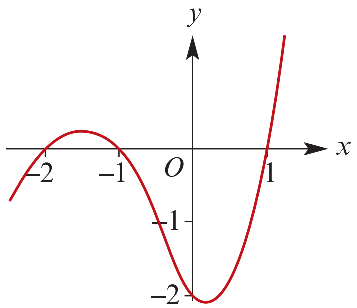


Note that (0, 0) is not a point of zero gradient.

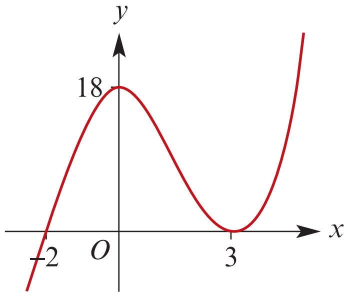
c



d



e



7 Find the rule of the cubic function for which the graph passes through the points with coordinates:

a $(0, 135), (1, 156), (2, 115), (3, 0)$

b $(-2, -203), (0, 13), (1, 25), (2, -11)$

8 Find the rule of the quartic function for which the graph passes through the points with coordinates:

a $(-1, 43), (0, 40), (2, 70), (6, 1618), (10, 670)$

b $(-3, 119), (-2, 32), (-1, 9), (0, 8), (1, 11)$

c $(-3, 6), (-1, 2), (1, 2), (3, 66), (6, 1227)$

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4H Solution of literal equations and systems of equations

Literal equations

We solved linear literal equations in Section 2B. We now look at non-linear equations. They certainly can be solved with a CAS calculator, but full setting out is shown here.

Example 37

Solve each of the following literal equations for x:

a $x^2 + kx + k = 0$

b $x^3 - 3ax^2 + 2a^2x = 0$

c $x(x^2 - a) = 0$, where $a > 0$

Solution

a The quadratic formula gives

$$x = \frac{-k \pm \sqrt{k^2 - 4k}}{2}$$

A real solution exists only for $k^2 - 4k \geq 0$, that is, for $k \geq 4$ or $k \leq 0$.

b

$$\begin{aligned} x^3 - 3ax^2 + 2a^2x &= 0 \\ x(x^2 - 3ax + 2a^2) &= 0 \\ x(x-a)(x-2a) &= 0 \end{aligned}$$

Hence $x = 0$ or $x = a$ or $x = 2a$.

c $x(x^2 - a) = 0$ implies $x = 0$ or $x = \sqrt{a}$ or $x = -\sqrt{a}$.

In the next example, we use the following two facts about power functions:

- If n is an odd natural number, then $b^n = a$ is equivalent to $b = a^{\frac{1}{n}}$.
- If n is an even natural number, then $b^n = a$ is equivalent to $b = \pm a^{\frac{1}{n}}$, where $a \geq 0$.

Note that care must be taken with even powers: for example, $x^2 = 2$ is equivalent to $x = \pm\sqrt{2}$.

Example 38

Solve each of the following equations for x:

a $ax^3 - b = c$

b $a(x+b)^3 = c$

c $x^4 = c$, where $c > 0$

d $ax^{\frac{1}{5}} = b$

e $x^5 - c = d$

Solution

a

$$\begin{aligned} ax^3 - b &= c \\ ax^3 &= b + c \\ x^3 &= \frac{b+c}{a} \\ \therefore \quad x &= \left(\frac{b+c}{a}\right)^{\frac{1}{3}} \end{aligned}$$

b

$$\begin{aligned} a(x+b)^3 &= c \\ (x+b)^3 &= \frac{c}{a} \\ x + b &= \left(\frac{c}{a}\right)^{\frac{1}{3}} \\ \therefore \quad x &= \left(\frac{c}{a}\right)^{\frac{1}{3}} - b \end{aligned}$$

c

$$x^4 = c$$

$$\therefore \quad x = \sqrt[4]{c} \quad \text{or} \quad x = -\sqrt[4]{c}$$

d

$$\begin{aligned} ax^{\frac{1}{5}} &= b \\ x^{\frac{1}{5}} &= \frac{b}{a} \\ \therefore \quad x &= \left(\frac{b}{a}\right)^5 \end{aligned}$$

e

$$\begin{aligned} x^5 - c &= d \\ x^5 &= c + d \\ \therefore \quad x &= (c+d)^{\frac{1}{5}} \end{aligned}$$

▶ Simultaneous equations

We now look at methods for finding the coordinates of the points of intersection of different graphs.

Example 39

Find the coordinates of the points of intersection of the parabola with equation $y = x^2 - 2x - 2$ and the straight line with equation $y = x + 4$.

Solution

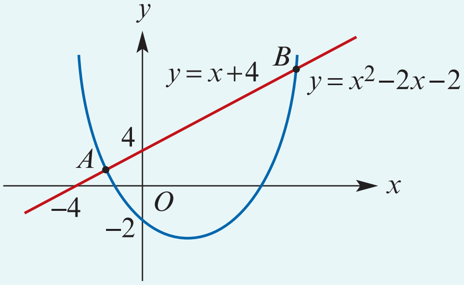
Equate the two expressions for y:

$$x^2 - 2x - 2 = x + 4$$

$$x^2 - 3x - 6 = 0$$

$$\therefore \quad x = \frac{3 \pm \sqrt{9 - 4 \times (-6)}}{2}$$

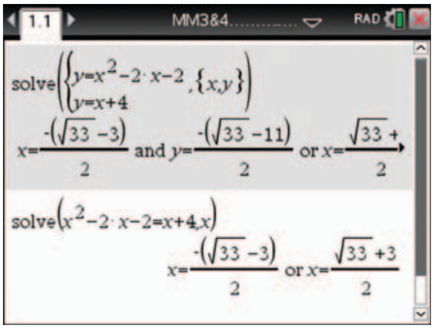
$$= \frac{3 \pm \sqrt{33}}{2}$$



The points of intersection are $A\left(\dfrac{3 - \sqrt{33}}{2}, \dfrac{11 - \sqrt{33}}{2}\right)$ and $B\left(\dfrac{3 + \sqrt{33}}{2}, \dfrac{11 + \sqrt{33}}{2}\right)$.

Using the TI-Nspire

- Use the simultaneous equations template (menu > **Algebra** > **Solve System of Equations** > **Solve System of Equations**) and complete as shown.
- Use the up arrow (▲) to move up to the answer and then use the right arrow (▶) to display the remaining part of the answer.
- Alternatively, equate the two expressions for y and solve for x as shown.



$$\text{solve}\left(\begin{cases} y=x^2-2\cdot x-2 \\ y=x+4 \end{cases}, \{x,y\}\right)$$

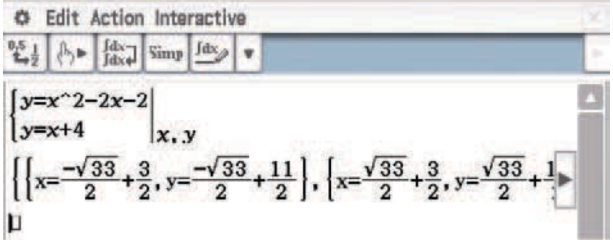
$$x=\frac{-(\sqrt{33}-3)}{2} \text{ and } y=\frac{-(\sqrt{33}-11)}{2} \text{ or } x=\frac{\sqrt{33}+}{2}$$

$$\text{solve}(x^2-2\cdot x-2=x+4,x)$$

$$x=\frac{-(\sqrt{33}-3)}{2} \text{ or } x=\frac{\sqrt{33}+3}{2}$$

Using the Casio ClassPad

- Select the simultaneous equations template {▬.
- Enter the equations $y = x^2 - 2x - 2$ and $y = x + 4$. Set the variables as x, y.



$$\begin{cases} y=x\^{}2-2x-2 \\ y=x+4 \end{cases}\Bigg|_{x,y}$$

$$\left\{\left\{x=\frac{-\sqrt{33}}{2}+\frac{3}{2}, y=\frac{-\sqrt{33}}{2}+\frac{11}{2}\right\}, \left\{x=\frac{\sqrt{33}}{2}+\frac{3}{2}, y=\frac{\sqrt{33}}{2}+\frac{1}{}\right.\right.$$

- Tap Rotate to rotate the screen, and tap the right-arrow button (▶) to view the solutions.

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Example 40

Find the points of intersection of the circle with equation $(x-4)^2+y^2=16$ and the line with equation $x-y=0$.

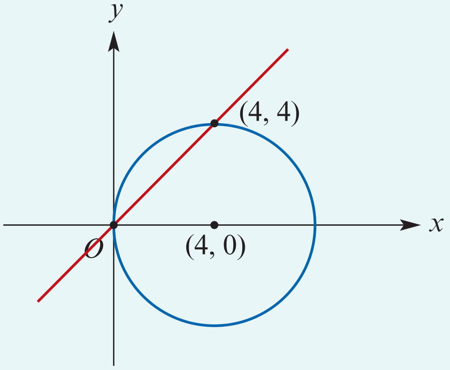
Solution

Rearrange $x-y=0$ to make y the subject.

Substitute $y=x$ into the equation of the circle:

$$\begin{aligned}(x-4)^2+x^2&=16\\ x^2-8x+16+x^2&=16\\ 2x^2-8x&=0\\ 2x(x-4)&=0\\ \therefore\quad x=0 \text{ or } x&=4\end{aligned}$$

The points of intersection are $(0,0)$ and $(4,4)$.



Example 41

Find the point of contact of the line with equation $\frac{1}{9}x+y=\frac{2}{3}$ and the curve with equation $xy=1$.

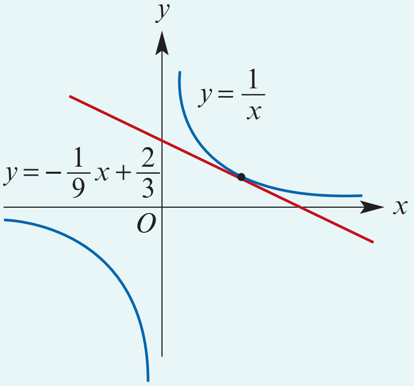
Solution

Rewrite the equations as $y=-\frac{1}{9}x+\frac{2}{3}$ and $y=\frac{1}{x}$.

Equate the expressions for y:

$$\begin{aligned}-\frac{1}{9}x+\frac{2}{3}&=\frac{1}{x}\\ -x^2+6x&=9\\ x^2-6x+9&=0\\ (x-3)^2&=0\\ \therefore\quad x&=3\end{aligned}$$

The point of intersection is $(3,\frac{1}{3})$.

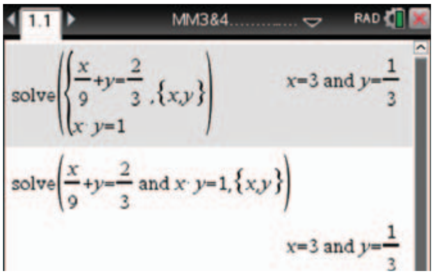


Using the TI-Nspire

Two methods are shown:

- Use the simultaneous equations template (menu > **Algebra** > **Solve System of Equations** > **Solve System of Equations**).
- Alternatively, use menu > **Algebra** > **Solve**.

Note: The multiplication sign is required between x and y.



$\text{solve}\left(\begin{cases}\frac{x}{9}+y=\frac{2}{3}\\ x\cdot y=1\end{cases},\{x,y\}\right)$ $\quad x=3 \text{ and } y=\frac{1}{3}$

$\text{solve}\left(\frac{x}{9}+y=\frac{2}{3} \text{ and } x\cdot y=1,\{x,y\}\right)$

$x=3 \text{ and } y=\frac{1}{3}$

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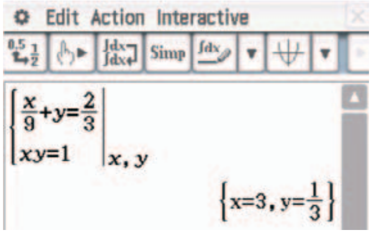
Using the Casio ClassPad

- Select the simultaneous equations template {⊟.
- Enter the equations $\frac{x}{9} + y = \frac{2}{3}$ and $xy = 1$ and set the variables as x, y.

Note: Tap ⊟ to enter the fractions.



$$\begin{cases} \frac{x}{9}+y=\frac{2}{3} \\ xy=1 \end{cases}\Bigg|_{x,\,y}$$

$$\left\{x=3,\ y=\frac{1}{3}\right\}$$

Exercise 4H

Example 37 **1** Solve each of the following literal equations for x:

a $kx^2 + x + k = 0$ **b** $x^3 - 7ax^2 + 12a^2x = 0$ **c** $x(x^3 - a) = 0$

d $x^2 - kx + k = 0$ **e** $x^3 - ax = 0$ **f** $x^4 - a^4 = 0$

g $(x - a)^5(x - b) = 0$ **h** $(a - x)^4(a - x^3)(x^2 - a) = 0$

Example 38 **2** Solve each of the following equations for x:

a $ax^3 + b = 2c$ **b** $ax^2 - b = c$, where $a, b, c > 0$

c $a - bx^2 = c$, where $a > c$ and $b > 0$ **d** $x^{\frac{1}{3}} = a$

e $x^{\frac{1}{n}} + c = a$, where $n \in \mathbb{N}$ and $a > c$ **f** $a(x - 2b)^3 = c$

g $ax^{\frac{1}{3}} = b$ **h** $x^3 - c = d$

Example 39 **3** Find the coordinates of the points of intersection for each of the following:

a $y = x^2$, $y = x$ **b** $y - 2x^2 = 0$, $y - x = 0$ **c** $y = x^2 - x$, $y = 2x + 1$

Example 40 **4** Find the coordinates of the points of intersection for each of the following:

a $x^2 + y^2 = 178$, $x + y = 16$ **b** $x^2 + y^2 = 125$, $x + y = 15$ **c** $x^2 + y^2 = 185$, $x - y = 3$

d $x^2 + y^2 = 97$, $x + y = 13$ **e** $x^2 + y^2 = 106$, $x - y = 4$

Example 41 **5** Find the coordinates of the points of intersection for each of the following:

a $x + y = 28$, $xy = 187$ **b** $x + y = 51$, $xy = 518$ **c** $x - y = 5$, $xy = 126$

6 Find the coordinates of the points of intersection of the straight line with equation $y = 2x$ and the circle with equation $(x - 5)^2 + y^2 = 25$.

7 Find the coordinates of the points of intersection of the curves with equations $y = \frac{1}{x - 2} + 3$ and $y = x$.

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8 Find the coordinates of the points of intersection of the line with equation $\frac{y}{4} - \frac{x}{5} = 1$ and the circle with equation $x^2 + 4x + y^2 = 12$.

9 Find the coordinates of the points of intersection of the curve $y = \frac{1}{x+2} - 3$ and the line $y = -x$.

10 Find the coordinates of the point where the line with equation $4y = 9x + 4$ touches the parabola with equation $y^2 = 9x$.

11 Find the coordinates of the point where the line with equation $y = 2x + 3\sqrt{5}$ touches the circle $x^2 + y^2 = 9$.

12 Find the coordinates of the point where the straight line with equation $y = \frac{1}{4}x + 1$ touches the curve with equation $y = -\frac{1}{x}$.

13 Find the coordinates of the points of intersection of the curve with equation $y = \frac{2}{x-2}$ and the line $y = x - 1$.

14 Solve the simultaneous equations:

a $5x - 4y = 7$ and $xy = 6$

b $2x + 3y = 37$ and $xy = 45$

c $5x - 3y = 18$ and $xy = 24$

15 What is the condition for $x^2 + ax + b$ to be divisible by $x + c$?

16 Solve the simultaneous equations $y = x + 2$ and $y = \frac{160}{x}$.

17 Find the equations of the lines that pass through the point $(1, 7)$ and touch the parabola $y = -3x^2 + 5x + 2$.

Hint: Form a quadratic equation and consider when the discriminant Δ is zero.

18 Find the values of m for which the line $y = mx - 8$ intersects the parabola $y = x^2 - 5x + m$ twice.

19 The line $y = x + c$ meets the hyperbola $y = \frac{9}{2-x}$ once. Find the possible values of c.

20 **a** Solve the simultaneous equations $y = mx$ and $y = \frac{1}{x} + 5$ for x in terms of m.

b Find the value of m for which the graphs of $y = mx$ and $y = \frac{1}{x} + 5$ touch, and give the coordinates of this point.

c For which values of m do the graphs not meet?

21 Show that, if the line with equation $y = kx + b$ touches the curve $y = x^2 + x + 4$, then $k^2 - 2k + 4b - 15 = 0$. Hence find the equations of such lines that also pass through the point $(0, 3)$.

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Chapter summary

Spreadsheet
AS
Nrich

Quadratic polynomials

- Turning point form
  - By completing the square, all quadratic functions in polynomial form $y = ax^2 + bx + c$ may be transposed into turning point form $y = a(x - h)^2 + k$.
  - The graph of $y = a(x - h)^2 + k$ is a parabola congruent to the graph of $y = ax^2$. The vertex (or turning point) is the point (h, k). The axis of symmetry is $x = h$.
- Axis of symmetry

  The axis of symmetry of the graph of the quadratic function $y = ax^2 + bx + c$ is the line with equation $x = -\dfrac{b}{2a}$.
- Quadratic formula

  The solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are given by the **quadratic formula**

  $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

  From the formula it can be seen that:
  - If $b^2 - 4ac > 0$, there are two solutions.
  - If $b^2 - 4ac = 0$, there is one solution.
  - If $b^2 - 4ac < 0$, there are no real solutions.

  The quantity $\Delta = b^2 - 4ac$ is called the **discriminant** of the quadratic $ax^2 + bx + c$.

Polynomials in general

- A **polynomial function** is a function that can be written in the form

  $$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

  where $n \in \mathbb{N} \cup \{0\}$ and the coefficients $a_0, \dots, a_n$ are real numbers with $a_n \neq 0$.
  The **leading term** is $a_n x^n$ (the term of highest index) and the **constant term** is a_0 (the term not involving x).
- The **degree of a polynomial** is the index n of the leading term.
  - Polynomials of degree 1 are called **linear** functions.
  - Polynomials of degree 2 are called **quadratic** functions.
  - Polynomials of degree 3 are called **cubic** functions.
  - Polynomials of degree 4 are called **quartic** functions.
- The sum, difference and product of two polynomials is a polynomial. Division does not always lead to another polynomial.
- Two polynomials P and Q are equal only if their corresponding coefficients are equal. Two cubic polynomials, $P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ and $Q(x) = b_3x^3 + b_2x^2 + b_1x + b_0$, are equal if and only if $a_3 = b_3$, $a_2 = b_2$, $a_1 = b_1$ and $a_0 = b_0$.

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- Division of polynomials

  When we divide the polynomial $P(x)$ by the polynomial $D(x)$ we obtain two polynomials, $Q(x)$ the **quotient** and $R(x)$ the **remainder**, such that

  $$P(x) = D(x)Q(x) + R(x)$$

  and either $R(x) = 0$ or $R(x)$ has degree less than $D(x)$.

  Two methods for dividing polynomials are **long division** and **equating coefficients**.

- Remainder theorem

  When $P(x)$ is divided by $\beta x + \alpha$, the remainder is $P\left(-\frac{\alpha}{\beta}\right)$.

- Factor theorem
  - If $\beta x + \alpha$ is a factor of $P(x)$, then $P\left(-\frac{\alpha}{\beta}\right) = 0$.
  - Conversely, if $P\left(-\frac{\alpha}{\beta}\right) = 0$, then $\beta x + \alpha$ is a factor of $P(x)$.

- A cubic polynomial can be factorised by using the factor theorem to find the first linear factor and then using polynomial division or the method of equating coefficients to complete the factorisation.

- Rational-root theorem

  Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of degree n with all the coefficients a_i integers. Let α and β be integers such that the highest common factor of α and β is 1 (i.e. α and β are relatively prime). If $\beta x + \alpha$ is a factor of $P(x)$, then β divides a_n and α divides a_0.

- Difference and sum of two cubes
  - $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$
  - $x^3 + a^3 = (x + a)(x^2 - ax + a^2)$

Technology-free questions

1 Sketch the graph of each of the following quadratic functions. Clearly indicate coordinates of the vertex and the axis intercepts.

a $h(x) = 3(x - 1)^2 + 2$ **b** $h(x) = (x - 1)^2 - 9$ **c** $f(x) = x^2 - x + 6$

d $f(x) = x^2 - x - 6$ **e** $f(x) = 2x^2 - x + 5$ **f** $h(x) = 2x^2 - x - 1$

2 The points with coordinates (1, 1) and (2, 5) lie on a parabola with equation of the form $y = ax^2 + b$. Find the values of a and b.

3 Solve the equation $3x^2 - 2x - 10 = 0$ by using the quadratic formula.

4 Sketch the graph of each of the following. State the coordinates of the point of zero gradient and the axis intercepts.

a $f(x) = 2(x - 1)^3 - 16$ **b** $g(x) = -(x + 1)^3 + 8$ **c** $h(x) = -(x + 2)^3 - 1$

d $f(x) = (x + 3)^3 - 1$ **e** $f(x) = 1 - (2x - 1)^3$

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5 Express each of the following in turning point form:

a $x^2 + 4x$ **b** $3x^2 + 6x$ **c** $x^2 - 4x + 6$

d $2x^2 - 6x - 4$ **e** $2x^2 - 7x - 4$ **f** $-x^2 + 3x - 4$

6 Draw a sign diagram for each of the following:

a $y = (x + 2)(2 - x)(x + 1)$ **b** $y = (x - 3)(x + 1)(x - 1)$

c $y = x^3 + 7x^2 + 14x + 8$ **d** $y = 3x^3 + 10x^2 + x - 6$

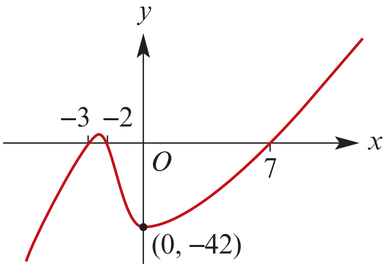
7 Without actually dividing, find the remainder when the first polynomial is divided by the second:

a $x^3 + 3x^2 - 4x + 2,\ \ x + 1$

b $x^3 - 3x^2 - x + 6,\ \ x - 2$

c $2x^3 + 3x^2 - 3x - 2,\ \ x + 2$

8 Determine the rule for the cubic function shown in the graph.



9 Factorise each of the following:

a $x^3 + 2x^2 - 5x - 6$ **b** $x^3 - 3x^2 - x + 3$

c $x^4 - x^3 - 7x^2 + x + 6$ **d** $x^3 + 2x^2 - 4x + 1$

10 Find the quotient and remainder when $x^2 + 4$ is divided by $x^2 - 2x + 2$.

11 Find the value of a for which $x - 2$ is a factor of $3x^3 + ax^2 + x - 2$.

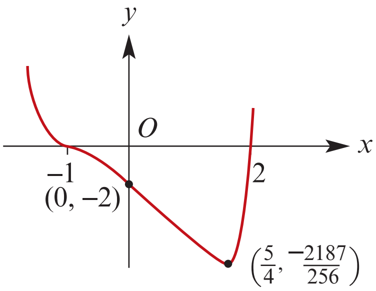
12 The graph of $f(x) = (x + 1)^3(x - 2)$ is shown. Sketch the graph of:

a $y = f(x - 1)$

b $y = f(x + 1)$

c $y = f(2x)$

d $y = f(x) + 2$



13 For what value of k is $2x^2 - kx + 8$ a perfect square?

14 Find the coordinates of the points of intersection of the graph of $y = 2x + 3$ with the graph of $y = x^2 + 3x - 9$.

15 Find constants a, b and c such that $3x^2 - 5x + 1 = a(x + b)^2 + c$ holds for all values of x.

16 Expand $(3 + 4x)^3$.

17 Given that $x^3 - 2x^2 + 5 = ax(x - 1)^2 + b(x - 1) + c$ for all real numbers x, find the values of a, b and c.

18 Find the values of p for which the equation $4x^2 - 2px + p + 3 = 0$ has no real solutions.

19 Find the rule for the cubic function, the graph of which passes through the points $(1, 1)$, $(2, 4)$, $(3, 9)$ and $(0, 6)$.

Multiple-choice questions

1 By completing the square, the expression $5x^2 - 10x - 2$ can be written in turning point form $a(x - h)^2 + k$ as

A $(5x + 1)^2 + 5$ **B** $(5x - 1)^2 - 5$ **C** $5(x - 1)^2 - 5$
D $5(x + 1)^2 - 2$ **E** $5(x - 1)^2 - 7$

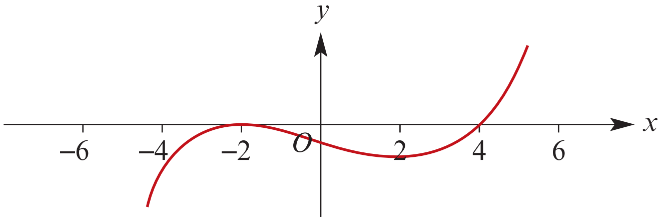
2 For which value(s) of m does the equation $mx^2 + 6x - 3 = 0$ have two real solutions?

A $m = -3$ **B** $m = 3$ **C** $m = 0$ **D** $m > -3$ **E** $m < -3$

3 $x^3 + 27$ is equal to

A $(x + 3)^3$ **B** $(x - 3)^3$ **C** $(x + 3)(x^2 - 6x + 9)$
D $(x - 3)(x^2 + 3x + 9)$ **E** $(x + 3)(x^2 - 3x + 9)$

4 The equation of the graph shown on the right is



A $y = x(x - 2)(x + 4)$
B $y = x(x + 2)(x - 4)$
C $y = (x + 2)^2(x - 4)$
D $y = (x + 2)(x - 4)^2$
E $y = (x + 2)^2(x - 4)^2$

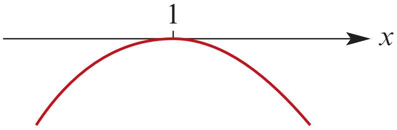
5 If $x - 1$ is a factor of $x^3 + 3x^2 - 2ax + 1$, then the value of a is

A 2 **B** 5 **C** $\frac{2}{5}$ **D** $-\frac{2}{5}$ **E** $\frac{5}{2}$

6 $6x^2 - 8xy - 8y^2$ is equal to

A $(3x + 2y)(2x - 4y)$ **B** $(3x - 2y)(6x + 4y)$ **C** $(6x - 4y)(x + 2y)$
D $(3x - 2y)(2x + 4y)$ **E** $(6x + y)(x - 8y)$

7 The diagram shows a part of the graph of a cubic polynomial function f, near the point $(1, 0)$. Which of the following could be the rule for f?



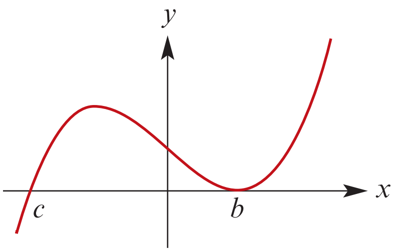
A $f(x) = x^2(x - 1)$ **B** $f(x) = (x - 1)^3$ **C** $f(x) = -x(x - 1)^2$

D $f(x) = x(x - 1)^2$ **E** $f(x) = -x(x + 1)^2$

8 The coordinates of the turning point of the graph of the function $p(x) = 3((x - 2)^2 + 4)$ are

A $(-2, 12)$ **B** $(-2, 4)$ **C** $(2, -12)$ **D** $(2, 4)$ **E** $(2, 12)$

9 The diagram shows part of the graph of a polynomial function. A possible equation for the graph is



A $y = (x + c)(x - b)^2$

B $y = (x - b)(x - c)^2$

C $y = (x - c)(b - x)^2$

D $y = -(x - c)(b - x)^2$

E $y = (x + b)^2(x - c)$

10 The number of solutions of the equation $(x^2 + a)(x - b)(x + c) = 0$, where $a, b, c \in \mathbb{R}^+$, is

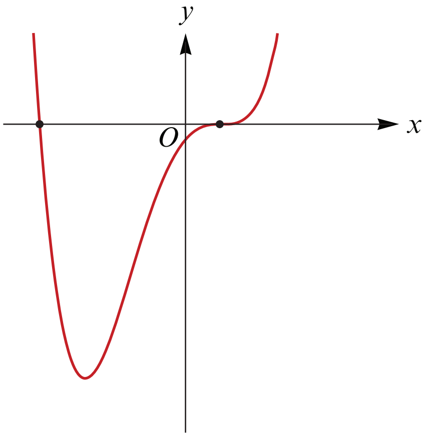
A 0 **B** 1 **C** 2 **D** 3 **E** 4

11 The graph of $y = kx - 3$ meets the graph of $y = -x^2 + 2x - 12$ at two distinct points for

A $k \in [-4, 8]$ **B** $k \in \{-4, -8\}$ **C** $k \in (-\infty, -4) \cup (8, \infty)$

D $k \in (-4, 8)$ **E** $k \in (-\infty, -8) \cup (4, \infty)$

12 The function f is a quartic polynomial. Its graph is shown on the right. It has x-axis intercepts at $(a, 0)$ and $(b, 0)$, where $a > 0$ and $b < 0$. A possible rule for this function is



A $f(x) = (x - a)^2(x + b)^2$

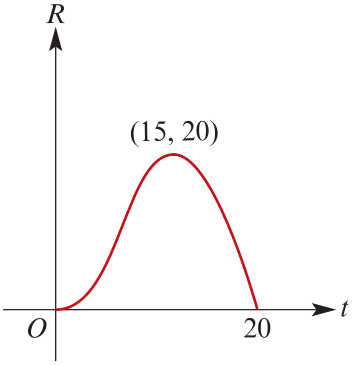
B $f(x) = (x - a)^3(x - b)$

C $f(x) = (x - a)(x - b)^2$

D $f(x) = (x + a)^2(x - b)^2$

E $f(x) = (x - b)^3(x - a)$

Extended-response questions



1 The rate of flow of water, R mL/min, into a vessel is described by the quartic expression

$$R = kt^3(20 - t), \quad \text{for } 0 \le t \le 20$$

where t minutes is the time elapsed from the beginning of the flow. The graph is shown.

a Find the value of k.

b Find the rate of flow when $t = 10$.

c The flow is adjusted so that the new expression for the flow is

$$R_{\text{new}} = 2kt^3(20 - t), \quad \text{for } 0 \le t \le 20$$

i Sketch the graph of R_{new} against t for $0 \le t \le 20$.

ii Find the rate of flow when $t = 10$.

d Water is allowed to run from the vessel and it is found that the rate of flow from the vessel is given by

$$R_{\text{out}} = -k(t - 20)^3(40 - t), \quad \text{for } 20 \le t \le 40$$

i Sketch the graph of R_{out} against t for $20 \le t \le 40$.

ii Find the rate of flow when $t = 30$.

Hints: The graph of R_{new} against t is given by a dilation of factor 2 from the x-axis. The graph of R_{out} against t is given by the translation with rule $(t, R) \to (t + 20, R)$ followed by a reflection in the t-axis.

2 A large gas container is being deflated. The volume V (in m^3) at time t hours is given by

$$V = 4(9 - t)^3, \quad \text{for } 0 \le t \le 9$$

a Find the volume when:

i $t = 0$ **ii** $t = 9$

b Sketch the graph of V against t for $0 \le t \le 9$.

c At what time is the volume 512 m^3?

3 A hemispherical bowl of radius 6 cm contains water. The volume of water in the hemispherical bowl, where the depth of the water is x cm, is given by

$$V = \frac{1}{3}\pi x^2(18 - x) \text{ cm}^3$$

a Find the volume of water when:

i $x = 2$ **ii** $x = 3$ **iii** $x = 4$

b Find the volume when the hemispherical bowl is full.

c Sketch the graph of V against x.

d Find the depth of water when the volume is equal to $\dfrac{325\pi}{3}$ cm^3.

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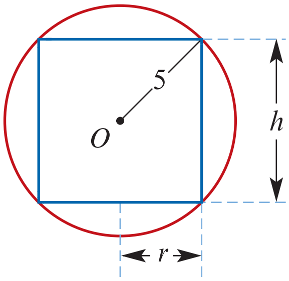
4 A metal worker is required to cut a circular cylinder from a solid sphere of radius 5 cm. A cross-section of the sphere and the cylinder is shown in the diagram.



a Express r in terms of h, where r cm is the radius of the cylinder and h cm is the height of the cylinder. Hence show that the volume, V cm^3, of the cylinder is given by $V = \frac{1}{4}\pi h(100 - h^2)$.

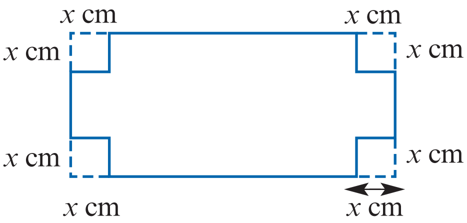
b Sketch the graph of V against h for $0 < h < 10$.

Hint: The coordinates of the maximum point are approximately $(5.77, 302.3)$.

c Find the volume of the cylinder if $h = 6$.

d Find the height and radius of the cylinder if the volume of the cylinder is 48π cm^3.

5 An open tank is to be made from a sheet of metal 84 cm by 40 cm by cutting congruent squares of side length x cm from each of the corners.



a Find the volume, V cm^3, of the box in terms of x.

b State the maximal domain for V when it is considered as a function of x.

c Plot the graph of V against x using a calculator.

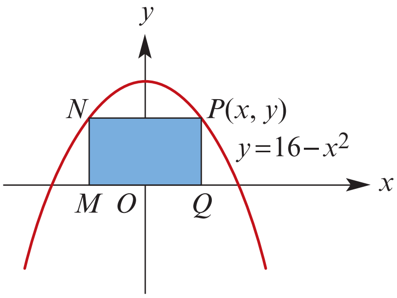
d Find the volume of the tank when:

i $x = 2$ **ii** $x = 6$ **iii** $x = 8$ **iv** $x = 10$

e Find the value(s) of x, correct to two decimal places, for which the capacity of the tank is 10 litres.

f Find, correct to two decimal places, the maximum capacity of the tank in cubic centimetres.

6 A rectangle is defined by vertices N and $P(x, y)$ on the curve with equation $y = 16 - x^2$ and vertices M and Q on the x-axis.



a **i** Find the area, A, of the rectangle in terms of x.

ii State the implied domain for the function defined by the rule given in part **i**.

b **i** Find the value of A when $x = 3$.

ii Find the value of x, correct to two decimal places, when $A = 25$.

c A cuboid has volume V given by the rule $V = xA$.

i Find V in terms of x.

ii Find the value of x, correct to two decimal places, such that $V = 100$.

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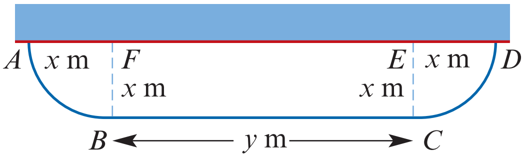
7 The plan of a garden adjoining a wall is shown. The rectangle $BCEF$ is of length y m and width x m. The borders of the two end sections are quarter circles of radius x m and centres at E and F.



A fence is erected along the curves AB and CD and the straight line BC.

a Find the area, A m^2, of the garden in terms of x and y.

b If the length of the fence is 100 m, find:

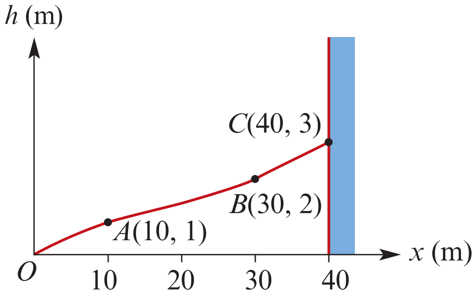
- **i** y in terms of x
- **ii** A in terms of x
- **iii** the maximal domain of the function with the rule obtained in part **ii**.

c Find, correct to two decimal places, the value(s) of x if the area of the garden is to be 1000 m^2.

d It is decided to build the garden up to a height of $\frac{x}{50}$ metres. If the length of the fence is 100 m, find correct to two decimal places:

- **i** the volume, V m^3, of soil needed in terms of x
- **ii** the volume of soil needed for a garden of area 1000 m^2
- **iii** the value(s) of x for which 500 m^3 of soil is required.

8 A mound of earth is piled up against a wall. The cross-section is as shown. The coordinates of several points on the surface are given.



a Find the rule of the cubic function for which the graph passes through the points O, A, B and C.

b For what value of x is the height of the mound 1.5 metres?

c The coefficient of x^3 for the function is 'small'. Consider the quadratic formed when the x^3 term is deleted. Compare the graph of the resulting quadratic function with the graph of the cubic function.

d The mound moves and the curve describing the cross-section now passes through the points $O(0, 0)$, $A(10, 0.3)$, $B(30, 2.7)$ and $D(40, 2.8)$. Find the rule of the cubic function for which the graph passes through these points.

e Let $y = f(x)$ be the function obtained in part **a**.

- **i** Sketch the graph of the piecewise-defined function

$$g(x) = \begin{cases} f(x) & \text{for } 0 \le x \le 40 \\ f(80 - x) & \text{for } 40 < x \le 80 \end{cases}$$

- **ii** Comment on the appearance of the graph of $y = g(x)$.

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5

Exponential and logarithmic functions

Objectives

- To graph **exponential** and **logarithmic functions** and transformations of these functions.
- To introduce **Euler's number** *e*.
- To revise the **index** and **logarithm laws**.
- To solve **exponential** and **logarithmic equations**.
- To find rules for the graphs of exponential and logarithmic functions.
- To find inverses of exponential and logarithmic functions.
- To apply exponential functions in modelling growth and decay.

Our work on functions is continued in this chapter. Many of the concepts introduced in Chapters 1 and 3 – domain, range, transformations and inverse functions – are used in the context of exponential and logarithmic functions.

An **exponential function** has a rule of the form $f(x) = ka^x$, where k is a non-zero constant and the base a is a positive real number other than 1.

Exponential functions were introduced in Mathematical Methods Units 1 & 2 and it was shown that there are practical situations where these functions can be applied, including radioactive decay and population growth. Some of these applications are further investigated in this chapter.

We also introduce the exponential function $f(x) = e^x$, which has many interesting properties. In particular, this function is its own derivative. That is, $f'(x) = f(x)$.

Here we define the number e as

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

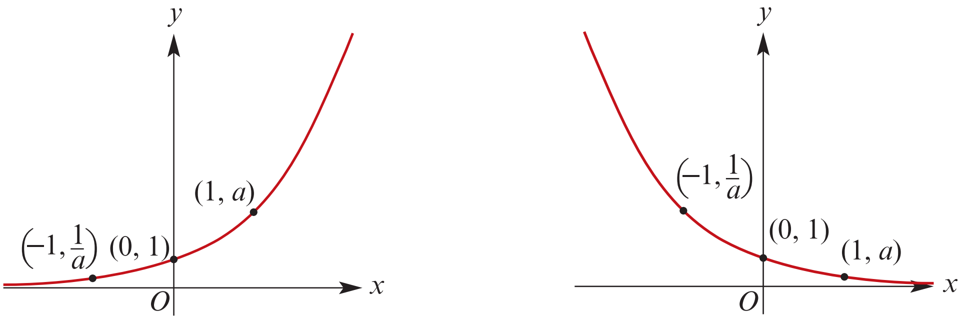
We will show that limits such as this arise in the consideration of compound interest.

5A Exponential functions

The function $f(x) = a^x$, where $a \in \mathbb{R}^+ \setminus \{1\}$, is an **exponential function**. The shape of the graph depends on whether $a > 1$ or $0 < a < 1$.

Graph of $f(x) = a^x$ for $a > 1$

Graph of $f(x) = a^x$ for $0 < a < 1$



- Key values are $f(-1) = \frac{1}{a}$, $f(0) = 1$ and $f(1) = a$.
- The maximal domain is $\mathbb{R}$ and the range is $\mathbb{R}^+$.
- The x-axis is a horizontal asymptote.

An exponential function with $a > 1$ is strictly increasing, and an exponential function with $0 < a < 1$ is strictly decreasing. In both cases, the function is one-to-one.

Graphing transformations of $f(x) = a^x$

Translations

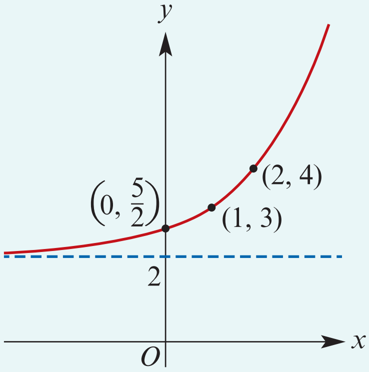
If the translation $(x, y) \to (x + h, y + k)$ is applied to the graph of $y = a^x$, then the image has equation $y = a^{x-h} + k$.

- The horizontal asymptote of the image has equation $y = k$.
- The range of the image is (k, ∞).

Example 1

Sketch the graph and state the range of $y = 2^{x-1} + 2$.

Solution



The range of the function is $(2, \infty)$.

Explanation

The graph of $y = 2^x$ is translated 1 unit in the positive direction of the x-axis and 2 units in the positive direction of the y-axis.

The mapping is $(x, y) \to (x + 1, y + 2)$.

Translation of key points:

- $(-1, \frac{1}{2}) \to (0, \frac{5}{2})$
- $(0, 1) \to (1, 3)$
- $(1, 2) \to (2, 4)$

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Reflections

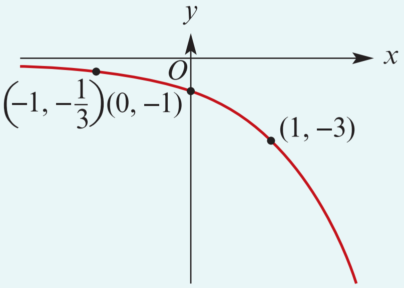
If a **reflection in the x-axis**, given by the mapping $(x, y) \to (x, -y)$, is applied to the graph of $y = a^x$, then the image has equation $y = -a^x$.

- The horizontal asymptote of the image has equation $y = 0$.
- The range of the image is $(-\infty, 0)$.

Example 2

Sketch the graph of $y = -3^x$.

Solution



Explanation

The graph of $y = 3^x$ is reflected in the x-axis.

The mapping is $(x, y) \to (x, -y)$.

Reflection of key points:

- $(-1, \frac{1}{3}) \to (-1, -\frac{1}{3})$
- $(0, 1) \to (0, -1)$
- $(1, 3) \to (1, -3)$

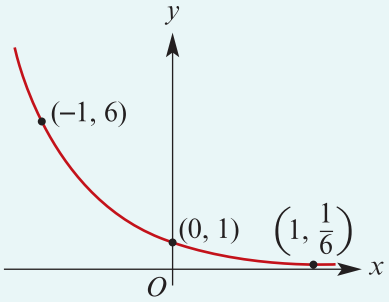
If a **reflection in the y-axis**, given by the mapping $(x, y) \to (-x, y)$, is applied to the graph of $y = a^x$, then the image has equation $y = a^{-x}$. This can also be written as $y = \dfrac{1}{a^x}$ or $y = \left(\dfrac{1}{a}\right)^x$.

- The horizontal asymptote of the image has equation $y = 0$.
- The range of the image is $(0, \infty)$.

Example 3

Sketch the graph of $y = 6^{-x}$.

Solution



Explanation

The graph of $y = 6^x$ is reflected in the y-axis.

The mapping is $(x, y) \to (-x, y)$.

Reflection of key points:

- $(-1, \frac{1}{6}) \to (1, \frac{1}{6})$
- $(0, 1) \to (0, 1)$
- $(1, 6) \to (-1, 6)$

Dilations

For $k > 0$, if a **dilation of factor k from the x-axis**, given by the mapping $(x, y) \to (x, ky)$, is applied to the graph of $y = a^x$, then the image has equation $y = ka^x$.

- The horizontal asymptote of the image has equation $y = 0$.
- The range of the image is $(0, \infty)$.

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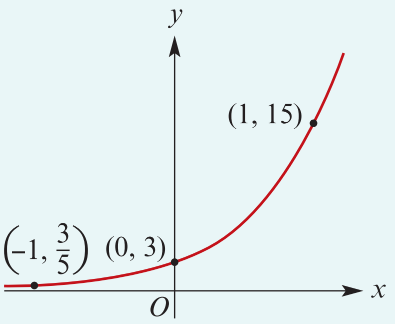
Example 4

Sketch the graph of each of the following:

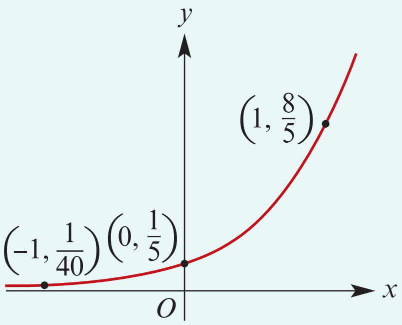
a $y = 3 \times 5^x$ **b** $y = 0.2 \times 8^x$

Solution

a



b



Explanation

The graph of $y = 5^x$ is dilated by factor 3 from the x-axis.

The mapping is $(x, y) \to (x, 3y)$.

Dilation of key points:

- $(-1, \frac{1}{5}) \to (-1, \frac{3}{5})$
- $(0, 1) \to (0, 3)$
- $(1, 5) \to (1, 15)$

The graph of $y = 8^x$ is dilated by factor $\frac{1}{5}$ from the x-axis.

The mapping is $(x, y) \to (x, \frac{1}{5}y)$.

Dilation of key points:

- $(-1, \frac{1}{8}) \to (-1, \frac{1}{40})$
- $(0, 1) \to (0, \frac{1}{5})$
- $(1, 8) \to (1, \frac{8}{5})$

For $k > 0$, if a **dilation of factor k from the y-axis**, given by the mapping $(x, y) \to (kx, y)$, is applied to the graph of $y = a^x$, then the image has equation $y = a^{\frac{x}{k}}$.

- The horizontal asymptote of the image has equation $y = 0$.
- The range of the image is $(0, \infty)$.

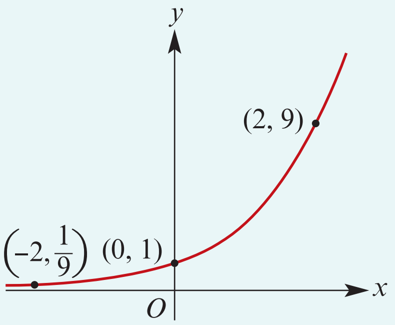
Example 5

Sketch the graph of each of the following:

a $y = 9^{\frac{x}{2}}$ **b** $y = 3^{2x}$

Solution

a



Explanation

The graph of $y = 9^x$ is dilated by factor 2 from the y-axis.

The mapping is $(x, y) \to (2x, y)$.

Dilation of key points:

- $(-1, \frac{1}{9}) \to (-2, \frac{1}{9})$
- $(0, 1) \to (0, 1)$
- $(1, 9) \to (2, 9)$

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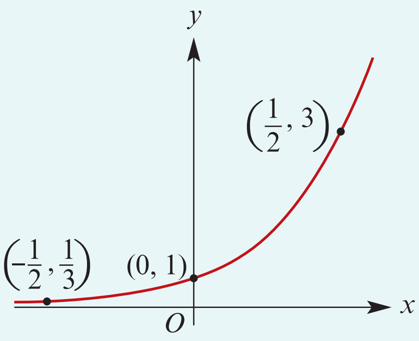
b



The graph of $y = 3^x$ is dilated by factor $\frac{1}{2}$ from the y-axis.

The mapping is $(x, y) \to (\frac{1}{2}x, y)$.

Dilation of key points:

- $(-1, \frac{1}{3}) \to (-\frac{1}{2}, \frac{1}{3})$
- $(0, 1) \to (0, 1)$
- $(1, 3) \to (\frac{1}{2}, 3)$

Note: Since $9^{\frac{x}{2}} = \left(9^{\frac{1}{2}}\right)^x = 3^x$, the graph of $y = 9^{\frac{x}{2}}$ is the same as the graph of $y = 3^x$. Similarly, the graph of $y = 3^{2x}$ is the same as the graph of $y = 9^x$.

A translation parallel to the x-axis results in a dilation from the x-axis. For example, if the graph of $y = 5^x$ is translated 3 units in the positive direction of the x-axis, then the image is the graph of $y = 5^{x-3}$, which can be written $y = 5^{-3} \times 5^x$. Hence, a translation of 3 units in the positive direction of the x-axis is equivalent to a dilation of factor 5^{-3} from the x-axis.

Combinations of transformations

We have seen translations, reflections and dilations applied to exponential graphs. In the following example we consider combinations of these transformations.

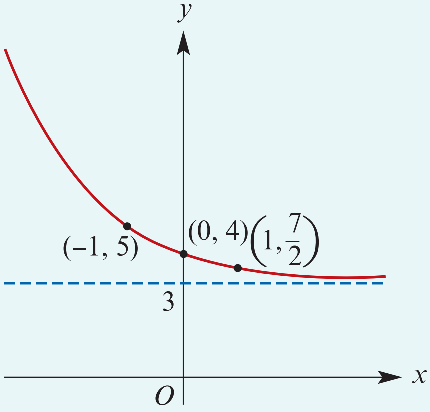
Example 6

Sketch the graph and state the range of each of the following:

a $y = 2^{-x} + 3$ **b** $y = 4^{3x} - 1$ **c** $y = -10^{x-1} - 2$

Solution

a



Graph of $y = 2^{-x} + 3$:

- The asymptote has equation $y = 3$.
- The y-axis intercept is $2^0 + 3 = 4$.
- The range of the function is $(3, \infty)$.

Explanation

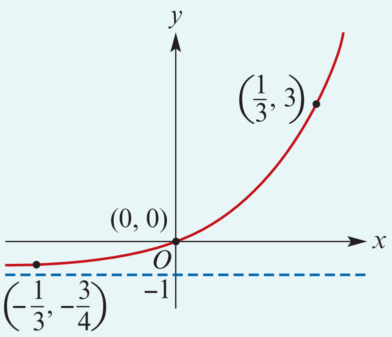
The graph of $y = 2^{-x} + 3$ is obtained from the graph of $y = 2^x$ by a reflection in the y-axis followed by a translation 3 units in the positive direction of the y-axis.

The mapping is $(x, y) \to (-x, y + 3)$.

For example:

$(-1, \frac{1}{2}) \to (1, \frac{7}{2})$
$(0, 1) \to (0, 4)$
$(1, 2) \to (-1, 5)$

b



Graph of $y = 4^{3x} - 1$:

- The asymptote has equation $y = -1$.
- The y-axis intercept is $4^0 - 1 = 0$.
- The range of the function is $(-1, \infty)$.

The graph of $y = 4^{3x} - 1$ is obtained from the graph of $y = 4^x$ by a dilation of factor $\frac{1}{3}$ from the y-axis followed by a translation 1 unit in the negative direction of the y-axis.

The mapping is $(x, y) \to (\frac{1}{3}x, y - 1)$.

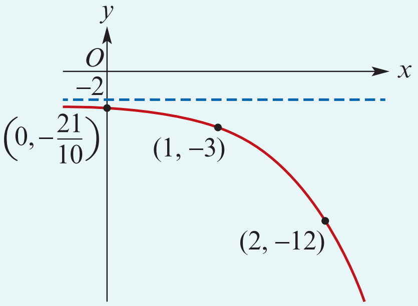
For example:

$(-1, \frac{1}{4}) \to (-\frac{1}{3}, -\frac{3}{4})$

$(0, 1) \to (0, 0)$

$(1, 4) \to (\frac{1}{3}, 3)$

c



Graph of $y = -10^{x-1} - 2$:

- The asymptote has equation $y = -2$.
- The y-axis intercept is $-10^{-1} - 2 = -\dfrac{21}{10}$.
- The range of the function is $(-\infty, -2)$.

The graph of $y = -10^{x-1} - 2$ is obtained from the graph of $y = 10^x$ by a reflection in the x-axis followed by a translation 1 unit in the positive direction of the x-axis and 2 units in the negative direction of the y-axis.

The mapping is $(x, y) \to (x + 1, -y - 2)$.

For example:

$(-1, \frac{1}{10}) \to (0, -\frac{21}{10})$

$(0, 1) \to (1, -3)$

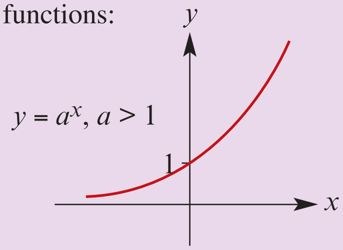
$(1, 10) \to (2, -12)$

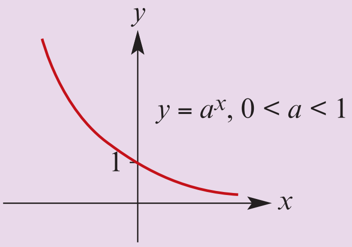
Note: We can use the method for determining transformations for each of the graphs in Example 6. Here we show the method for part **c**:

- Write the equation as $y' = -10^{x'-1} - 2$.
- Rearrange to $-y' - 2 = 10^{x'-1}$.
- We choose to write $y = -y' - 2$ and $x = x' - 1$.
- Hence $y' = -y - 2$ and $x' = x + 1$.

Section summary

- Graphs of exponential functions:





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- For $a \in \mathbb{R}^+ \setminus \{1\}$, the graph of $y = a^x$ has the following properties:
  - The x-axis is an asymptote.
  - The y-axis intercept is 1.
  - The y-values are always positive.
  - There is no x-axis intercept.
- Transformations can be applied to exponential functions. For example, the graph of

$$y = a^{b(x-h)} + k, \quad \text{where } b > 0$$

can be obtained from the graph of $y = a^x$ by a dilation of factor $\frac{1}{b}$ from the y-axis followed by the translation $(x, y) \to (x + h, y + k)$.

Exercise 5A

Example 1

1 For each of the following functions, sketch the graph (labelling the asymptote) and state the range:

a $y = 2^{x+1} - 2$ **b** $y = 2^{x-3} - 1$ **c** $y = 2^{x+2} - 1$ **d** $y = 2^{x-2} + 2$

Example 2, 3

2 For each of the following, use the one set of axes to sketch the two graphs (labelling asymptotes):

a $y = 2^x$ and $y = 3^x$ **b** $y = 2^{-x}$ and $y = 3^{-x}$

c $y = 5^x$ and $y = -5^x$ **d** $y = 1.5^x$ and $y = -1.5^x$

Example 4, 5

3 For each of the following functions, sketch the graph (labelling the asymptote) and state the range:

a $y = 3 \times 2^x$ **b** $y = \frac{1}{2} \times 5^x$ **c** $y = 2^{3x}$ **d** $y = 2^{\frac{x}{3}}$

Example 6

4 Sketch the graph and state the range of each of the following:

a $y = 3^{-x} + 2$ **b** $y = 2^{5x} - 4$ **c** $y = -10^{x-2} - 2$

5 For each of the following functions, sketch the graph (labelling the asymptote) and state the range:

a $y = 3^x$ **b** $y = 3^x + 1$ **c** $y = 1 - 3^x$

d $y = (\frac{1}{3})^x$ **e** $y = 3^{-x} + 2$ **f** $y = (\frac{1}{3})^x - 1$

6 For each of the following functions, sketch the graph (labelling the asymptote) and state the range:

a $y = (\frac{1}{2})^{x-2}$ **b** $y = (\frac{1}{2})^x - 1$ **c** $y = (\frac{1}{2})^{x-2} + 1$

7 For $f(x) = 2^x$, sketch the graph of each of the following, labelling asymptotes where appropriate:

a $y = f(x + 1)$ **b** $y = f(x) + 1$ **c** $y = f(-x) + 2$

d $y = -f(x) - 1$ **e** $y = f(3x)$ **f** $y = f\left(\frac{x}{2}\right)$

g $y = 2f(x - 1) + 1$ **h** $y = f(x - 2)$

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8 For each of the following functions, sketch the graph (labelling the asymptote) and state the range:

a $y = 10^x - 1$ **b** $y = 10^{\frac{x}{10}} + 1$ **c** $y = 2 \times 10^x - 20$

d $y = 1 - 10^{-x}$ **e** $y = 10^{x+1} + 3$ **f** $y = 2 \times 10^{\frac{x}{10}} + 4$

9 A bank offers cash loans at 0.04% interest per day, compounded daily. A loan of \$10 000 is taken and the interest payable at the end of x days is given by $C_1 = 10\,000\,[(1.0004)^x - 1]$.

a Plot the graph of C_1 against x.

b Find the interest at the end of:

i 100 days **ii** 300 days.

c After how many days is the interest payable \$1000?

d A loan company offers \$10 000 and charges a fee of \$4.25 per day. The amount charged after x days is given by $C_2 = 4.25x$.

i Plot the graph of C_2 against x (using the same window as in part **a**).

ii Find the smallest value of x for which $C_2 < C_1$.

10 If you invest \$100 at an interest rate of 2% per day, compounded daily, then after x days the amount of money you have (in dollars) is given by $y = 100(1.02)^x$. For how many days would you have to invest to double your money?

11 **a** **i** Graph $y = 2^x$, $y = 3^x$ and $y = 5^x$ on the same set of axes.

ii For what values of x is $2^x > 3^x > 5^x$?

iii For what values of x is $2^x < 3^x < 5^x$?

iv For what values of x is $2^x = 3^x = 5^x$?

b Repeat part **a** for $y = (\frac{1}{2})^x$, $y = (\frac{1}{3})^x$ and $y = (\frac{1}{5})^x$.

c Use your answers to parts **a** and **b** to sketch the graph of $y = a^x$ for:

i $a > 1$ **ii** $a = 1$ **iii** $0 < a < 1$

5B The exponential function $f(x) = e^x$

In the previous section, we explored the family of exponential functions $f(x) = a^x$, where $a \in \mathbb{R}^+ \setminus \{1\}$. One particular member of this family is of great importance in mathematics.

This function has the rule $f(x) = e^x$, where e is Euler's number, named after the eighteenth century Swiss mathematician Leonhard Euler.

Euler's number is defined as follows.

Euler's number

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

To see what the value of e might be, we could try large values of n and use a calculator to evaluate $(1 + \frac{1}{n})^n$, as shown in the table on the right.

As n is taken larger and larger, it can be seen that $(1 + \frac{1}{n})^n$ approaches a limiting value ($\approx 2.718\,28$).

n	$(1 + \frac{1}{n})^n$
100	$(1.01)^{100} = 2.704\,813\ldots$
1000	$(1.001)^{1000} = 2.716\,923\ldots$
10 000	$(1.0001)^{10\,000} = 2.718\,145\ldots$
100 000	$(1.000\,01)^{100\,000} = 2.718\,268\ldots$
1 000 000	$(1.000\,001)^{1\,000\,000} = 2.718\,280\ldots$

Like π, the number e is irrational:

$$e = 2.718\,281\,828\,459\,045\ldots$$

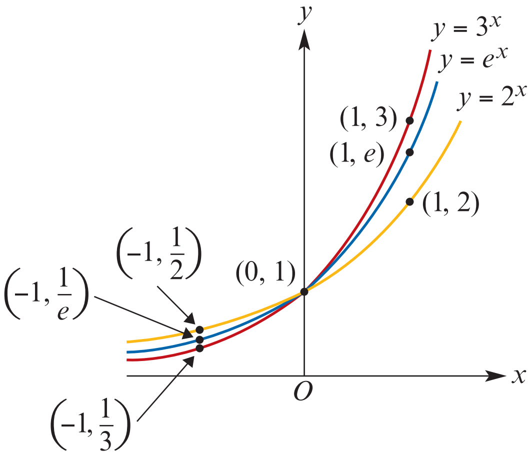
The function $f(x) = e^x$ is very important in mathematics. In Chapter 9 you will find that it has the remarkable property that $f'(x) = f(x)$. That is, the derivative of e^x is e^x.

Note: The function e^x can be found on your calculator.

Graphing $f(x) = e^x$

The graph of $y = e^x$ is as shown.

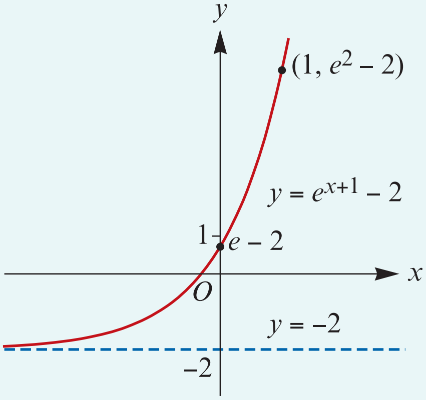
The graphs of $y = 2^x$ and $y = 3^x$ are shown on the same set of axes.



Example 7

Sketch the graph of $f: \mathbb{R} \to \mathbb{R}, f(x) = e^{x+1} - 2$.

Solution



The asymptote has equation $y = -2$.
The y-axis intercept is $e - 2$.

Explanation

To find the transformation:

- Write the image as $y' + 2 = e^{x'+1}$.
- We can choose $y = y' + 2$ and $x = x' + 1$.
- Hence $y' = y - 2$ and $x' = x - 1$.

The mapping is

$$(x, y) \to (x - 1, y - 2)$$

which is a translation of 1 unit in the negative direction of the x-axis and 2 units in the negative direction of the y-axis.

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▶ Compound interest

Assume that you invest $P at an annual interest rate r. If the interest is compounded only once per year, then the balance of your investment after t years is given by $A = P(1 + r)^t$.

Now assume that the interest is compounded n times per year. The interest rate in each period is $\frac{r}{n}$. The balance at the end of one year is $P\left(1 + \frac{r}{n}\right)^n$, and the balance at the end of t years is given by

$$A = P\left(1 + \frac{r}{n}\right)^{nt} = P\left(\left(1 + \frac{r}{n}\right)^{\frac{n}{r}}\right)^{rt}$$

We recognise that

$$\lim_{\frac{n}{r}\to\infty}\left(1 + \frac{r}{n}\right)^{\frac{n}{r}} = e$$

So, as $n \to \infty$, we can write $A = Pe^{rt}$.

For example, if $1000 is invested for one year at 5%, the resulting amount is $1050. However, if the interest is compounded 'continuously', then the amount is given by

$$A = Pe^{rt} = 1000 \times e^{0.05} = 1000 \times 1.051\,271\ldots \approx 1051.27$$

That is, the balance after one year is $1051.27.

Section summary

Euler's number is the natural base for exponential functions:

$$\begin{aligned} e &= \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n \\ &= 2.718\,281\ldots \end{aligned}$$

Exercise 5B

Skillsheet

Example 7

1 Sketch the graph of each of the following and state the range:

a $f(x) = e^x + 1$

b $f(x) = 1 - e^x$

c $f(x) = 1 - e^{-x}$

d $f(x) = e^{-2x}$

e $f(x) = e^{x-1} - 2$

f $f(x) = 2e^x$

g $h(x) = 2(1 + e^x)$

h $h(x) = 2(1 - e^{-x})$

i $g(x) = 2e^{-x} + 1$

j $h(x) = 2e^{x-1}$

k $f(x) = 3e^{x+1} - 2$

l $h(x) = 2 - 3e^x$

2 For each of the following, give a sequence of transformations that maps the graph of $y = e^x$ to the graph of $y = f_1(x)$:

a $f_1(x) = e^{x+2} - 3$

b $f_1(x) = 3e^{x+1} - 4$

c $f_1(x) = 5e^{2x+1}$

d $f_1(x) = 2 - e^{x-1}$

e $f_1(x) = 3 - 2e^{x+2}$

f $f_1(x) = 4e^{2x} - 1$

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3 Find the rule of the image when the graph of $f(x) = e^x$ undergoes each of the following sequences of transformations:

- **a** a dilation of factor 2 from the x-axis, followed by a reflection in the x-axis, followed by a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis
- **b** a dilation of factor 2 from the x-axis, followed by a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis, followed by a reflection in the x-axis
- **c** a reflection in the x-axis, followed by a dilation of factor 2 from the x-axis, followed by a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis
- **d** a reflection in the x-axis, followed by a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis, followed by a dilation of factor 2 from the x-axis
- **e** a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis, followed by a dilation of factor 2 from the x-axis, followed by a reflection in the x-axis
- **f** a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis, followed by a reflection in the x-axis, followed by a dilation of factor 2 from the x-axis.

4 For each of the following, give a sequence of transformations that maps the graph of $y = f_1(x)$ to the graph of $y = e^x$:

- **a** $f_1(x) = e^{x+2} - 3$
- **b** $f_1(x) = 3e^{x+1} - 4$
- **c** $f_1(x) = 5e^{2x+1}$
- **d** $f_1(x) = 2 - e^{x-1}$
- **e** $f_1(x) = 3 - 2e^{x+2}$
- **f** $f_1(x) = 4e^{2x} - 1$

5 Solve each of the following equations using a calculator. Give answers correct to three decimal places.

- **a** $e^x = x + 2$
- **b** $e^{-x} = x + 2$
- **c** $x^2 = e^x$
- **d** $x^3 = e^x$

6 **a** Using a calculator, plot the graph of $y = f(x)$ where $f(x) = e^x$.

b Using the same screen, plot the graphs of:

- **i** $y = f(x - 2)$
- **ii** $y = f\left(\frac{x}{3}\right)$
- **iii** $y = f(-x)$

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5C Exponential equations

One method for solving exponential equations is to use the one-to-one property of exponential functions:

$$a^x = a^y \text{ implies } x = y, \quad \text{for } a \in \mathbb{R}^+ \setminus \{1\}$$

Example 8

Find the value of x for which:

a $4^x = 256$ **b** $3^{x-1} = 81$

Solution

a
$$\begin{aligned} 4^x &= 256 \\ 4^x &= 4^4 \\ \therefore \quad x &= 4 \end{aligned}$$

b
$$\begin{aligned} 3^{x-1} &= 81 \\ 3^{x-1} &= 3^4 \\ \therefore \quad x - 1 &= 4 \\ x &= 5 \end{aligned}$$

When solving an exponential equation, you may also need to use the index laws.

Index laws

For all positive numbers a and b and all real numbers x and y:

- $a^x \times a^y = a^{x+y}$
- $a^x \div a^y = a^{x-y}$
- $(a^x)^y = a^{xy}$
- $(ab)^x = a^x b^x$
- $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$
- $a^{-x} = \dfrac{1}{a^x}$
- $a^x = \dfrac{1}{a^{-x}}$
- $a^0 = 1$

Note: More generally, each index law applies for real numbers a and b provided both sides of the equation are defined. For example: $a^m \times a^n = a^{m+n}$ for $a \in \mathbb{R}$ and $m, n \in \mathbb{Z}$.

Example 9

Find the value of x for which $5^{2x-4} = 25^{-x+2}$.

Solution

$$\begin{aligned} 5^{2x-4} &= 25^{-x+2} \\ &= (5^2)^{-x+2} \\ &= 5^{-2x+4} \\ \therefore \quad 2x - 4 &= -2x + 4 \\ 4x &= 8 \\ x &= 2 \end{aligned}$$

Explanation

Express both sides of the equation as powers with base 5.

Use the fact that $5^a = 5^b$ implies $a = b$.

To solve the equations in the next example, we must recognise that they will become quadratic equations once we make a substitution.

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Example 10

Solve for x:

a $9^x = 12 \times 3^x - 27$

b $3^{2x} = 27 - 6 \times 3^x$

Solution

a We can write the equation as

$$(3^x)^2 = 12 \times 3^x - 27$$

Let $y = 3^x$. The equation becomes

$$\begin{aligned} y^2 &= 12y - 27 \\ y^2 - 12y + 27 &= 0 \\ (y-3)(y-9) &= 0 \end{aligned}$$

$$\begin{aligned} \therefore \quad y = 3 \quad &\text{or} \quad y = 9 \\ 3^x = 3 \quad &\text{or} \quad 3^x = 3^2 \\ x = 1 \quad &\text{or} \quad x = 2 \end{aligned}$$

b We can write the equation as

$$(3^x)^2 = 27 - 6 \times 3^x$$

Let $y = 3^x$. The equation becomes

$$\begin{aligned} y^2 &= 27 - 6y \\ y^2 + 6y - 27 &= 0 \\ (y-3)(y+9) &= 0 \end{aligned}$$

$$\begin{aligned} \therefore \quad y = 3 \quad &\text{or} \quad y = -9 \\ 3^x = 3 \quad &\text{or} \quad 3^x = -9 \end{aligned}$$

The only solution is $x = 1$, since $3^x > 0$ for all x.

Section summary

- One method for solving an exponential equation, without using a calculator, is first to express both sides of the equation as powers with the same base and then to equate the indices (since $a^x = a^y$ implies $x = y$, for any base $a \in \mathbb{R}^+ \setminus \{1\}$).
  For example: $2^{x+1} = 8 \Leftrightarrow 2^{x+1} = 2^3 \Leftrightarrow x + 1 = 3 \Leftrightarrow x = 2$
- Equations such as $3^{2x} - 6 \times 3^x - 27 = 0$ can be solved by making a substitution. In this case, substitute $y = 3^x$ to obtain a quadratic equation in y.

Exercise 5C

1 Simplify the following expressions:

a $3x^2y^3 \times 2x^4y^6$ **b** $\dfrac{12x^8}{4x^2}$ **c** $18x^2y^3 \div (3x^4y)$

d $(4x^4y^2)^2 \div (2(x^2y)^4)$ **e** $(4x^0)^2$ **f** $15(x^5y^{-2})^4 \div (3(x^4y)^{-2})$

g $\dfrac{3(2x^2y^3)^4}{2x^3y^2}$ **h** $(8x^3y^6)^{\frac{1}{3}}$ **i** $\dfrac{x^2 + y^2}{x^{-2} + y^{-2}}$

Example 8

2 Solve for x in each of the following:

a $3^x = 81$ **b** $81^x = 9$ **c** $2^x = 256$

d $625^x = 5$ **e** $32^x = 8$ **f** $5^x = 125$

g $16^x = 1024$ **h** $2^{-x} = \dfrac{1}{64}$ **i** $5^{-x} = \dfrac{1}{625}$

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Example 9

3 Solve for n in each of the following:

a $5^{2n} \times 25^{2n-1} = 625$ **b** $4^{2n-2} = 1$ **c** $4^{2n-1} = \frac{1}{256}$

d $\frac{3^{n-2}}{9^{2-n}} = 27$ **e** $2^{2n-2} \times 4^{-3n} = 64$ **f** $2^{n-4} = 8^{4-n}$

g $27^{n-2} = 9^{3n+2}$ **h** $8^{6n+2} = 8^{4n-1}$ **i** $125^{4-n} = 5^{6-2n}$

j $2^{n-1} \times 4^{2n+1} = 16$ **k** $(27 \times 3^n)^n = 27^n \times 3^{\frac{1}{4}}$

Example 10

4 Solve for x:

a $3^{2x} - 2(3^x) - 3 = 0$ **b** $5^{2x} - 23(5^x) - 50 = 0$ **c** $5^{2x} - 10(5^x) + 25 = 0$

d $2^{2x} = 6(2^x) - 8$ **e** $8(3^x) - 6 = 2(3^{2x})$ **f** $2^{2x} - 20(2^x) = -64$

g $4^{2x} - 5(4^x) = -4$ **h** $3(3^{2x}) = 28(3^x) - 9$ **i** $7(7^{2x}) = 8(7^x) - 1$

5D Logarithms

Consider the statement

$$2^3 = 8$$

This may be written in an alternative form:

$$\log_2 8 = 3$$

which is read as ‘the logarithm of 8 to the base 2 is equal to 3’.

For $a \in \mathbb{R}^+ \setminus \{1\}$, the **logarithm function** with base a is defined as follows:

$$a^x = y \quad \text{is equivalent to} \quad \log_a y = x$$

Note: Since a^x is positive, the expression $\log_a y$ is only defined when y is positive.

Further examples:

- $3^2 = 9$ is equivalent to $\log_3 9 = 2$
- $10^4 = 10\,000$ is equivalent to $\log_{10} 10\,000 = 4$
- $a^0 = 1$ is equivalent to $\log_a 1 = 0$

Example 11

Without the aid of a calculator, evaluate the following:

a $\log_2 32$ **b** $\log_3 81$

Solution

a Let $\log_2 32 = x$

Then $2^x = 32$

$2^x = 2^5$

Therefore $x = 5$, giving $\log_2 32 = 5$.

b Let $\log_3 81 = x$

Then $3^x = 81$

$3^x = 3^4$

Therefore $x = 4$, giving $\log_3 81 = 4$.

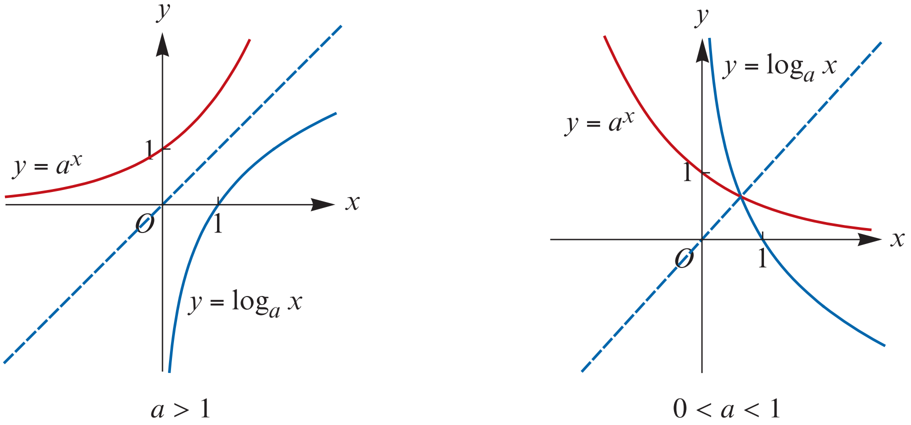
Note: To find $\log_2 32$, we ask 'What power of 2 gives 32?'
To find $\log_3 81$, we ask 'What power of 3 gives 81?'

Inverse functions

For each base $a \in \mathbb{R}^+ \setminus \{1\}$, the exponential function $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = a^x$ is one-to-one and so has an inverse function.

> Let $a \in \mathbb{R}^+ \setminus \{1\}$. The inverse of the exponential function $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = a^x$ is the logarithmic function $f^{-1}\colon \mathbb{R}^+ \to \mathbb{R}$, $f^{-1}(x) = \log_a x$.
>
> - $\log_a(a^x) = x$ for all $x \in \mathbb{R}$
> - $a^{\log_a x} = x$ for all $x \in \mathbb{R}^+$

Because they are inverse functions, the graphs of $y = a^x$ and $y = \log_a x$ are reflections of each other in the line $y = x$.



$a > 1$ $\quad\quad$ $0 < a < 1$

The natural logarithm

Earlier in the chapter we defined the number e and the important function $f(x) = e^x$. The inverse of this function is $f^{-1}(x) = \log_e x$. Because the logarithm function with base e is known as the **natural logarithm**, the expression $\log_e x$ is also written as $\ln x$.

The common logarithm

The function $f(x) = \log_{10} x$ has both historical and practical importance. When logarithms were used as a calculating device, it was often base 10 that was used. By simplifying calculations, logarithms contributed to the advancement of science, and especially of astronomy. In schools, books of tables of logarithms were provided for calculations and this was done up to the 1970s.

Base 10 logarithms are used for scales in science such as the Richter scale, decibels and pH.

You can understand the practicality of base 10 by observing:

- $\log_{10} 10 = 1$
- $\log_{10} 0.1 = -1$
- $\log_{10} 100 = 2$
- $\log_{10} 0.01 = -2$
- $\log_{10} 1000 = 3$
- $\log_{10} 0.001 = -3$
- $\log_{10} 10\,000 = 4$
- $\log_{10} 0.0001 = -4$

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Laws of logarithms

The index laws are used to establish rules for computations with logarithms.

Law 1: Logarithm of a product

The logarithm of a product is the sum of their logarithms:

$$\log_a(mn) = \log_a m + \log_a n$$

Proof Let $\log_a m = x$ and $\log_a n = y$, where m and n are positive real numbers. Then $a^x = m$ and $a^y = n$, and therefore

$$mn = a^x \times a^y = a^{x+y} \qquad \text{(using the first index law)}$$

Hence $\log_a(mn) = x + y = \log_a m + \log_a n$.

For example:

$$\begin{aligned}\log_{10} 200 + \log_{10} 5 &= \log_{10}(200 \times 5)\\ &= \log_{10} 1000 = 3\end{aligned}$$

Law 2: Logarithm of a quotient

The logarithm of a quotient is the difference of their logarithms:

$$\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$$

Proof Let $\log_a m = x$ and $\log_a n = y$, where m and n are positive real numbers. Then as before $a^x = m$ and $a^y = n$, and therefore

$$\frac{m}{n} = \frac{a^x}{a^y} = a^{x-y} \qquad \text{(using the second index law)}$$

Hence $\log_a\left(\frac{m}{n}\right) = x - y = \log_a m - \log_a n$.

For example:

$$\begin{aligned}\log_2 32 - \log_2 8 &= \log_2\left(\frac{32}{8}\right)\\ &= \log_2 4 = 2\end{aligned}$$

Law 3: Logarithm of a power

$$\log_a(m^p) = p\log_a m$$

Proof Let $\log_a m = x$. Then $a^x = m$, and therefore

$$m^p = (a^x)^p = a^{xp} \qquad \text{(using the third index law)}$$

Hence $\log_a(m^p) = xp = p\log_a m$.

For example:

$$\log_2 32 = \log_2(2^5) = 5$$

Law 4: Logarithm of $\frac{1}{m}$

$$\log_a(m^{-1}) = -\log_a m$$

Proof Use logarithm law 3 with $p = -1$.

For example:

$$\log_a(\tfrac{1}{2}) = \log_a(2^{-1}) = -\log_a 2$$

Law 5

$$\log_a 1 = 0 \quad \text{and} \quad \log_a a = 1$$

Proof Since $a^0 = 1$, we have $\log_a 1 = 0$.

Since $a^1 = a$, we have $\log_a a = 1$.

Example 12

Express the following as the logarithm of a single term:

$$2\log_e 3 + \log_e 16 - 2\log_e\left(\frac{6}{5}\right)$$

Solution

$$\begin{aligned}2\log_e 3 + \log_e 16 - 2\log_e\left(\frac{6}{5}\right) &= \log_e(3^2) + \log_e 16 - \log_e\left(\frac{6}{5}\right)^2\\ &= \log_e 9 + \log_e 16 - \log_e\left(\frac{36}{25}\right)\\ &= \log_e\left(9 \times 16 \times \frac{25}{36}\right)\\ &= \log_e 100\end{aligned}$$

Logarithmic equations

Example 13

Solve each of the following equations for x:

a $\log_2 x = 5$ **b** $\log_2(2x-1) = 4$ **c** $\log_e(3x+1) = 0$

Solution

a
$$\begin{aligned}\log_2 x &= 5\\ x &= 2^5\\ \therefore\quad x &= 32\end{aligned}$$

b
$$\begin{aligned}\log_2(2x-1) &= 4\\ 2x - 1 &= 2^4\\ 2x &= 17\\ \therefore\quad x &= \frac{17}{2}\end{aligned}$$

c
$$\begin{aligned}\log_e(3x+1) &= 0\\ 3x + 1 &= e^0\\ 3x &= 1 - 1\\ \therefore\quad x &= 0\end{aligned}$$

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Example 14

Solve each of the following equations for x:

a $\log_e(x-1) + \log_e(x+2) = \log_e(6x-8)$

b $\log_2 x - \log_2(7-2x) = \log_2 6$

Solution

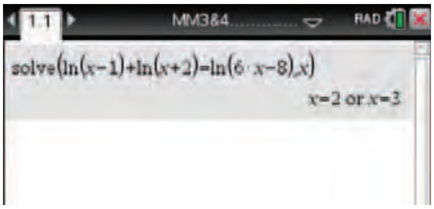
a
$$\begin{aligned} \log_e(x-1) + \log_e(x+2) &= \log_e(6x-8) \\ \log_e((x-1)(x+2)) &= \log_e(6x-8) \\ x^2 + x - 2 &= 6x - 8 \\ x^2 - 5x + 6 &= 0 \\ (x-3)(x-2) &= 0 \\ \therefore \quad x = 3 \text{ or } x &= 2 \end{aligned}$$

Note: The solutions must satisfy $x - 1 > 0$, $x + 2 > 0$ and $6x - 8 > 0$. Therefore both of these solutions are allowable.

b
$$\begin{aligned} \log_2 x - \log_2(7-2x) &= \log_2 6 \\ \log_2\left(\frac{x}{7-2x}\right) &= \log_2 6 \\ \frac{x}{7-2x} &= 6 \\ x &= 42 - 12x \\ 13x &= 42 \\ \therefore \quad x &= \frac{42}{13} \end{aligned}$$

Using the TI-Nspire

- Use **solve** from the **Algebra** menu as shown.
- Note that $\ln(x) = \log_e(x)$. The logarithm with base e is available on the keypad by pressing ctrl e^x.



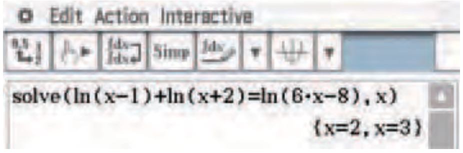
solve(ln(x−1)+ln(x+2)=ln(6·x−8),x)

x=2 or x=3

Note: Logarithms with other bases are obtained by pressing the **log** key (ctrl 10^x) and completing the template.

Using the Casio ClassPad

- For a logarithm with base e, use ln from the Math1 keyboard. Note that $\ln(x) = \log_e(x)$.
- Enter and highlight the equation $\ln(x-1) + \ln(x+2) = \ln(6x-8)$.
- Select **Interactive** > **Equation/Inequality** > **solve**. Ensure the variable is set to x.



solve(ln(x−1)+ln(x+2)=ln(6·x−8),x)

{x=2,x=3}

Note: For logarithms with other bases, tap log■□ and complete the template.

Example 15

Solve each of the following equations for x:

a $\log_e(2x+1) - \log_e(x-1) = 4$

b $\log_e(x-1) + \log_e(x+1) = 1$

Solution

a

$$\begin{aligned}\log_e(2x+1)-\log_e(x-1)&=4\\ \log_e\left(\frac{2x+1}{x-1}\right)&=4\\ \frac{2x+1}{x-1}&=e^4\\ 2x+1&=e^4(x-1)\\ (2-e^4)x&=-(e^4+1)\\ \therefore\quad x&=\frac{e^4+1}{e^4-2}\end{aligned}$$

b

$$\begin{aligned}\log_e(x-1)+\log_e(x+1)&=1\\ \log_e((x-1)(x+1))&=1\\ \log_e(x^2-1)&=1\\ x^2-1&=e\\ \therefore\quad x&=\pm\sqrt{e+1}\end{aligned}$$

But the original equation is not defined for $x = -\sqrt{e+1}$ and so the only solution is $x = \sqrt{e+1}$.

Example 16

Solve the equation $\log_x 27 = \frac{3}{2}$ for x.

Solution

$\log_x 27 = \frac{3}{2}$ is equivalent to

$$\begin{aligned}x^{\frac{3}{2}}&=27\\ (\sqrt{x})^3&=3^3\\ \sqrt{x}&=3\\ \therefore\quad x&=9\end{aligned}$$

Section summary

- For $a \in \mathbb{R}^+ \setminus \{1\}$, the logarithm function base a is defined as follows:

  $a^x = y$ is equivalent to $\log_a y = x$

- To evaluate $\log_a y$ ask the question: 'What power of a gives y?'
- For $a \in \mathbb{R}^+ \setminus \{1\}$, the inverse of the exponential function $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = a^x$ is the logarithmic function $f^{-1}\colon \mathbb{R}^+ \to \mathbb{R}$, $f^{-1}(x) = \log_a x$.
  - $\log_a(a^x) = x$ for all x
  - $a^{\log_a x} = x$ for all positive values of x
- Laws of logarithms
  1. $\log_a(mn) = \log_a m + \log_a n$
  2. $\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$
  3. $\log_a(m^p) = p\log_a m$
  4. $\log_a(m^{-1}) = -\log_a m$
  5. $\log_a 1 = 0$ and $\log_a a = 1$

Exercise 5D

Skillsheet

Example 11

1 Evaluate each of the following:

a $\log_{10} 1000$ **b** $\log_2\left(\frac{1}{16}\right)$ **c** $\log_{10} 0.001$

d $\log_2 64$ **e** $\log_{10} 1\,000\,000$ **f** $\log_2\left(\frac{1}{128}\right)$

Example 12 **2** Express each of the following as the logarithm of a single term:

a $\log_e 2 + \log_e 3$ **b** $\log_e 32 - \log_e 8$

c $\log_e 10 + \log_e 100 + \log_e 1000$ **d** $\log_e\left(\frac{1}{2}\right) + \log_e 14$

e $\log_e\left(\frac{1}{3}\right) + \log_e\left(\frac{1}{4}\right) + \log_e\left(\frac{1}{5}\right)$ **f** $\log_e(uv) + \log_e(uv^2) + \log_e(uv^3)$

g $2\log_e x + 5\log_e x$ **h** $\log_e(x + y) + \log_e(x - y) - \log_e(x^2 - y^2)$

Example 13 **3** Solve each of the following equations for x:

a $\log_{10} x = 2$ **b** $2\log_2 x = 8$ **c** $\log_e(x - 5) = 0$

d $\log_2 x = 6$ **e** $2\log_e(x + 5) = 6$ **f** $\log_e(2x) = 0$

g $\log_e(2x + 3) = 0$ **h** $\log_{10} x = -3$ **i** $2\log_2(x - 4) = 10$

Example 14 **4** Solve each of the following equations for x:

a $\log_{10} x = \log_{10} 3 + \log_{10} 5$ **b** $\log_e x = \log_e 15 - \log_e 3$

c $\log_e x = \frac{2}{3}\log_e 8$ **d** $\log_e x + \log_e(2x - 1) = 0$

e $2\log_e x - \log_e(x - 1) = \log_e(x + 3)$

5 Express each of the following as the logarithm of a single term:

a $\log_{10} 9 + \log_{10} 3$ **b** $\log_2 24 - \log_2 6$ **c** $\frac{1}{2}\log_{10} a - \frac{1}{2}\log_{10} b$

d $1 + \log_{10} a - \frac{1}{3}\log_{10} b$ **e** $\frac{1}{2}\log_{10} 36 - \frac{1}{3}\log_{10} 27 - \frac{2}{3}\log_{10} 64$

6 Without using your calculator, evaluate each of the following:

a $\log_{10} 5 + \log_{10} 2$ **b** $\log_{10} 5 + 3\log_{10} 2 - \log_{10} 4$

c $\log_2 \sqrt{2} + \log_2 1 + 2\log_2 2$ **d** $2\log_{10} 5 + 2\log_{10} 2 + 1$

e $4\log_{10} 2 - \log_{10} 16$

7 Simplify the following expressions:

a $\log_3\left(\frac{1}{3^x}\right)$ **b** $\log_2 x - 2\log_2 y + \log_2(xy^2)$ **c** $\log_e(x^2 - y^2) - \log_e(x - y) - \log_e(x + y)$

Example 15 **8** Solve each of the following equations for x:

a $\log_e(x^2 - 2x + 8) = 2\log_e x$ **b** $\log_e(5x) - \log_e(3 - 2x) = 1$

9 Solve each of the following equations for x:

a $\log_e x + \log_e(3x + 1) = 1$ **b** $8e^{-x} - e^x = 2$

Example 16 **10** Solve each of the following equations for x:

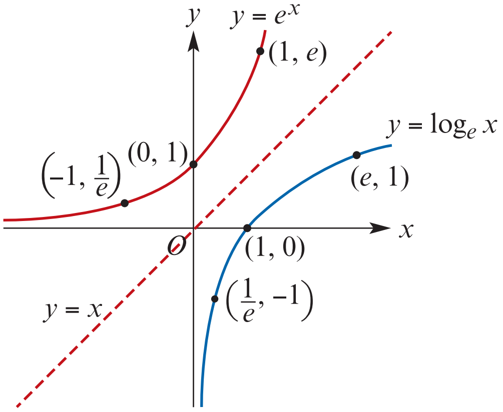
a $\log_x 81 = 4$ **b** $\log_x\left(\frac{1}{32}\right) = 5$

11 Solve $2\log_e x + \log_e 4 = \log_e(9x - 2)$.

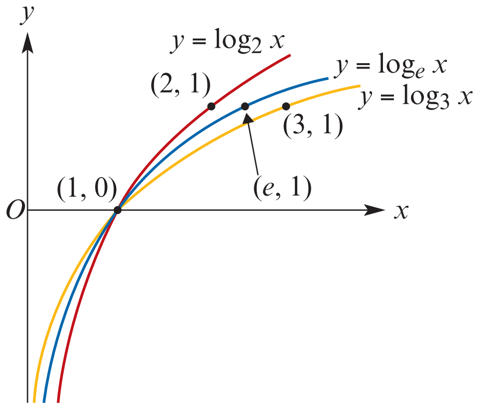
12 Given that $\log_a N = \frac{1}{2}(\log_a 24 - \log_a 0.375 - 6\log_a 3)$, find the value of N.

5E Graphing logarithmic functions

The graphs of $y = e^x$ and its inverse function $y = \log_e x$ are shown on the one set of axes.



The graphs of $y = \log_2 x$, $y = \log_e x$ and $y = \log_3 x$ are shown on the one set of axes.



For each base $a \in \mathbb{R}^+ \setminus \{1\}$, the graph of the logarithmic function $f(x) = \log_a x$ has the following features:

- Key values are $f\left(\frac{1}{a}\right) = -1$, $f(1) = 0$ and $f(a) = 1$.
- The maximal domain is $\mathbb{R}^+$ and the range is $\mathbb{R}$.
- The y-axis is a vertical asymptote.

A logarithmic function with $a > 1$ is strictly increasing, and a logarithmic function with $0 < a < 1$ is strictly decreasing. In both cases, the function is one-to-one.

▶ Graphing transformations of $f(x) = \log_a x$

We now look at transformations applied to the graph of $f(x) = \log_a x$ where $a > 1$. We make the following general observations:

- The graph of $y = \log_a(mx - n)$, where $m > 0$, has a vertical asymptote $x = \frac{n}{m}$ and implied domain $\left(\frac{n}{m}, \infty\right)$. The x-axis intercept is $\frac{1 + n}{m}$.

Example 17

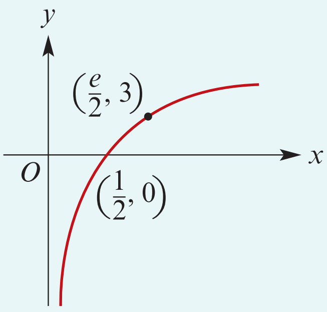
Sketch the graph of $y = 3\log_e(2x)$.

Solution

This is obtained from the graph of $y = \log_e x$ by a dilation of factor 3 from the x-axis and a dilation of factor $\frac{1}{2}$ from the y-axis.

The mapping is $(x, y) \to (\frac{1}{2}x, 3y)$.

- $(1, 0) \to (\frac{1}{2}, 0)$
- $(e, 1) \to (\frac{1}{2}e, 3)$



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Example 18

Sketch the graph and state the implied domain of each of the following:

a $y = \log_2(x-5)+1$ **b** $y = -\log_3(x+4)$

Solution

a The graph of $y = \log_2(x-5)+1$ is obtained from the graph of $y = \log_2 x$ by a translation of 5 units in the positive direction of the x-axis and 1 unit in the positive direction of the y-axis.

The mapping is $(x, y) \to (x+5, y+1)$.

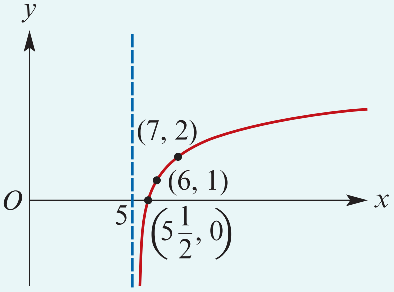
- $(1, 0) \to (6, 1)$
- $(2, 1) \to (7, 2)$

The asymptote has equation $x = 5$.

When $y = 0$,

$$\begin{aligned} \log_2(x-5)+1 &= 0 \\ \log_2(x-5) &= -1 \\ x-5 &= 2^{-1} \\ \therefore \quad x &= 5\tfrac{1}{2} \end{aligned}$$

The domain of the function is $(5, \infty)$.



b The graph of $y = -\log_3(x+4)$ is obtained from the graph of $y = \log_3 x$ by a reflection in the x-axis and a translation of 4 units in the negative direction of the x-axis.

The mapping is $(x, y) \to (x-4, -y)$.

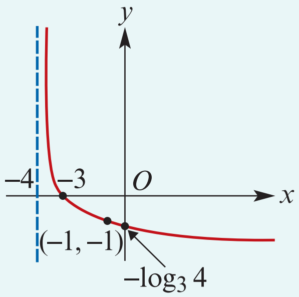
- $(1, 0) \to (-3, 0)$
- $(3, 1) \to (-1, -1)$

The asymptote has equation $x = -4$.

When $x = 0$,

$$\begin{aligned} y &= -\log_3(0+4) \\ &= -\log_3 4 \end{aligned}$$

The domain of the function is $(-4, \infty)$.



Example 19

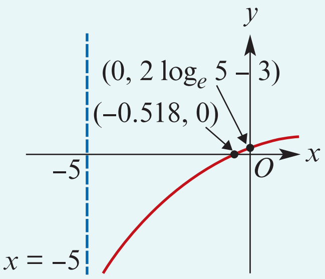
Sketch the graph of $y = 2\log_e(x+5)-3$ and state the implied domain.

Solution

The graph of $y = 2\log_e(x+5)-3$ is obtained from the graph of $y = \log_e x$ by a dilation of factor 2 from the x-axis followed by a translation of 5 units in the negative direction of the x-axis and 3 units in the negative direction of the y-axis.

The equation of the asymptote is $x = -5$.

The domain of the function is $(-5, \infty)$.



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Axis intercepts

When $x = 0$, $\quad y = 2\log_e(0+5) - 3$

$= 2\log_e 5 - 3$

When $y = 0$,

$$\begin{aligned} 2\log_e(x+5) - 3 &= 0 \\ \log_e(x+5) &= \frac{3}{2} \\ x + 5 &= e^{\frac{3}{2}} \\ \therefore \quad x &= e^{\frac{3}{2}} - 5 \end{aligned}$$

► Exponential and logarithmic graphs with different bases

It is often useful to know how to go from one base to another.

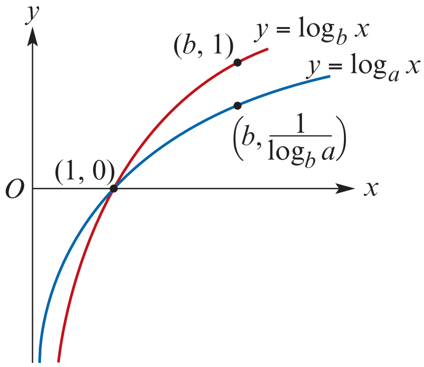
To change the base of $\log_a x$ from a to b (where $a, b > 0$ and $a, b \neq 1$), we use the definition that $y = \log_a x$ implies $a^y = x$. Taking $\log_b$ of both sides:

$$\begin{aligned} \log_b(a^y) &= \log_b x \\ y\log_b a &= \log_b x \\ y &= \frac{\log_b x}{\log_b a} \end{aligned}$$

Since $y = \log_a x$, this gives:

$$\log_a x = \frac{\log_b x}{\log_b a}$$

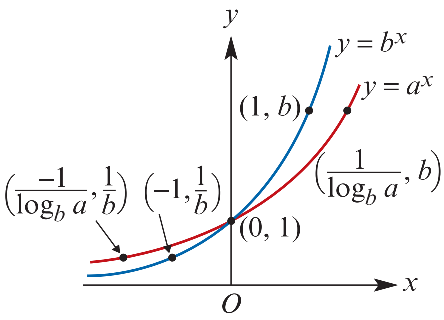
Hence the graph of $y = \log_a x$ can be obtained from the graph of $y = \log_b x$ by a dilation of factor $\dfrac{1}{\log_b a}$ from the x-axis.



Using properties of inverses, we can write $a = b^{\log_b a}$. This gives:

$$a^x = b^{(\log_b a)x}$$

Hence the graph of $y = a^x$ can be obtained from the graph of $y = b^x$ by a dilation of factor $\dfrac{1}{\log_b a}$ from the y-axis.



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Example 20

Find a transformation that takes the graph of $y = 2^x$ to the graph of $y = e^x$.

Solution

We can write $e = 2^{\log_2 e}$ and so

$$\begin{aligned} e^x &= (2^{\log_2 e})^x \\ &= 2^{(\log_2 e)x} \end{aligned}$$

The graph of $y = e^x$ is the image of the graph of $y = 2^x$ under a dilation of factor $\dfrac{1}{\log_2 e}$ from the y-axis.

Exercise 5E

Example 17

1 Sketch the graph of each of the following:

a $y = 2\log_e(3x)$
b $y = 4\log_e(5x)$
c $y = 2\log_e(4x)$
d $y = 3\log_e\left(\dfrac{x}{2}\right)$

Example 18

2 For each of the following functions, sketch the graph (labelling axis intercepts and asymptotes) and state the maximal domain and range:

a $y = 2\log_e(x - 3)$
b $y = \log_e(x + 3) - 2$
c $y = 2\log_e(x + 1) - 1$
d $y = 2 + \log_e(3x - 2)$
e $y = -2\log_e(x + 2)$
f $y = -2\log_e(x - 2)$
g $y = 1 - \log_e(x + 1)$
h $y = \log_e(2 - x)$
i $y + 1 = \log_e(4 - 3x)$

Example 19

3 Sketch the graph of each of the following. Label the axis intercepts and asymptotes. State the implied domain of each function.

a $y = \log_2(2x)$
b $y = \log_{10}(x - 5)$
c $y = -\log_{10} x$
d $y = \log_{10}(-x)$
e $y = \log_{10}(5 - x)$
f $y = 2\log_2(2x) + 2$
g $y = -2\log_2(3x)$
h $y = \log_{10}(-x - 5) + 2$
i $y = 4\log_2(-3x)$
j $y = 2\log_2(2 - x) - 6$
k $y = \log_e(2x - 1)$
l $y = -\log_e(3 - 2x)$

4 Solve each of the following equations using a calculator. Give answers correct to three decimal places.

a $-x + 2 = \log_e x$
b $\dfrac{1}{3}\log_e(2x + 1) = -\dfrac{1}{2}x + 1$

5 **a** Using a calculator, plot the graph of $y = f(x)$ where $f(x) = \log_e x$.

b Using the same screen, plot the graphs of:

i $y = f(-x)$ **ii** $y = -f(x)$ **iii** $y = f\left(\dfrac{x}{3}\right)$ **iv** $y = f(3x)$

Example 20

6 Find a transformation that takes the graph of $y = 3^x$ to the graph of $y = e^x$.

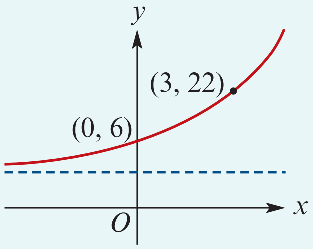
7 Find a transformation that takes the graph of $y = e^x$ to the graph of $y = 2^x$.

5F Determining rules for graphs of exponential and logarithmic functions

In previous chapters, we have determined the rules for graphs of various types of functions, including polynomial functions. In this chapter, we consider similar questions for exponential and logarithmic functions.

Example 21

The rule for the function with the graph shown is of the form $y = ae^x + b$. Find the values of a and b.



Solution

When $x = 0$, $y = 6$ and when $x = 3$, $y = 22$:

$$6 = ae^0 + b \qquad (1)$$
$$22 = ae^3 + b \qquad (2)$$

Subtract (1) from (2):

$$16 = a(e^3 - e^0)$$
$$16 = a(e^3 - 1)$$
$$\therefore \quad a = \frac{16}{e^3 - 1}$$

From equation (1):

$$b = 6 - a$$
$$= 6 - \frac{16}{e^3 - 1}$$
$$= \frac{6e^3 - 22}{e^3 - 1}$$

The function has rule $y = \left(\dfrac{16}{e^3 - 1}\right)e^x + \dfrac{6e^3 - 22}{e^3 - 1}$.

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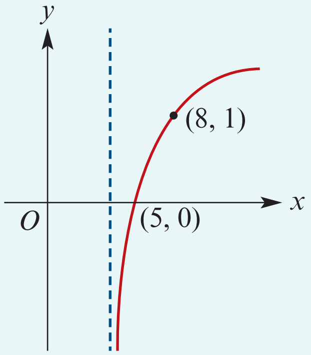
Example 22

The rule for the function with the graph shown is of the form $y = a\log_e(x+b)$. Find the values of a and b.



Solution

$$0 = a\log_e(5+b) \qquad (1)$$
$$1 = a\log_e(8+b) \qquad (2)$$

From (1): $\log_e(5+b) = 0$

$$5 + b = e^0$$
$$\therefore \quad b = -4$$

Substitute in (2): $1 = a\log_e 4$

$$\therefore \quad a = \frac{1}{\log_e 4}$$

The rule is $y = \dfrac{1}{\log_e 4}\log_e(x-4)$.

Explanation

Form two equations in a and b by substituting into the rule $y = a\log_e(x+b)$:

- $y = 0$ when $x = 5$
- $y = 1$ when $x = 8$

Example 23

Given that $y = Ae^{bt}$ with $y = 6$ when $t = 1$ and $y = 8$ when $t = 2$, find A and b.

Solution

$$6 = Ae^{b} \qquad (1)$$
$$8 = Ae^{2b} \qquad (2)$$

Divide (2) by (1): $\dfrac{4}{3} = e^b$

$$\therefore \quad b = \log_e \frac{4}{3}$$

Substitute in (1): $6 = Ae^{\log_e \frac{4}{3}}$

$$6 = \frac{4}{3}A$$
$$\therefore \quad A = \frac{18}{4} = \frac{9}{2}$$

The rule is $y = \dfrac{9}{2}e^{(\log_e \frac{4}{3})t}$.

Explanation

Form two equations in A and b by substituting into the rule $y = Ae^{bt}$:

- $y = 6$ when $t = 1$
- $y = 8$ when $t = 2$

$e^{\log_e \frac{4}{3}} = \dfrac{4}{3}$ since $e^{\log_e a} = a$ for all $a > 0$

Note that $y = \dfrac{9}{2}\left(e^{\log_e \frac{4}{3}}\right)^t = \dfrac{9}{2}\left(\dfrac{4}{3}\right)^t$

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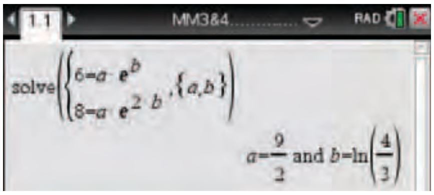
Using the TI-Nspire

Use menu > **Algebra** > **Solve System of Equations** > **Solve System of Equations** and complete as shown.

Note: Do not use the ‘e’ from the alpha keys; it will be treated as a variable.

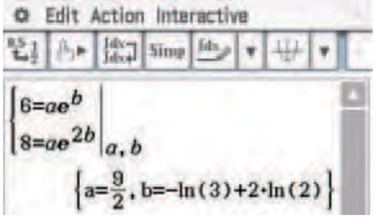


$\text{solve}\left(\begin{cases}6=a\cdot e^{b}\\8=a\cdot e^{2\cdot b}\end{cases},\{a,b\}\right)$

$a=\frac{9}{2}$ and $b=\ln\left(\frac{4}{3}\right)$

Using the Casio ClassPad

- Select the simultaneous equations template {■.
- Enter the equations as shown: select e■ from the Math1 keyboard and select the parameters a, b from the Var keyboard.



$\begin{cases}6=ae^{b}\\8=ae^{2b}\end{cases}\Big|_{a,\,b}$

$\left\{a=\frac{9}{2},\ b=-\ln(3)+2\cdot\ln(2)\right\}$

Exercise 5F

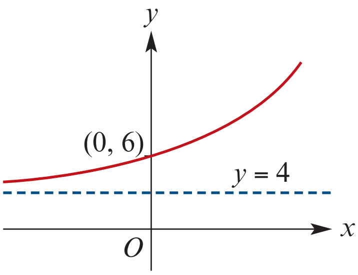
Example 21 **1** An exponential function has rule $y = a \times e^x + b$ and the points with coordinates $(0, 5)$ and $(4, 11)$ are on the graph of the function. Find the values of a and b.

Example 22 **2** A logarithmic function has rule $y = a\log_e(x + b)$ and the points with coordinates $(5, 0)$ and $(10, 2)$ are on the graph of the function. Find the values of a and b.

3 The graph shown has rule

$$y = ae^x + b$$

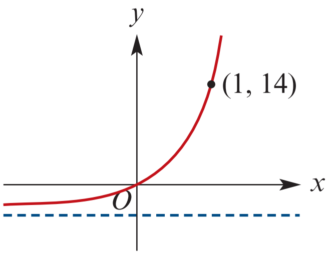
Find the values of a and b.



4 The rule for the function for which the graph is shown is of the form

$$y = ae^x + b$$

Find the values of a and b.

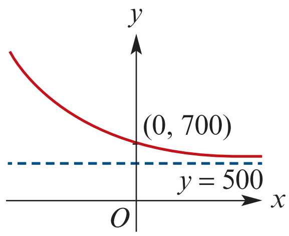


Example 23 **5** Find the values of a and b such that the graph of $y = ae^{-bx}$ goes through the points $(3, 50)$ and $(6, 10)$.

6 The rule for the function f is of the form

$$f(x) = ae^{-x} + b$$

Find the values of a and b.

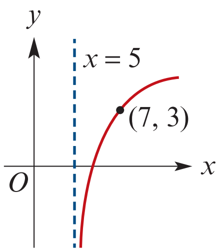


7 Find the values of a and b such that the graph of $y = a\log_2 x + b$ goes through the points $(8, 10)$ and $(32, 14)$.

8 The rule of the graph shown is of the form

$$y = a\log_2(x - b)$$

Find the values of a and b.



9 Find the values of a and b such that the graph of $y = ae^{bx}$ goes through the points $(3, 10)$ and $(6, 50)$.

10 Find the values of a and b such that the graph of $y = a\log_2(x - b)$ passes through the points $(5, 2)$ and $(7, 4)$.

11 The points $(3, 10)$ and $(5, 12)$ lie on the graph of $y = a\log_e(x - b) + c$. The graph has a vertical asymptote with equation $x = 1$. Find the values of a, b and c.

12 The graph of the function with rule $f(x) = a\log_e(-x) + b$ passes though the points $(-2, 6)$ and $(-4, 8)$. Find the values of a and b.

5G Solution of exponential equations using logarithms

Example 24

If $\log_2 6 = k\log_2 3 + 1$, find the value of k.

Solution

$$\begin{aligned} \log_2 6 &= k\log_2 3 + 1 \\ &= \log_2(3^k) + \log_2 2 \\ &= \log_2(2 \times 3^k) \\ \therefore \quad 6 &= 2 \times 3^k \\ 3 &= 3^k \\ k &= 1 \end{aligned}$$

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Example 25

Solve for x if $2^x = 11$, expressing the answer to two decimal places.

Solution

$$2^x = 11 \iff x = \log_2 11$$
$$= 3.45943\ldots$$

Therefore $x \approx 3.46$ correct to two decimal places.

Example 26

Solve $3^{2x-1} = 28$, expressing the answer to three decimal places.

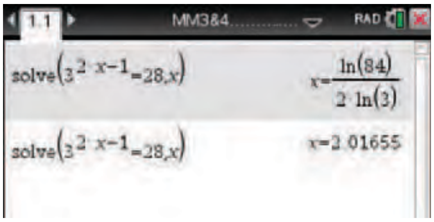
Solution

$$3^{2x-1} = 28 \iff 2x - 1 = \log_3 28$$

Thus
$$\begin{aligned} 2x - 1 &= \log_3 28 \\ 2x &= \log_3 28 + 1 \\ x &= \frac{1}{2}(\log_3 28 + 1) \\ &\approx 2.017 \quad \text{correct to three decimal places} \end{aligned}$$

Using the TI-Nspire

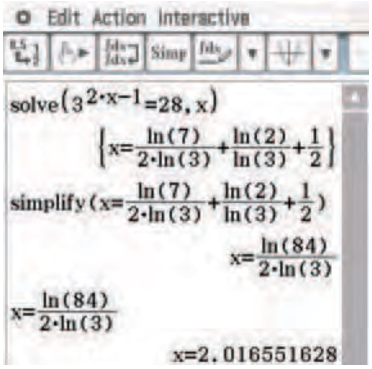
- Use menu > **Algebra** > **Solve** and complete as shown.
- Convert to a decimal answer using ctrl enter or menu > **Actions** > **Convert to Decimal**.
- Round to three decimal places as required: $x = 2.017$.



```
solve(3^(2·x-1)=28,x)        x=ln(84)/(2·ln(3))
solve(3^(2·x-1)=28,x)        x=2.01655
```

Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, enter and highlight the equation $3^{2x-1} = 28$.
- Go to **Interactive** > **Equation/Inequality** > **solve** and tap OK.
- Copy and paste the answer into the next entry line and go to **Interactive** > **Transformation** > **simplify** to obtain a simplified exact answer.
- Highlight the answer and tap $0.5 \leftrightarrow \frac{1}{2}$ to obtain the decimal approximation.



```
solve(3^(2·x-1)=28,x)
        {x=ln(7)/(2·ln(3))+ln(2)/ln(3)+1/2}
simplify(x=ln(7)/(2·ln(3))+ln(2)/ln(3)+1/2)
        x=ln(84)/(2·ln(3))
x=ln(84)/(2·ln(3))
        x=2.016551628
```

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Example 27

Solve the inequality $0.7^x \ge 0.3$.

Solution

Taking $\log_{10}$ of both sides:

$$\log_{10}(0.7^x) \ge \log_{10} 0.3$$
$$x \log_{10} 0.7 \ge \log_{10} 0.3$$
$$\therefore \quad x \le \frac{\log_{10} 0.3}{\log_{10} 0.7} \qquad \text{(direction of inequality reversed since } \log_{10} 0.7 < 0\text{)}$$

Alternatively, we can solve the inequality $0.7^x \ge 0.3$ directly as follows:

Note that $0 < 0.7 < 1$ and thus $y = 0.7^x$ is strictly decreasing. Therefore the inequality $0.7^x \ge 0.3$ holds for $x \le \log_{0.7} 0.3$.

Section summary

- If $a \in \mathbb{R}^+ \setminus \{1\}$ and $x \in \mathbb{R}$, then the statements $a^x = b$ and $\log_a b = x$ are equivalent. This defining property of logarithms may be used in the solution of exponential equations and inequalities. For example:
  - $2^x = 5 \Leftrightarrow x = \log_2 5$
  - $2^x \ge 5 \Leftrightarrow x \ge \log_2 5$
  - $0.3^x = 5 \Leftrightarrow x = \log_{0.3} 5$
  - $0.3^x \ge 5 \Leftrightarrow x \le \log_{0.3} 5$
- An exponential inequality may also be solved by taking $\log_a$ of both sides. For $a > 1$, the direction of the inequality stays the same (as $y = \log_a x$ is strictly increasing). For $0 < a < 1$, the direction of the inequality reverses (as $y = \log_a x$ is strictly decreasing).

Exercise 5G

Example 24

1 **a** If $\log_2 8 = k \log_2 7 + 2$, find the value of k.

b If $\log_2 7 - x \log_2 7 = 4$, find the value of x.

c If $\log_e 7 - x \log_e 14 = 1$, find the value of x.

Example 25, 26

2 Use your calculator to solve each of the following equations, correct to two decimal places:

a $2^x = 6$
b $3^x = 0.7$
c $3^x = 11$
d $4^x = 5$
e $2^{-x} = 5$
f $0.2^x = 3$
g $5^x = 3^{x-1}$
h $8^x = 2005^{x+1}$
i $3^{x-1} = 8$
j $0.3^{x+2} = 0.7$
k $2^{x-1} = 3^{x+1}$
l $1.4^{x+2} = 25(0.9^x)$
m $5^x = 2^{2x-2}$
n $2^{\frac{1}{2}(x+2)} = 3^{x-1}$
o $2^{x+1} \times 3^{x-1} = 100$

3 Solve for x using a calculator. Express your answer correct to two decimal places.

a $2^x < 7$
b $3^x > 6$
c $0.2^x > 3$
d $3^{x-2} \le 8$
e $0.2^x \le 0.4$

4 Solve each of the following equations for x. Give exact answers.

a $2^x = 5$ **b** $3^{2x-1} = 8$ **c** $7^{3x+1} = 20$

d $3^x = 7$ **e** $3^x = 6$ **f** $5^x = 6$

g $3^{2x} - 3^{x+2} + 8 = 0$ **h** $5^{2x} - 4 \times 5^x - 5 = 0$

Example 27

5 Solve each of the following inequalities for x. Give exact answers.

a $7^x > 52$ **b** $3^{2x-1} < 40$ **c** $4^{3x+1} \geq 5$

d $3^{x-5} \leq 30$ **e** $3^x < 106$ **f** $5^x < 0.6$

6 **a** If $a \log_2 7 = 3 - \log_6 14$, find the value of a, correct to three significant figures.

b If $\log_3 18 = \log_{11} k$, find the value of k, correct to one decimal place.

7 Prove that if $\log_r p = q$ and $\log_q r = p$, then $\log_q p = pq$.

8 If $u = \log_9 x$, find in terms of u:

a x **b** $\log_9(3x)$ **c** $\log_x 81$

9 Solve the equation $\log_5 x = 16 \log_x 5$.

10 Given that $q^p = 25$, find $\log_5 q$ in terms of p.

5H Inverses

We have observed that $f(x) = \log_a x$ and $g(x) = a^x$ are inverse functions. In this section, this observation is used to find inverses of related functions and to transform equations. An important consequence is the following:

$$\log_a(a^x) = x \qquad \text{for all } x \in \mathbb{R}$$

$$a^{\log_a x} = x \qquad \text{for all } x \in \mathbb{R}^+$$

Example 28

Find the inverse of the function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = e^x + 2$ and state the domain and range of the inverse function.

Solution

Recall that the transformation 'reflection in the line $y = x$' is given by the mapping $(x, y) \to (y, x)$. Consider

$$\begin{aligned} x &= e^y + 2 \\ x - 2 &= e^y \\ \therefore \quad y &= \log_e(x - 2) \end{aligned}$$

Thus the inverse function has rule $f^{-1}(x) = \log_e(x - 2)$.

Domain of f^{-1} = range of $f = (2, \infty)$.

Range of f^{-1} = domain of $f = \mathbb{R}$.

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Example 29

Rewrite the equation $y = 2\log_e(x) + 3$ with x as the subject.

Solution

$$\begin{aligned} y &= 2\log_e(x) + 3 \\ \frac{y-3}{2} &= \log_e x \\ \therefore \quad x &= e^{\frac{y-3}{2}} \end{aligned}$$

Example 30

Find the inverse of the function $f\colon (1, \infty) \to \mathbb{R}$, $f(x) = 2\log_e(x - 1) + 3$. State the domain and range of the inverse.

Solution

Solve $x = 2\log_e(y - 1) + 3$ for y:

$$\begin{aligned} \frac{x-3}{2} &= \log_e(y-1) \\ y - 1 &= e^{\frac{x-3}{2}} \\ \therefore \quad y &= e^{\frac{x-3}{2}} + 1 \end{aligned}$$

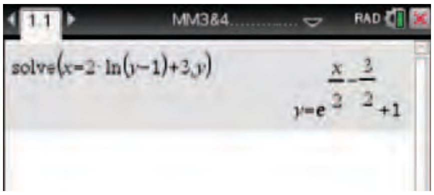
Hence $f^{-1}(x) = e^{\frac{x-3}{2}} + 1$.

Domain of f^{-1} = range of $f = \mathbb{R}$.

Range of f^{-1} = domain of $f = (1, \infty)$.

Using the TI-Nspire

Use **solve** from the **Algebra** menu as shown.

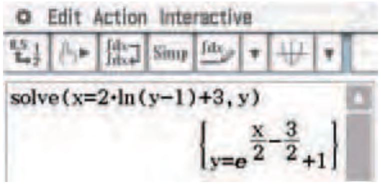


solve(x=2·ln(y−1)+3,y)

$y=e^{\frac{x}{2}-\frac{3}{2}}+1$

Using the Casio ClassPad

- Enter and highlight $x = 2\ln(y - 1) + 3$.
- Select **Interactive** > **Equation/Inequality** > **solve** and ensure the variable is set to y.



solve(x=2·ln(y−1)+3, y)

$\left\{y=e^{\frac{x}{2}-\frac{3}{2}}+1\right\}$

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Example 31

Rewrite the equation $P = Ae^{kt}$ with t as the subject.

Solution

$$P = Ae^{kt}$$

Take logarithms with base e of both sides:

$$\begin{aligned} \log_e P &= \log_e(Ae^{kt}) \\ &= \log_e A + \log_e(e^{kt}) \\ &= \log_e A + kt \\ \therefore \quad t &= \frac{1}{k}(\log_e P - \log_e A) \\ &= \frac{1}{k}\log_e\left(\frac{P}{A}\right) \end{aligned}$$

Section summary

Let $a \in \mathbb{R}^+ \setminus \{1\}$. The functions $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = a^x$ and $g\colon \mathbb{R}^+ \to \mathbb{R}$, $g(x) = \log_a x$ are inverse functions. That is, $g = f^{-1}$.

- $\log_a(a^x) = x$ for all x
- $a^{\log_a x} = x$ for all positive values of x

Exercise 5H

Example 28

1 Find the inverse of the function $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = e^x - 2$ and state the domain and range of the inverse function.

2 On the one set of axes, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$, where $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = e^{-x} + 3$.

3 On the one set of axes, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$, where $f\colon (1, \infty) \to \mathbb{R}$, $f(x) = \log_e(x - 1)$.

Example 29

4 Rewrite the equation $y = 3\log_e(x) - 4$ with x as the subject.

Example 30

5 Find the inverse of each of the following functions and state the domain and range in each case:

a $f\colon \mathbb{R}^+ \to \mathbb{R}$, $f(x) = \log_e(2x)$

b $f\colon \mathbb{R}^+ \to \mathbb{R}$, $f(x) = 3\log_e(2x) + 1$

c $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = e^x + 2$

d $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = e^{x+2}$

e $f\colon (-\frac{1}{2}, \infty) \to \mathbb{R}$, $f(x) = \log_e(2x + 1)$

f $f\colon (-\frac{2}{3}, \infty) \to \mathbb{R}$, $f(x) = 4\log_e(3x + 2)$

g $f\colon (-1, \infty) \to \mathbb{R}$, $f(x) = \log_{10}(x + 1)$

h $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 2e^{x-1}$

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6 The function f has the rule $f(x) = 1 - e^{-x}$.

- **a** Sketch the graph of f.
- **b** Find the domain of f^{-1} and find $f^{-1}(x)$.
- **c** Sketch the graph of f^{-1} on the same set of axes as the graph of f.

7 Let $f: \mathbb{R} \to \mathbb{R}$ where $f(x) = 5e^{2x} - 3$.

- **a** Sketch the graph of f.
- **b** Find the inverse function f^{-1}.
- **c** Sketch the graph of f^{-1} on the same set of axes as the graph of f.

8 Let $f: \mathbb{R}^+ \to \mathbb{R}$ where $f(x) = 2\log_e(x) + 1$.

- **a** Sketch the graph of f.
- **b** Find the inverse function f^{-1} and state the range.
- **c** Sketch the graph of f^{-1} on the same set of axes as the graph of f.

Example 31

9 Rewrite the equation $P = Ae^{-kt} + b$ with t as the subject.

10 For each of the following formulas, make the pronumeral in brackets the subject:

- **a** $y = 2\log_e(x) + 5$ (x)
- **b** $P = Ae^{-6x}$ (x)
- **c** $y = ax^n$ (n)
- **d** $y = 5 \times 10^x$ (x)
- **e** $y = 5 - 3\log_e(2x)$ (x)
- **f** $y = 6x^{2n}$ (n)
- **g** $y = \log_e(2x - 1)$ (x)
- **h** $y = 5(1 - e^{-x})$ (x)

11 For $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 2e^x - 4$:

- **a** Find the inverse function f^{-1}.
- **b** Find the coordinates of the points of intersection of the graphs of $y = f(x)$ and $y = f^{-1}(x)$.

12 For $f: (-3, \infty) \to \mathbb{R}$, $f(x) = 2\log_e(x + 3) + 4$:

- **a** Find the inverse function f^{-1}.
- **b** Find the coordinates of the points of intersection of the graphs of $y = f(x)$ and $y = f^{-1}(x)$.

13

- **a** Using a calculator, for each of the following plot the graphs of $y = f(x)$ and $y = g(x)$, together with the line $y = x$, on the one set of axes:
  - **i** $f(x) = \log_e x$ and $g(x) = e^x$
  - **ii** $f(x) = 2\log_e(x) + 3$ and $g(x) = e^{\frac{x-3}{2}}$
  - **iii** $f(x) = \log_{10} x$ and $g(x) = 10^x$
- **b** Use your answers to part **a** to comment on the relationship in general between $f(x) = a\log_b(x) + c$ and $g(x) = b^{\frac{x-c}{a}}$.

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5I Exponential growth and decay

We will show in Chapter 11 that, if the rate at which a quantity increases or decreases is proportional to its current value, then the quantity obeys the **law of exponential change**.

Let A be the quantity at time t. Then

$$A = A_0 e^{kt}$$

where A_0 is the initial quantity and k is the **rate constant**.

If $k > 0$, the model represents **growth**:

- growth of cells
- population growth
- continuously compounded interest

If $k < 0$, the model represents **decay**:

- radioactive decay
- cooling of materials

An equivalent way to write this model is as $A = A_0 b^t$, where we take $b = e^k$. In this form, growth corresponds to $b > 1$ and decay corresponds to $b < 1$.

Cell growth

Suppose a particular type of bacteria cell divides into two new cells every T_D minutes. Let N_0 be the initial number of cells of this type. After t minutes the number of cells, N, is given by

$$N = N_0 2^{\frac{t}{T_D}}$$

where T_D is called the **generation time**.

Example 32

What is the generation time of a bacterial population that increases from 5000 cells to 100 000 cells in four hours of growth?

Solution

In this example, $N_0 = 5000$ and $N = 100\,000$ when $t = 240$.

Hence $100\,000 = 5000 \times 2^{\frac{240}{T_D}}$

$$20 = 2^{\frac{240}{T_D}}$$

Thus $T_D = \dfrac{240}{\log_2 20} \approx 55.53$ (correct to two decimal places).

The generation time is approximately 55.53 minutes.

Radioactive decay

Radioactive materials decay such that the amount of radioactive material, A, present at time t (in years) is given by

$$A = A_0 e^{-kt}$$

where A_0 is the initial amount and k is a positive constant that depends on the type of material. A radioactive substance is often described in terms of its **half-life**, which is the time required for half the material to decay.

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Example 33

After 1000 years, a sample of radium-226 has decayed to 64.7% of its original mass. Find the half-life of radium-226.

Solution

We use the formula $A = A_0e^{-kt}$. When $t = 1000$, $A = 0.647A_0$. Thus

$$\begin{aligned} 0.647A_0 &= A_0e^{-1000k} \\ 0.647 &= e^{-1000k} \\ -1000k &= \log_e 0.647 \\ k &= \frac{-\log_e 0.647}{1000} \approx 0.000435 \end{aligned}$$

To find the half-life, we consider when $A = \frac{1}{2}A_0$:

$$\begin{aligned} A_0e^{-kt} &= \tfrac{1}{2}A_0 \\ e^{-kt} &= \tfrac{1}{2} \\ -kt &= \log_e(\tfrac{1}{2}) \\ t &= -\frac{\log_e(\frac{1}{2})}{k} \approx 1591.95 \end{aligned}$$

The half-life of radium-226 is approximately 1592 years.

Population growth

It is sometimes possible to model population growth through exponential models.

Example 34

The population of a town was 8000 at the beginning of 2007 and 15 000 at the end of 2014. Assume that the growth is exponential.

a Find the population at the end of 2016.

b In what year will the population be double that of 2014?

Solution

Let P be the population at time t years (measured from 1 January 2007). Then

$$P = 8000e^{kt}$$

At the end of 2014, $t = 8$ and $P = 15\,000$. Therefore

$$\begin{aligned} 15\,000 &= 8000e^{8k} \\ \frac{15}{8} &= e^{8k} \\ k &= \frac{1}{8}\log_e\left(\frac{15}{8}\right) \approx 0.079 \end{aligned}$$

The rate of increase is 7.9% per annum.

Note: The approximation 0.079 was not used in the calculations which follow. The value for k was held in the calculator.

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a When $t = 10$,
$$\begin{aligned} P &= 8000e^{10k} \\ &\approx 17\,552.6049 \\ &\approx 17\,550 \end{aligned}$$

The population is approximately 17 550.

b When does $P = 30\,000$? Consider the equation

$$\begin{aligned} 30\,000 &= 8000e^{kt} \\ \frac{30\,000}{8000} &= e^{kt} \\ \frac{15}{4} &= e^{kt} \\ \therefore \quad t &= \frac{1}{k}\log_e\left(\frac{15}{4}\right) \\ &\approx 16.82 \end{aligned}$$

The population reaches 30 000 approximately 16.82 years after the beginning of 2007, i.e. during the year 2023.

Example 35

There are approximately ten times as many red kangaroos as grey kangaroos in a certain area. If the population of grey kangaroos increases at a rate of 11% per annum while that of the red kangaroos decreases at 5% per annum, find how many years must elapse before the proportions are reversed, assuming the same rates continue to apply.

Solution

Let G_0 be the population of grey kangaroos at the start.

Then the number of grey kangaroos after n years is $G = G_0(1.11)^n$, and the number of red kangaroos after n years is $R = 10G_0(0.95)^n$.

When the proportions are reversed:

$$\begin{aligned} G &= 10R \\ G_0(1.11)^n &= 10 \times 10G_0(0.95)^n \\ (1.11)^n &= 100(0.95)^n \end{aligned}$$

Taking $\log_e$ of both sides:

$$\begin{aligned} \log_e((1.11)^n) &= \log_e(100(0.95)^n) \\ n\log_e 1.11 &= \log_e 100 + n\log_e 0.95 \\ \therefore \quad n &= \frac{\log_e 100}{\log_e 1.11 - \log_e 0.95} \\ &\approx 29.6 \end{aligned}$$

i.e. the proportions of the kangaroo populations will be reversed after 30 years.

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Section summary

There are many situations in which a varying quantity can be modelled by an exponential function. Let A be the quantity at time t. Then

$$A = A_0 e^{kt}$$

where A_0 is the initial quantity and k is a constant. Growth corresponds to $k > 0$, and decay corresponds to $k < 0$.

Exercise 5I

Skillsheet
Example 32

1 A population of 1000 E. coli bacteria doubles every 15 minutes.

a Determine the formula for the number of bacteria at time t minutes.

b How long will it take for the population to reach 10 000? (Give your answer to the nearest minute.)

2 In the initial period of its life a particular species of tree grows in the manner described by the rule $d = d_0 10^{mt}$ where d is the diameter (in cm) of the tree t years after the beginning of this period. The diameter is 52 cm after 1 year, and 80 cm after 3 years. Calculate the values of the constants d_0 and m.

3 The number of people, N, who have a particular disease at time t years is given by $N = N_0 e^{kt}$.

a If the number is initially 20 000 and the number decreases by 20% each year, find:

i the value of N_0 **ii** the value of k.

b How long does it take until only 5000 people are infected?

Example 33

4 Polonium-210 is a radioactive substance. The decay of polonium-210 is described by the formula $M = M_0 e^{-kt}$, where M is the mass in grams of polonium-210 left after t days, and M_0 and k are constants. At $t = 0$, $M = 10$ g and at $t = 140$, $M = 5$ g.

a Find the values of M_0 and k.

b What will be the mass of the polonium-210 after 70 days?

c After how many days is the mass remaining 2 g?

5 A quantity A of radium at time t years is given by $A = A_0 e^{-kt}$, where k is a positive constant and A_0 is the amount of radium at time $t = 0$.

a Given that $A = \frac{1}{2}A_0$ when $t = 1690$ years, calculate k.

b After how many years does only 20% of the original amount remain? Give your answer to the nearest year.

6 The half-life of plutonium-239 is 24 000 years. If 20 grams are present now, how long will it take until only 20% of the original sample remains? (Give your answer to the nearest year.)

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7 Carbon-14 is a radioactive substance with a half-life of 5730 years. It is used to determine the age of ancient objects. A Babylonian cloth fragment now has 40% of the carbon-14 that it contained originally. How old is the fragment of cloth?

Example 34 **8** The population of a town was 10 000 at the beginning of 2002 and 15 000 at the end of 2014. Assume that the growth is exponential.

a Find the population at the end of 2017.

b In what year will the population be double that of 2014?

Example 35 **9** There are approximately five times as many magpies as currawongs in a certain area. If the population of currawongs increases at a rate of 12% per annum while that of the magpies decreases at 6% per annum, find how many years must elapse before the proportions are reversed, assuming the same rates continue to apply.

10 The pressure in the Earth's atmosphere decreases exponentially as you rise above the surface. The pressure in millibars at a height of h kilometres is given approximately by the function $P(h) = 1000 \times 10^{-0.05428h}$.

a Find the pressure at a height of 4 km. (Give your answer to the nearest millibar.)

b Find the height at which the pressure is 450 millibars. (Give your answer to the nearest metre.)

11 A biological culture contains 500 000 bacteria at 12 p.m. on Sunday. The culture increases by 10% every hour. At what time will the culture exceed 4 million bacteria?

12 When a liquid is placed into a refrigerator, its temperature T°C at time t minutes is given by the formula $T = T_0 e^{-kt}$. The temperature is initially 100°C and drops to 40°C in 5 minutes. Find the temperature of the liquid after 15 minutes.

13 The number of bacteria in a certain culture at time t weeks is given by the rule $N = N_0 e^{kt}$. If when $t = 2$, $N = 101$ and when $t = 4$, $N = 203$, calculate the values of N_0 and k.

14 Five kilograms of sugar is gradually dissolved in a vat of water. After t hours, the amount, S kg, of undissolved sugar remaining is given by $S = 5 \times e^{-kt}$.

a Calculate k given that $S = 3.2$ when $t = 2$.

b At what time will there be 1 kg of sugar remaining?

15 The number of bacteria, N, in a culture increases exponentially with time according to the rule $N = a \times b^t$, where time t is measured in hours. When observation started, there were 1000 bacteria, and 5 hours later there were 15 000 bacteria.

a Find the values of a and b.

b Find, to the nearest hour, when there were 5000 bacteria.

c Find, to the nearest hour, when the number of bacteria first exceeds 1 000 000.

d How many bacteria would there be 12 hours after the first observation?

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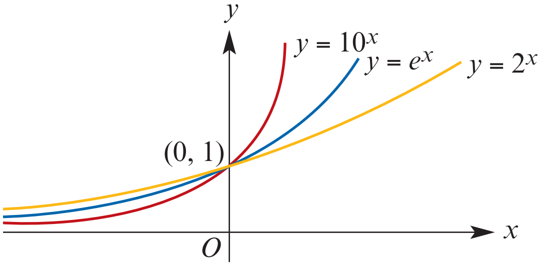
Chapter summary

Spreadsheet AS Nrich

- Sketch graphs of the form $y = a^x$ and transformations of these graphs.



- Index laws

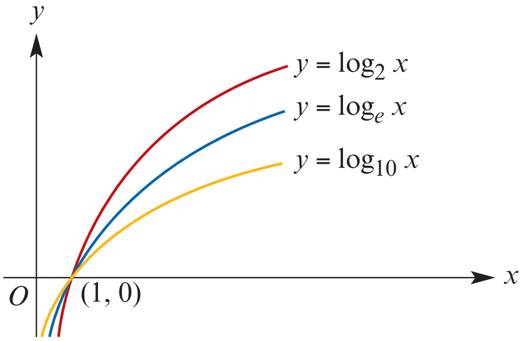
$$a^m \times a^n = a^{m+n} \qquad a^m \div a^n = a^{m-n} \qquad (a^m)^n = a^{mn}$$

- Logarithms

For $a \in \mathbb{R}^+ \setminus \{1\}$, the logarithm function with base a is defined as follows:

$$a^x = y \quad \text{is equivalent to} \quad \log_a y = x$$

- Sketch graphs of the form $y = \log_a x$ and transformations of these graphs.



- Logarithm laws

$$\log_a(mn) = \log_a m + \log_a n \qquad \log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$$

$$\log_a\left(\frac{1}{n}\right) = -\log_a n \qquad \log_a(m^p) = p\log_a m$$

- Change of base

$$\log_a x = \frac{\log_b x}{\log_b a} \quad \text{and} \quad a^x = b^{(\log_b a)x}$$

- Inverse functions

The inverse function of $f : \mathbb{R} \to \mathbb{R}$, $f(x) = a^x$ is $f^{-1} : \mathbb{R}^+ \to \mathbb{R}$, $f^{-1}(x) = \log_a x$.

  - $\log_a(a^x) = x$ for all $x \in \mathbb{R}$
  - $a^{\log_a x} = x$ for all $x \in \mathbb{R}^+$

- Law of exponential change

Assume that the rate at which the quantity A increases or decreases is proportional to its current value. Then the value of A at time t is given by

$$A = A_0e^{kt}$$

where A_0 is the initial quantity and k is a constant. Growth corresponds to $k > 0$, and decay corresponds to $k < 0$.

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Technology-free questions

1 Sketch the graph of each of the following. Label asymptotes and axis intercepts.

a $f(x) = e^x - 2$ **b** $g(x) = 10^{-x} + 1$ **c** $h(x) = \frac{1}{2}(e^x - 1)$
d $f(x) = 2 - e^{-x}$ **e** $f(x) = \log_e(2x + 1)$ **f** $h(x) = \log_e(x - 1) + 1$
g $g(x) = -\log_e(x - 1)$ **h** $f(x) = -\log_e(1 - x)$

2 **a** For $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = e^{2x} - 1$, find f^{-1}.
b For $f\colon (2, \infty) \to \mathbb{R}$, $f(x) = 3\log_e(x - 2)$, find f^{-1}.
c For $f\colon (-1, \infty) \to \mathbb{R}$, $f(x) = \log_{10}(x + 1)$, find f^{-1}.
d For $f\colon \mathbb{R}^+ \to \mathbb{R}$, $f(x) = 2^x + 1$, find f^{-1}.

3 For each of the following, find y in terms of x:

a $\log_e y = \log_e x + 2$ **b** $\log_{10} y = \log_{10} x + 1$ **c** $\log_2 y = 3\log_2 x + 4$
d $\log_{10} y = -1 + 5\log_{10} x$ **e** $\log_e y = 3 - \log_e x$ **f** $\log_e y = 2x - 3$

4 Solve each of the following equations for x, expressing your answers in terms of logarithms with base e:

a $3^x = 11$ **b** $2^x = 0.8$ **c** $2^x = 3^{x+1}$

5 Solve each of the following for x:

a $2^{2x} - 2^x - 2 = 0$ **b** $\log_e(3x - 1) = 0$
c $\log_{10}(2x) + 1 = 0$ **d** $10^{2x} - 7 \times 10^x + 12 = 0$

6 The graph of the function with rule $y = 3\log_2(x + 1) + 2$ intersects the axes at the points $(a, 0)$ and $(0, b)$. Find the exact values of a and b.

7 The graph of $y = 5\log_{10}(x + 1)$ passes through the point $(k, 6)$. Find the value of k.

8 Find the exact value of x for which $4e^{3x} = 287$.

9 Find the value of x in terms of a, where $3\log_a x = 3 + \log_a 8$.

10 For $f\colon (4, \infty) \to \mathbb{R}$, $f(x) = \log_3(x - 4)$, state the domain of the inverse function f^{-1}.

11 The graph of the function with rule $f(x) = e^{2x} - 3ke^x + 5$ intersects the axes at $(0, 0)$ and $(a, 0)$ and has a horizontal asymptote at $y = b$. Find the exact values of a, b and k.

12 Let $f\colon \mathbb{R} \to \mathbb{R}$ where $f(x) = e^{3x} - 4$.

a Find the rule and domain of the inverse function f^{-1}. **b** Find $f(-f^{-1}(3x))$.

13 Let $f(x) = e^x + e^{-x}$ and $g(x) = e^x - e^{-x}$.

a Show that f is an even function. **b** Find $f(u) + f(-u)$.
c Find $f(u) - f(-u)$. **d** Find $[f(u)]^2 - 2$.
e Show that g is an odd function.
f Find $f(x) + g(x)$, $f(x) - g(x)$ and $f(x) \cdot g(x)$.

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Multiple-choice questions

1 If $4\log_b(x^2) = \log_b 16 + 8$, then x is equal to

A b^4 **B** ± 6 **C** $\pm\sqrt{2}\,b$ **D** 2^6 **E** $2b^4$

2 The expression $\log_e(4e^{3x})$ is equal to

A $\log_e(e^{12x})$ **B** $\log_e(12) + x$ **C** $3x\log_e(4)$ **D** $\log_e(4) + 3x$ **E** $12x$

3 The expression $3^{\log_3(x-4)}$ is equal to

A $\dfrac{x}{4}$ **B** $x - 4$ **C** $3(x - 4)$ **D** $3^x - 3^4$ **E** $\log_3 x - \log_3 4$

4 Consider the three functions

$$f\colon A \to \mathbb{R},\ f(x) = e^{2x}, \quad g\colon B \to \mathbb{R},\ g(x) = \frac{1}{x+1}, \quad h\colon C \to \mathbb{R},\ h(x) = e^{2x} + \frac{1}{x+1}$$

where A, B and C are the largest domains for which f, g and h respectively are defined. Which one of the following statements is true?

A $A \neq C$ and $\operatorname{ran}(g) = \operatorname{ran}(h)$ **B** $A = B$ and $\operatorname{ran}(f) \neq \operatorname{ran}(h)$

C $A \neq C$ and $\operatorname{ran}(f) = \operatorname{ran}(h)$ **D** $B = C$ and $\operatorname{ran}(g) = \operatorname{ran}(h)$

E $B = C$ and $\operatorname{ran}(g) \neq \operatorname{ran}(h)$

5 If $x = 5$ is a solution of the equation $\log_{10}(kx - 3) = 2$, then the exact value of k is

A $\dfrac{103}{5}$ **B** $\dfrac{\log_{10} 2 + 3}{5}$ **C** 2 **D** 5 **E** 21

6 $3^{4\log_3(x)+\log_3(4x)}$ is equal to

A $8x$ **B** $x^4 + 4x$ **C** $4x^5$ **D** 3^{8x} **E** $\log_3(4x^5)$

7 The solution of the equation $3x = 10^{-0.3x}$ is closest to

A 0.83 **B** 0.28 **C** 0 **D** 0.30 **E** 0.91

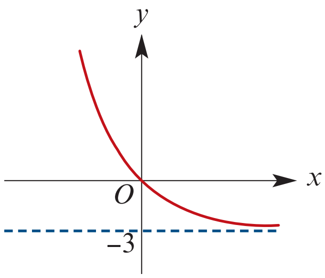
8 The graph of the function with rule $y = ae^{-x} + b$ is shown on the right.

The values of a and b respectively are

A $3,\ -3$ **B** $-3,\ 3$

C $-3,\ -3$ **D** $0,\ -3$

E $-3,\ 0$



9 Which one of the following statements is not true of the graph of the function $f\colon \mathbb{R}^+ \to \mathbb{R},\ f(x) = \log_5 x$?

A The domain is $\mathbb{R}^+$.

B The range is $\mathbb{R}$.

C It passes through the point $(5, 0)$.

D It has a vertical asymptote with equation $x = 0$.

E The slope of the tangent at any point on the graph is positive.

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10 If $3\log_2 x - 7\log_2(x-1) = 2 + \log_2 y$, then y is equal to

A $\dfrac{3x}{28(x-1)}$ **B** $\dfrac{1}{4x^4}$ **C** $3 - 4x$

D $\dfrac{x^3}{4(x-1)^7}$ **E** $x^3 - (x-1)^7 - 4$

11 The graph of the function $f(x) = e^{2x} - 12$ intersects the graph of $g(x) = -e^x$ where

A $x = \log_e 3$ **B** $x = \log_e 2$ **C** $x = \log_e 7$ **D** $x = \log_e 4$ **E** $x = \log_e 5$

12 Let the rule for a function g be $g(x) = \log_e((x-4)^2)$. For the function g, the maximal domain and range are

A $\mathbb{R},\ \mathbb{R}$ **B** $(4, \infty),\ \mathbb{R}^+$ **C** $\mathbb{R} \setminus \{4\},\ \mathbb{R}$

D $\mathbb{R} \setminus \{2\},\ \mathbb{R} \setminus \{0\}$ **E** $(-\infty, 4),\ \mathbb{R}$

13 The maximal domain D of the function $f\colon D \to \mathbb{R}$, $f(x) = \log_e((x-3)^2) + 6$ is

A $(3, \infty)$ **B** $[3, \infty)$ **C** $\mathbb{R} \setminus \{3\}$ **D** $\mathbb{R}^+$ **E** $(\infty, 3)$

14 The function $f\colon [a, \infty) \to \mathbb{R}$, $f(x) = \log_e(x^2)$ will have an inverse function if

A $a \in (-\infty, 0)$ **B** $a \in (-\infty, -1)$ **C** $a \in (-\infty, 1)$

D $a \in (0, \infty)$ **E** $a \in (-1, \infty)$

15 The inverse of the function $f\colon \mathbb{R}^+ \to \mathbb{R}$, $f(x) = e^{3x+4}$ is

A $f^{-1}\colon \mathbb{R} \to \mathbb{R},\ f^{-1}(x) = -3\log_e(3x-4)$ **B** $f^{-1}\colon (e^4, \infty) \to \mathbb{R},\ f^{-1}(x) = \dfrac{\log_e(x) - 4}{3}$

C $f^{-1}\colon \mathbb{R}^+ \to \mathbb{R},\ f^{-1}(x) = -3\log_e\left(\dfrac{x-4}{3}\right)$ **D** $f^{-1}\colon (e^4, \infty) \to \mathbb{R},\ f^{-1}(x) = \log_e(3x-4)$

E $f^{-1}\colon (-\frac{4}{3}, \infty) \to \mathbb{R},\ f^{-1}(x) = \log_e(3x) - 4$

16 If $f(x) = 2\log_e(3x)$ and $f(6x) = \log_e(y)$, then

A $y = 18x$ **B** $y = \dfrac{x}{3}$ **C** $y = 6x^2$ **D** $y = 324x^2$ **E** $y = 36x^2$

Extended-response questions

1 A liquid cools from its original temperature of 90°C to a temperature of T°C in x minutes. Given that $T = 90(0.98)^x$, find:

a the value of T when $x = 10$

b the value of x when $T = 27$.

2 The population of a village at the beginning of the year 1800 was 240. The population increased so that, after a period of n years, the new population was $240(1.06)^n$. Find:

a the population at the beginning of 1820

b the year in which the population first reached 2500.

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3 The value, \$$V$, of a particular car can be modelled by the equation $V = ke^{-\lambda t}$, where t years is the age of the car. The car's original price was \$22 497, and after 1 year it is valued at \$18 000.

a State the value of k and calculate λ, giving your answer to two decimal places.

b Find the value of the car when it is 3 years old.

4 The value, \$$M$, of a particular house during the period 1988 to 1994 can be modelled by the equation $M = Ae^{-pt}$, where t is the time in years after 1 January 1988. The value of the house on 1 January 1988 was \$65 000 and its value on 1 January 1989 was \$61 000.

a State the value of A and calculate the value of p, correct to two significant figures.

b What was the value of the house in 1993? Give your answer to the nearest \$100.

5 There are two species of insects living in a suburb: the *Asla bibla* and the *Cutus pius*. The number of *Asla bibla* alive at time t days after 1 January 2000 is given by

$$N_A(t) = 10\,000 + 1000t, \quad 0 \le t \le 15$$

The number of *Cutus pius* alive at time t days after 1 January 2000 is given by

$$N_C(t) = 8000 + 3 \times 2^t, \quad 0 \le t \le 15$$

a With a calculator, plot the graphs of $y = N_A(t)$ and $y = N_C(t)$ on the one screen.

b **i** Find the coordinates of the point of intersection of the two graphs.

ii At what time is $N_A(t) = N_C(t)$?

iii What is the number of each species of insect at this time?

c **i** Show that $N_A(t) = N_C(t)$ if and only if $t = 3\log_2 10 + \log_2\left(\frac{2+t}{3}\right)$.

ii Plot the graphs of $y = x$ and $y = 3\log_2 10 + \log_2\left(\frac{2+x}{3}\right)$ and find the coordinates of the point of intersection.

d It is found by observation that the model for *Cutus pius* does not quite work. It is known that the model for the population of *Asla bibla* is satisfactory. The form of the model for *Cutus pius* is $N_C(t) = 8000 + c \times 2^t$. Find the value of c, correct to two decimal places, if it is known that $N_A(15) = N_C(15)$.

6 The number of a type of bacteria is modelled by the formula $n = A(1 - e^{-Bt})$, where n is the size of the population at time t hours, and A and B are positive constants.

a When $t = 2$, $n = 10\,000$ and when $t = 4$, $n = 15\,000$.

i Show that $2e^{-4B} - 3e^{-2B} + 1 = 0$.

ii Use the substitution $a = e^{-2B}$ to show that $2a^2 - 3a + 1 = 0$.

iii Solve this equation for a.

iv Find the exact value of B.

v Find the exact value of A.

b Sketch the graph of n against t.

c After how many hours is the population of bacteria 18 000?

7 The barometric pressure P (in centimetres of mercury) at a height h km above sea level is given by $P = 75(10^{-0.15h})$. Find:

a P when $h = 0$ **b** P when $h = 10$ **c** h when $P = 60$.

8 A radioactive substance is decaying such that the amount, A g, at time t years is given by the formula $A = A_0 e^{kt}$. If when $t = 1$, $A = 60.7$ and when $t = 6$, $A = 5$, find the values of the constants A_0 and k.

9 In a chemical reaction the amount, x g, of a substance that has reacted is given by $x = 8(1 - e^{-0.2t})$, where t is the time in minutes from the beginning of the reaction.

a Sketch the graph of x against t.

b Find the amount of substance that has reacted after:

i 0 minutes **ii** 2 minutes **iii** 10 minutes.

c Find the time when exactly 7 g of the substance has reacted.

10 Newton's law of cooling for an object in a medium of constant temperature states

$$T - T_s = (T_0 - T_s)\, e^{-kt}$$

where:

- T is the temperature (in °C) of the object at time t (in minutes)
- T_s is the temperature of the surrounding medium
- T_0 is the initial temperature of the object.

An egg at 96°C is placed to cool in a sink of water at 15°C. After 5 minutes the egg's temperature is 40°C. (Assume that the temperature of the water does not change.)

a Find the value of k.

b Find the temperature of the egg when $t = 10$.

c How long does it take for the egg to reach a temperature of 30°C?

11 The population of a colony of small, interesting insects is modelled by the following function:

$$N(t) = \begin{cases} 20e^{0.2t} & \text{for } 0 \le t \le 50 \\ 20e^{10} & \text{for } 50 < t \le 70 \\ 10e^{10}(e^{70-t} + 1) & \text{for } t > 70 \end{cases}$$

where t is the number of days.

a Sketch the graph of $N(t)$ against t.

b Find:

i $N(10)$ **ii** $N(40)$ **iii** $N(60)$ **iv** $N(80)$

c Find the number of days for the population to reach:

i 2968 **ii** 21 932

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Chapter 6

Circular functions

Objectives

- To measure angles in **degrees** and **radians**.
- To define the circular functions **sine**, **cosine** and **tangent**.
- To explore the **symmetry properties** of circular functions.
- To find **exact values** of circular functions.
- To **sketch graphs** of circular functions.
- To **solve equations** involving circular functions.
- To apply circular functions in modelling **periodic motion**.

Following on from our study of polynomial, exponential and logarithmic functions, we meet a further three important functions in this chapter. Again we use the notation developed in Chapter 1 for describing functions and their properties.

In this chapter we revise and extend our consideration of the functions sine, cosine and tangent. The first two of these functions have the real numbers as their domain, and the third the real numbers without the odd multiples of $\frac{\pi}{2}$.

An important property of these three functions is that they are periodic. That is, they each repeat their values in regular intervals or periods. In general, a function f is **periodic** if there is a positive constant a such that $f(x + a) = f(x)$. The sine and cosine functions each have period 2π, while the tangent function has period π.

The sine and cosine functions are used to model wave motion, and are therefore central to the application of mathematics to any problem in which periodic motion is involved – from the motion of the tides and ocean waves to sound waves and modern telecommunications.

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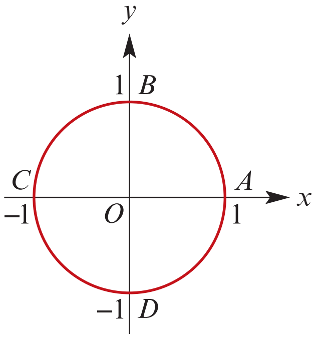
6A Measuring angles in degrees and radians



The diagram shows a **unit circle**, i.e. a circle of radius 1 unit.

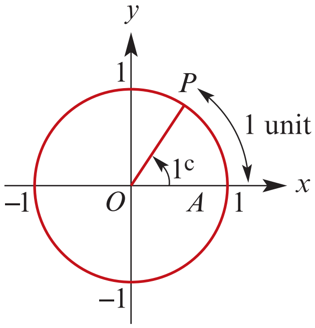
$$\begin{aligned}\text{The circumference of the unit circle} &= 2\pi \times 1\\ &= 2\pi \text{ units}\end{aligned}$$

Thus, the distance in an anticlockwise direction around the circle from

$$\begin{aligned} A \text{ to } B &= \frac{\pi}{2} \text{ units}\\ A \text{ to } C &= \pi \text{ units}\\ A \text{ to } D &= \frac{3\pi}{2} \text{ units}\end{aligned}$$

Definition of a radian

In moving around the circle a distance of 1 unit from A to P, the angle POA is defined. The measure of this angle is 1 radian.



One **radian** (written 1^c) is the angle subtended at the centre of the unit circle by an arc of length 1 unit.

Note: Angles formed by moving **anticlockwise** around the unit circle are defined as **positive**; those formed by moving **clockwise** are defined as **negative**.

Degrees and radians

The angle, in radians, swept out in one revolution of a circle is $2\pi^c$.

$$2\pi^c = 360^\circ$$

$$\therefore \quad \pi^c = 180^\circ$$

$$\therefore \quad 1^c = \frac{180^\circ}{\pi} \quad \text{or} \quad 1^\circ = \frac{\pi^c}{180}$$

Example 1

Convert 30° to radians.

Solution

$$1^\circ = \frac{\pi^c}{180}$$

$$\therefore \quad 30^\circ = \frac{30 \times \pi}{180} = \frac{\pi^c}{6}$$

Explanation

Multiply by $\frac{\pi}{180}$ and simplify by cancelling.

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Example 2

Convert $\frac{\pi^c}{4}$ to degrees.

Solution

$$1^c = \frac{180^\circ}{\pi}$$

$$\therefore \quad \frac{\pi^c}{4} = \frac{\pi \times 180}{4 \times \pi} = 45^\circ$$

Explanation

Multiply by $\frac{180}{\pi}$ and simplify by cancelling.

Note: Often the symbol for radians, ^{c}, is omitted.
For example, the angle 45° is written as $\frac{\pi}{4}$ rather than $\frac{\pi^c}{4}$.

Section summary

- One **radian** (written 1^c) is the angle subtended at the centre of the unit circle by an arc of length 1 unit.
- To convert:
  - degrees to radians, multiply by $\frac{\pi}{180}$
  - radians to degrees, multiply by $\frac{180}{\pi}$.

Exercise 6A

Example 1

1 Express the following angles in radian measure in terms of π:

a 50° **b** 136° **c** 250°
d 340° **e** 420° **f** 490°

Example 2

2 Express, in degrees, the angles with the following radian measures:

a $\frac{\pi}{3}$ **b** $\frac{5\pi}{6}$ **c** $\frac{4\pi}{3}$
d $\frac{7\pi}{9}$ **e** 3.5π **f** $\frac{7\pi}{5}$

3 Use a calculator to convert each of the following angles from radians to degrees:

a 0.8 **b** 1.64 **c** 2.5
d 3.96 **e** 4.18 **f** 5.95

4 Use a calculator to express each of the following in radian measure. (Give your answer correct to two decimal places.)

a 37° **b** 74° **c** 115°
d 122.25° **e** 340° **f** 132.5°

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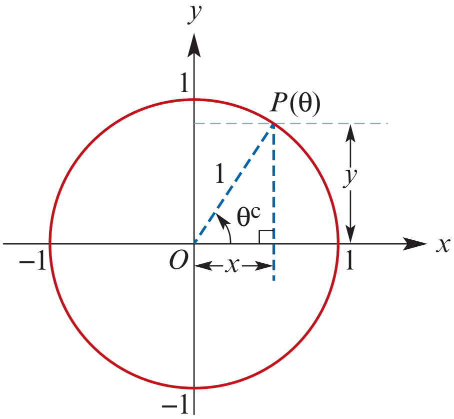
6B Defining circular functions: sine, cosine and tangent

The point P on the unit circle corresponding to an angle θ is written $P(\theta)$.

The x-coordinate of $P(\theta)$ is determined by the angle θ. Similarly, the y-coordinate of $P(\theta)$ is determined by the angle θ. So we can define two functions, called sine and cosine, as follows:



The x-coordinate of $P(\theta)$ is given by

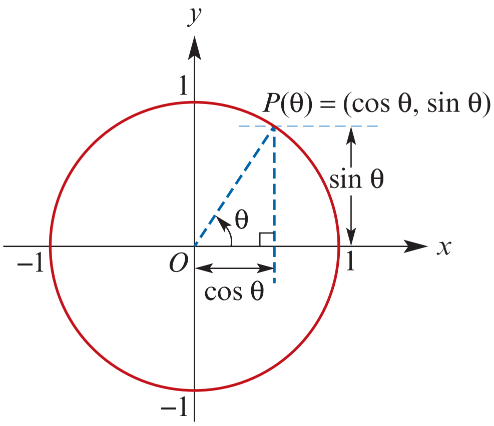
$$x = \text{cosine}\,\theta, \quad \text{for } \theta \in \mathbb{R}$$

The y-coordinate of $P(\theta)$ is given by

$$y = \text{sine}\,\theta, \quad \text{for } \theta \in \mathbb{R}$$

These functions are usually written in an abbreviated form as follows:

$$x = \cos\theta$$
$$y = \sin\theta$$



Hence the coordinates of $P(\theta)$ are $(\cos\theta, \sin\theta)$.

Note: Adding 2π to the angle results in a return to the same point on the unit circle. Thus $\cos(2\pi + \theta) = \cos\theta$ and $\sin(2\pi + \theta) = \sin\theta$.

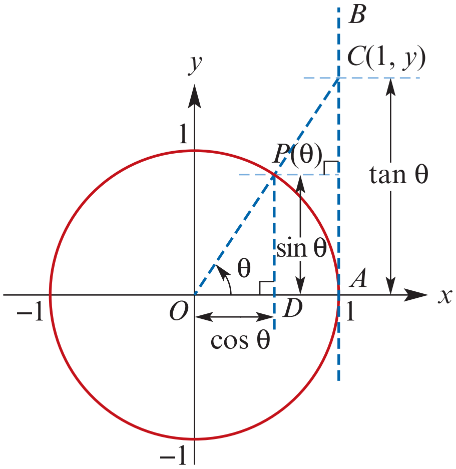
Again consider the unit circle.

If we draw a tangent to the unit circle at A, then the y-coordinate of C, the point of intersection of the line OP and the tangent, is called **tangent** θ (abbreviated to $\tan\theta$).

By considering the similar triangles OPD and OCA:

$$\frac{\tan\theta}{1} = \frac{\sin\theta}{\cos\theta}$$

$$\therefore \quad \tan\theta = \frac{\sin\theta}{\cos\theta}$$



Note that $\tan\theta$ is undefined when $\cos\theta = 0$. The domain of tan is $\mathbb{R} \setminus \{\,\theta : \cos\theta = 0\,\}$ and so $\tan\theta$ is undefined when $\theta = \pm\frac{\pi}{2}, \ \pm\frac{3\pi}{2}, \ \pm\frac{5\pi}{2}, \ \dots$

Note: Adding π to the angle does not change the line OP. Thus $\tan(\pi + \theta) = \tan\theta$.

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From the periodicity of the circular functions:

- $\sin(2k\pi + \theta) = \sin\theta$, for all integers k
- $\cos(2k\pi + \theta) = \cos\theta$, for all integers k
- $\tan(k\pi + \theta) = \tan\theta$, for all integers k.

Example 3

Evaluate each of the following:

a $\sin\left(\frac{3\pi}{2}\right)$ **b** $\sin\left(-\frac{3\pi}{2}\right)$ **c** $\cos\left(\frac{5\pi}{2}\right)$ **d** $\cos\left(-\frac{\pi}{2}\right)$

e $\cos\left(\frac{23\pi}{2}\right)$ **f** $\sin\left(\frac{55\pi}{2}\right)$ **g** $\tan(55\pi)$ **h** $\tan\left(\frac{15\pi}{2}\right)$

Solution	*Explanation*
a $\sin\left(\frac{3\pi}{2}\right) = -1$	since $P\left(\frac{3\pi}{2}\right)$ has coordinates $(0, -1)$.
b $\sin\left(-\frac{3\pi}{2}\right) = 1$	since $P\left(-\frac{3\pi}{2}\right)$ has coordinates $(0, 1)$.
c $\cos\left(\frac{5\pi}{2}\right) = \cos\left(2\pi + \frac{\pi}{2}\right) = 0$	since $P\left(\frac{\pi}{2}\right)$ has coordinates $(0, 1)$.
d $\cos\left(-\frac{\pi}{2}\right) = 0$	since $P\left(-\frac{\pi}{2}\right)$ has coordinates $(0, -1)$.
e $\cos\left(\frac{23\pi}{2}\right) = \cos\left(10\pi + \frac{3\pi}{2}\right) = 0$	since $P\left(\frac{3\pi}{2}\right)$ has coordinates $(0, -1)$.
f $\sin\left(\frac{55\pi}{2}\right) = \sin\left(26\pi + \frac{3\pi}{2}\right) = -1$	since $P\left(\frac{3\pi}{2}\right)$ has coordinates $(0, -1)$.
g $\tan(55\pi) = 0$	since $\tan(k\pi) = 0$, for any integer k.
h $\tan\left(\frac{15\pi}{2}\right)$ is undefined	since $\tan\left(\frac{k\pi}{2}\right)$ is undefined for any odd integer k.

Example 4

Evaluate using a calculator. (Give answers to two decimal places.)

a tan 1.3 **b** sin 1.8 **c** cos(−2.6) **d** sin 3.8

e tan(−2.8) **f** tan 59° **g** tan 138°

Solution

a $\tan 1.3 = 3.60$ **b** $\sin 1.8 = 0.97$

c $\cos(-2.6) = -0.86$ **d** $\sin 3.8 = -0.61$

e $\tan(-2.8) = 0.36$ **f** $\tan 59° = 1.66$

g $\tan 138° = -0.90$

Explanation

Your calculator should be in radian mode for **a**–**e** and in degree mode for **f** and **g**.

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▶ Exact values of circular functions

A calculator can be used to find the values of the circular functions for different values of θ. For many values of θ the calculator gives an approximation. We consider some values of θ such that sin, cos and tan can be calculated exactly.

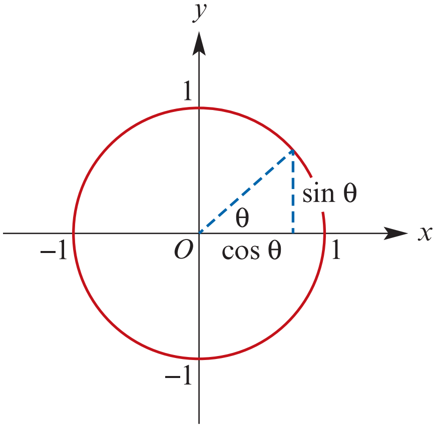
Exact values for 0 (0°) and $\frac{\pi}{2}$ (90°)

From the unit circle:

$$\sin 0^\circ = 0 \qquad \sin 90^\circ = 1$$

$$\cos 0^\circ = 1 \qquad \cos 90^\circ = 0$$

$$\tan 0^\circ = 0 \qquad \tan 90^\circ \text{ is undefined}$$



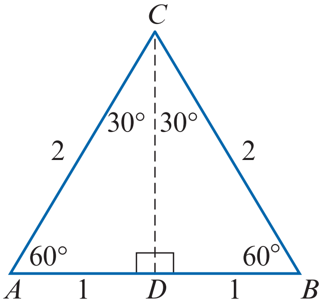
Exact values for $\frac{\pi}{6}$ (30°) and $\frac{\pi}{3}$ (60°)

Consider an equilateral triangle ABC of side length 2 units.
In $\triangle ACD$, by Pythagoras' theorem, $CD = \sqrt{AC^2 - AD^2} = \sqrt{3}$.

$$\sin 30^\circ = \frac{AD}{AC} = \frac{1}{2} \qquad \sin 60^\circ = \frac{CD}{AC} = \frac{\sqrt{3}}{2}$$

$$\cos 30^\circ = \frac{CD}{AC} = \frac{\sqrt{3}}{2} \qquad \cos 60^\circ = \frac{AD}{AC} = \frac{1}{2}$$

$$\tan 30^\circ = \frac{AD}{CD} = \frac{1}{\sqrt{3}} \qquad \tan 60^\circ = \frac{CD}{AD} = \frac{\sqrt{3}}{1} = \sqrt{3}$$



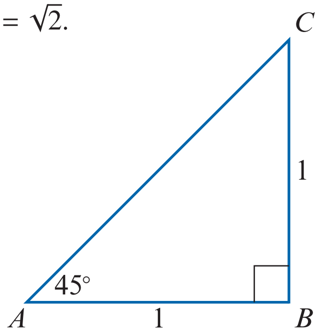
Exact values for $\frac{\pi}{4}$ (45°)

For the triangle ABC shown on the right, we have $AC = \sqrt{1^2 + 1^2} = \sqrt{2}$.

$$\sin 45^\circ = \frac{BC}{AC} = \frac{1}{\sqrt{2}}$$

$$\cos 45^\circ = \frac{AB}{AC} = \frac{1}{\sqrt{2}}$$

$$\tan 45^\circ = \frac{BC}{AB} = 1$$



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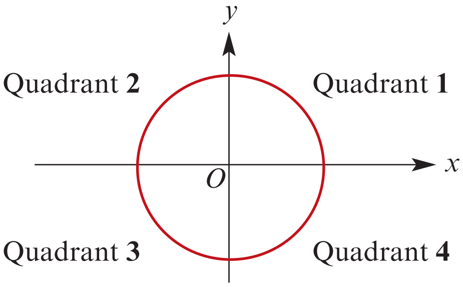
Symmetry properties of circular functions

The coordinate axes divide the unit circle into four quadrants. The quadrants can be numbered, anticlockwise from the positive direction of the x-axis, as shown.

Using symmetry, we can determine relationships between the circular functions for angles in different quadrants:



Quadrant 2

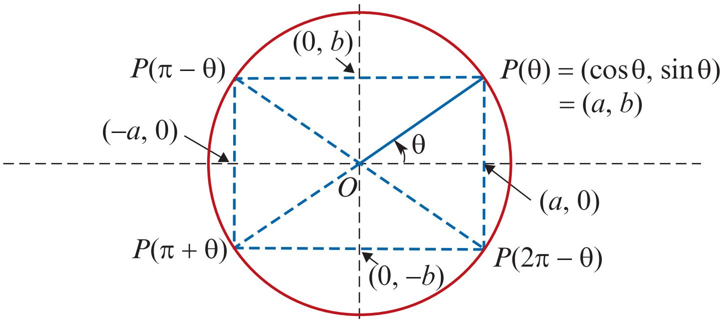
By symmetry:

$\sin(\pi - \theta) = b = \sin\theta$

$\cos(\pi - \theta) = -a = -\cos\theta$

$\tan(\pi - \theta) = \dfrac{b}{-a} = -\tan\theta$

Quadrant 1



Quadrant 3

$\sin(\pi + \theta) = -b = -\sin\theta$

$\cos(\pi + \theta) = -a = -\cos\theta$

$\tan(\pi + \theta) = \dfrac{-b}{-a} = \tan\theta$

Quadrant 4

$\sin(2\pi - \theta) = -b = -\sin\theta$

$\cos(2\pi - \theta) = a = \cos\theta$

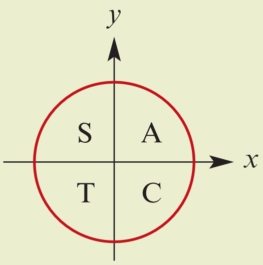
$\tan(2\pi - \theta) = \dfrac{-b}{a} = -\tan\theta$

Note: These relationships are true for all values of θ.

Signs of circular functions

Using the symmetry properties, the signs of sin, cos and tan for the four quadrants can be summarised as follows:

1st quadrant	all are positive	(A)
2nd quadrant	sin is positive	(S)
3rd quadrant	tan is positive	(T)
4th quadrant	cos is positive	(C)



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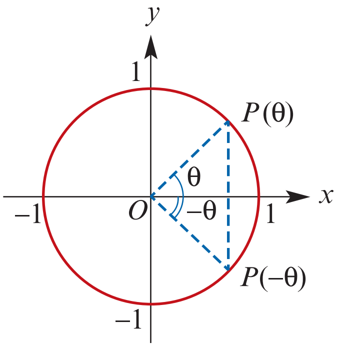
Negative of angles

By symmetry:

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

$$\tan(-\theta) = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$$



Therefore:

- sin is an odd function
- cos is an even function
- tan is an odd function.

Example 5

Evaluate:

a $\cos\left(\frac{5\pi}{4}\right)$ **b** $\sin\left(\frac{11\pi}{6}\right)$ **c** $\cos\left(\frac{200\pi}{3}\right)$ **d** $\tan\left(\frac{52\pi}{6}\right)$

Solution

a $$\cos\left(\frac{5\pi}{4}\right) = \cos\left(\pi + \frac{\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

b $$\sin\left(\frac{11\pi}{6}\right) = \sin\left(2\pi - \frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$$

c $$\cos\left(\frac{200\pi}{3}\right) = \cos\left(66\pi + \frac{2\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right) = \cos\left(\pi - \frac{\pi}{3}\right) = -\cos\left(\frac{\pi}{3}\right) = -\frac{1}{2}$$

d $$\tan\left(\frac{52\pi}{6}\right) = \tan\left(8\pi + \frac{2\pi}{3}\right) = \tan\left(\frac{2\pi}{3}\right) = \tan\left(\pi - \frac{\pi}{3}\right) = -\tan\left(\frac{\pi}{3}\right) = -\sqrt{3}$$

Example 6

If $\sin x = 0.6$, find the value of:

a $\sin(\pi - x)$ **b** $\sin(\pi + x)$ **c** $\sin(2\pi - x)$ **d** $\sin(-x)$

Solution

a $$\sin(\pi - x) = \sin x = 0.6$$

b $$\sin(\pi + x) = -\sin x = -0.6$$

c $$\sin(2\pi - x) = -\sin x = -0.6$$

d $$\sin(-x) = -\sin x = -0.6$$

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Example 7

If $\cos x^\circ = 0.8$, find the value of:

a $\cos(180 - x)^\circ$ **b** $\cos(180 + x)^\circ$ **c** $\cos(360 - x)^\circ$ **d** $\cos(-x)^\circ$

Solution

a
$$\begin{aligned} &\cos(180 - x)^\circ \\ &= -\cos x^\circ \\ &= -0.8 \end{aligned}$$

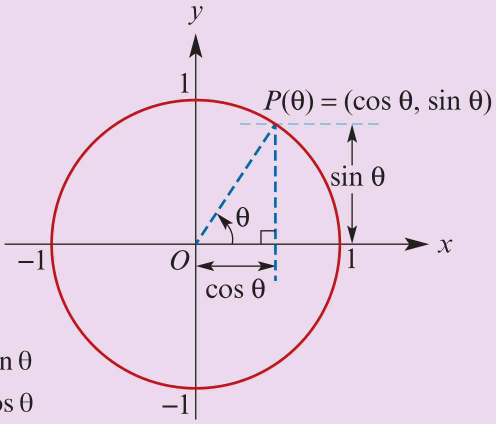
b
$$\begin{aligned} &\cos(180 + x)^\circ \\ &= -\cos x^\circ \\ &= -0.8 \end{aligned}$$

c
$$\begin{aligned} &\cos(360 - x)^\circ \\ &= \cos x^\circ \\ &= 0.8 \end{aligned}$$

d
$$\begin{aligned} &\cos(-x)^\circ \\ &= \cos x^\circ \\ &= 0.8 \end{aligned}$$

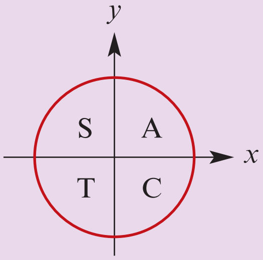
Section summary

- $P(\theta) = (\cos\theta, \sin\theta)$
- $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ for $\cos\theta \neq 0$
- The circular functions are periodic:
  - $\sin(2\pi + \theta) = \sin\theta$
  - $\cos(2\pi + \theta) = \cos\theta$
  - $\tan(\pi + \theta) = \tan\theta$
- Negative of angles:
  - sin is an odd function, i.e. $\sin(-\theta) = -\sin\theta$
  - cos is an even function, i.e. $\cos(-\theta) = \cos\theta$
  - tan is an odd function, i.e. $\tan(-\theta) = -\tan\theta$



- Memory aids:

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$ (30°)	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$ (45°)	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$ (60°)	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$ (90°)	1	0	undefined



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Exercise 6B

Example 3

1 Evaluate each of the following:

a $\sin(3\pi)$ **b** $\cos\left(-\frac{5\pi}{2}\right)$ **c** $\sin\left(\frac{7\pi}{2}\right)$ **d** $\cos(3\pi)$

e $\sin(-4\pi)$ **f** $\tan(-\pi)$ **g** $\tan(2\pi)$ **h** $\tan(-2\pi)$

i $\cos(23\pi)$ **j** $\cos\left(\frac{49\pi}{2}\right)$ **k** $\cos(35\pi)$ **l** $\cos\left(\frac{-45\pi}{2}\right)$

m $\tan(24\pi)$ **n** $\cos(20\pi)$

Example 4

2 Evaluate each of the following using a calculator. (Give answers correct to two decimal places.)

a $\sin 1.7$ **b** $\sin 2.6$ **c** $\sin 4.2$ **d** $\cos 0.4$

e $\cos 2.3$ **f** $\cos(-1.8)$ **g** $\sin(-1.7)$ **h** $\sin(-3.6)$

i $\tan 1.6$ **j** $\tan(-1.2)$ **k** $\tan 3.9$ **l** $\tan(-2.5)$

Example 5

3 Write down the exact values of:

a $\sin\left(\frac{3\pi}{4}\right)$ **b** $\cos\left(\frac{2\pi}{3}\right)$ **c** $\cos\left(\frac{7\pi}{6}\right)$ **d** $\sin\left(\frac{5\pi}{6}\right)$

e $\cos\left(\frac{4\pi}{3}\right)$ **f** $\sin\left(\frac{5\pi}{4}\right)$ **g** $\sin\left(\frac{7\pi}{4}\right)$ **h** $\cos\left(\frac{5\pi}{3}\right)$

i $\cos\left(\frac{11\pi}{3}\right)$ **j** $\sin\left(\frac{200\pi}{3}\right)$ **k** $\cos\left(-\frac{11\pi}{3}\right)$ **l** $\sin\left(\frac{25\pi}{3}\right)$

m $\sin\left(-\frac{13\pi}{4}\right)$ **n** $\cos\left(-\frac{20\pi}{3}\right)$ **o** $\sin\left(\frac{67\pi}{4}\right)$ **p** $\cos\left(\frac{68\pi}{3}\right)$

q $\tan\left(\frac{11\pi}{3}\right)$ **r** $\tan\left(\frac{200\pi}{3}\right)$ **s** $\tan\left(-\frac{11\pi}{6}\right)$ **t** $\tan\left(\frac{25\pi}{3}\right)$

u $\tan\left(-\frac{13\pi}{4}\right)$ **v** $\tan\left(-\frac{25\pi}{6}\right)$

Example 6

4 If $\sin\theta = 0.52$, $\cos x = 0.68$ and $\tan\alpha = 0.4$, find the value of:

a $\sin(\pi - \theta)$ **b** $\cos(\pi + x)$ **c** $\sin(2\pi + \theta)$ **d** $\tan(\pi + \alpha)$

e $\sin(\pi + \theta)$ **f** $\cos(2\pi - x)$ **g** $\tan(2\pi - \alpha)$ **h** $\cos(\pi - x)$

i $\sin(-\theta)$ **j** $\cos(-x)$ **k** $\tan(-\alpha)$

5 If $\sin\theta = 0.4$, $\cos x = 0.7$ and $\tan\alpha = 1.2$, find the value of:

a $\sin(\pi - \theta)$ **b** $\cos(\pi + x)$ **c** $\sin(2\pi + \theta)$ **d** $\tan(\pi + \alpha)$

e $\sin(\pi + \theta)$ **f** $\cos(2\pi - x)$ **g** $\tan(2\pi - \alpha)$ **h** $\cos(\pi - x)$

i $\sin(-\theta)$ **j** $\cos(-x)$ **k** $\tan(-\alpha)$

Example 7

6 Without using a calculator, evaluate the sin, cos and tan of each of the following:

a 150° **b** 225° **c** 405° **d** −120° **e** −315° **f** −30°

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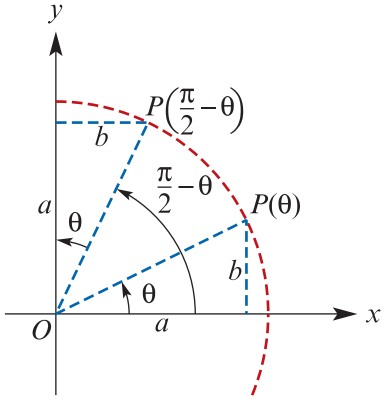
6C Further symmetry properties and the Pythagorean identity

► Complementary relationships

From the diagram to the right:

$$\sin\left(\frac{\pi}{2} - \theta\right) = a = \cos\theta$$

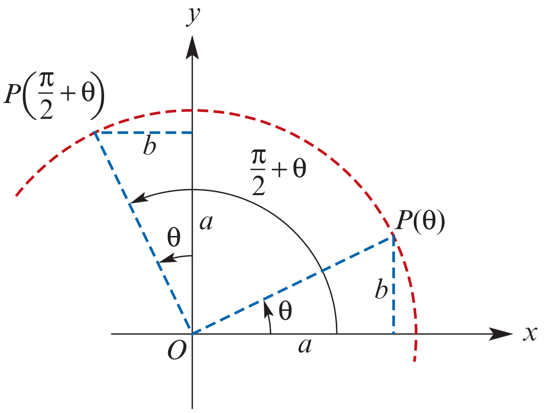
$$\cos\left(\frac{\pi}{2} - \theta\right) = b = \sin\theta$$


y
x
O
$P\left(\frac{\pi}{2}-\theta\right)$
$P(\theta)$
$\frac{\pi}{2}-\theta$
θ
θ
a
b
b
a

From the diagram to the right:

$$\sin\left(\frac{\pi}{2} + \theta\right) = a = \cos\theta$$

$$\cos\left(\frac{\pi}{2} + \theta\right) = -b = -\sin\theta$$


y
x
O
$P\left(\frac{\pi}{2}+\theta\right)$
$P(\theta)$
$\frac{\pi}{2}+\theta$
θ
θ
b
a
b
a

Example 8

If $\sin\theta = 0.3$ and $\cos\psi = 0.8$, find the value of:

a $\sin\left(\frac{\pi}{2} - \psi\right)$

b $\cos\left(\frac{\pi}{2} + \theta\right)$

Solution

a $\sin\left(\frac{\pi}{2} - \psi\right) = \cos\psi = 0.8$

b $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta = -0.3$

► The Pythagorean identity

Consider a point, $P(\theta)$, on the unit circle.

By Pythagoras' theorem,

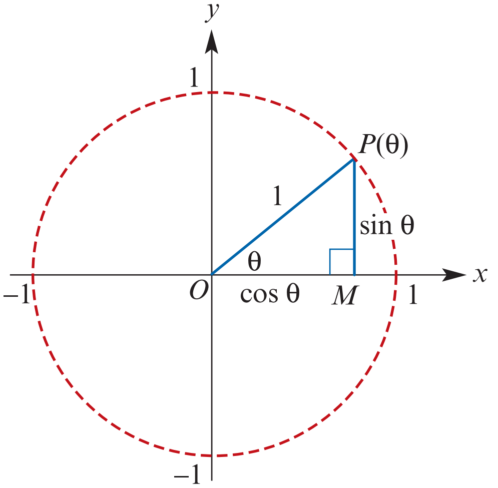
$$OP^2 = OM^2 + MP^2$$

$$\therefore \quad 1 = (\cos\theta)^2 + (\sin\theta)^2$$

Now $(\cos\theta)^2$ and $(\sin\theta)^2$ may be written as $\cos^2\theta$ and $\sin^2\theta$. Thus we obtain:

$$\cos^2\theta + \sin^2\theta = 1$$

This holds for all values of θ, and is called the **Pythagorean identity**.


y
x
1
−1
−1
1
O
$P(\theta)$
1
sin θ
θ
cos θ
M

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Example 9

Given that $\sin x = \frac{3}{5}$ and $\frac{\pi}{2} < x < \pi$, find:

a $\cos x$ **b** $\tan x$

Solution

a Substitute $\sin x = \frac{3}{5}$ into the Pythagorean identity:

$$\begin{aligned} \cos^2 x + \sin^2 x &= 1 \\ \cos^2 x + \frac{9}{25} &= 1 \\ \cos^2 x &= 1 - \frac{9}{25} \\ &= \frac{16}{25} \end{aligned}$$

Therefore $\cos x = \pm\frac{4}{5}$. But x is in the 2nd quadrant, and so $\cos x = -\frac{4}{5}$.

b Using part **a**, we have

$$\begin{aligned} \tan x &= \frac{\sin x}{\cos x} \\ &= \frac{3}{5} \div \left(-\frac{4}{5}\right) \\ &= \frac{3}{5} \times \left(-\frac{5}{4}\right) \\ &= -\frac{3}{4} \end{aligned}$$

Section summary

- Complementary relationships

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$$

- Pythagorean identity

$$\cos^2\theta + \sin^2\theta = 1$$

Exercise 6C

Example 8

1 If $\sin x = 0.3$, $\cos\alpha = 0.6$ and $\tan\theta = 0.7$, find the value of:

a $\cos(-\alpha)$ **b** $\sin\left(\frac{\pi}{2} + \alpha\right)$ **c** $\tan(-\theta)$

d $\cos\left(\frac{\pi}{2} - x\right)$ **e** $\sin(-x)$ **f** $\tan\left(\frac{\pi}{2} - \theta\right)$

g $\cos\left(\frac{\pi}{2} + x\right)$ **h** $\sin\left(\frac{\pi}{2} - \alpha\right)$ **i** $\sin\left(\frac{3\pi}{2} + \alpha\right)$

j $\cos\left(\frac{3\pi}{2} - x\right)$ **k** $\tan\left(\frac{3\pi}{2} - \theta\right)$ **l** $\cos\left(\frac{5\pi}{2} - x\right)$

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Example 9

2 **a** Given that $\cos x = \frac{3}{5}$ and $\frac{3\pi}{2} < x < 2\pi$, find $\sin x$ and $\tan x$.

b Given that $\sin x = \frac{5}{13}$ and $\frac{\pi}{2} < x < \pi$, find $\cos x$ and $\tan x$.

c Given that $\cos x = \frac{1}{5}$ and $\frac{3\pi}{2} < x < 2\pi$, find $\sin x$ and $\tan x$.

d Given that $\sin x = -\frac{12}{13}$ and $\pi < x < \frac{3\pi}{2}$, find $\cos x$ and $\tan x$.

e Given that $\cos x = \frac{4}{5}$ and $\frac{3\pi}{2} < x < 2\pi$, find $\sin x$ and $\tan x$.

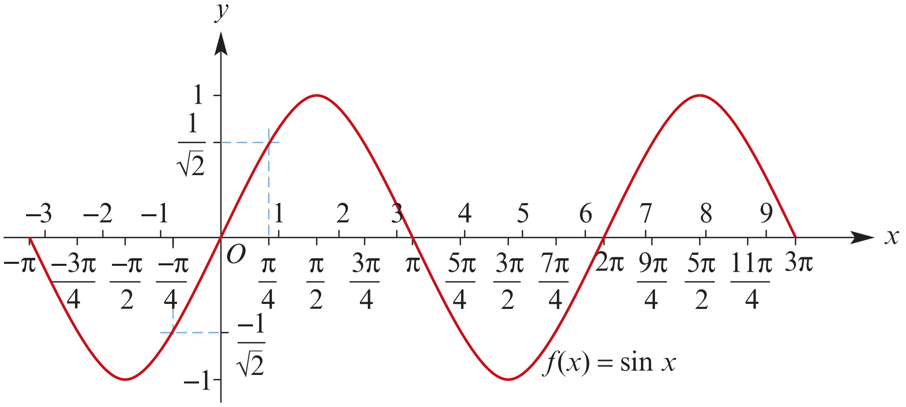
f Given that $\sin x = -\frac{5}{13}$ and $\pi < x < \frac{3\pi}{2}$, find $\cos x$ and $\tan x$.

g Given that $\cos x = \frac{8}{10}$ and $\frac{3\pi}{2} < x < 2\pi$, find $\sin x$ and $\tan x$.

6D Graphs of sine and cosine

Graph of the sine function

A calculator can be used to plot the graph of $f(x) = \sin x$ for $-\pi \le x \le 3\pi$. Note that radian mode must be selected.



Observations from the graph of $y = \sin x$

- The graph repeats itself after an interval of 2π units.
  A function which repeats itself regularly is called a **periodic** function, and the interval between the repetitions is called the **period** of the function (also called the wavelength).
  Thus $y = \sin x$ has a period of 2π units.
- The maximum and minimum values of $\sin x$ are 1 and -1 respectively.
  The distance between the mean position and the maximum position is called the **amplitude**. The graph of $y = \sin x$ has an amplitude of 1.

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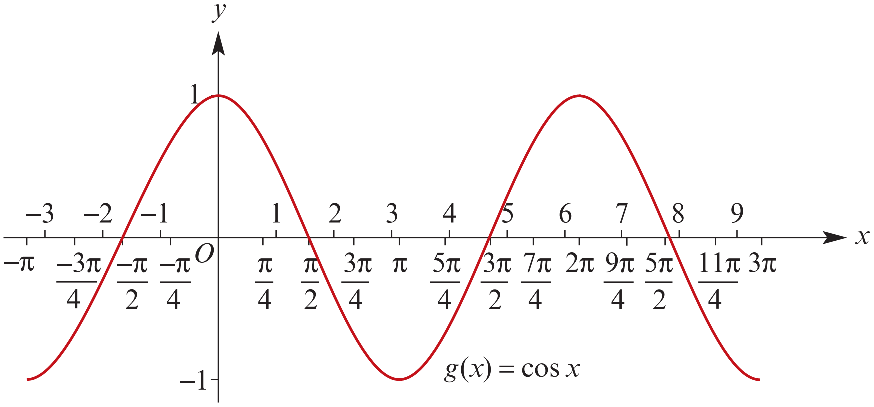
Graph of the cosine function

The graph of $g(x) = \cos x$ is shown below for $-\pi \le x \le 3\pi$.



Observations from the graph of $y = \cos x$

- The period is 2π.
- The amplitude is 1.
- The graph of $y = \cos x$ is the graph of $y = \sin x$ translated $\frac{\pi}{2}$ units in the negative direction of the x-axis.

Sketch graphs of $y = a\sin(nt)$ and $y = a\cos(nt)$

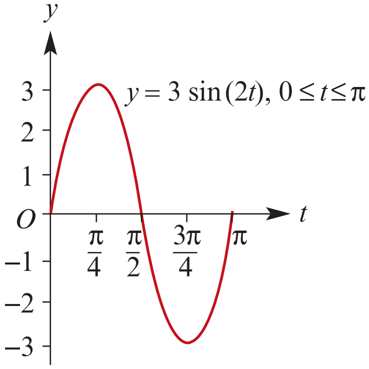
The graphs of functions of the forms $y = a\sin(nt)$ and $y = a\cos(nt)$ are transformations of the graphs of $y = \sin t$ and $y = \cos t$ respectively. We first consider the case where a and n are positive numbers.

Transformations: dilations

Graph of $y = 3\sin(2t)$ The image of the graph of $y = \sin t$ under a dilation of factor 3 from the t-axis and a dilation of factor $\frac{1}{2}$ from the y-axis is $y = 3\sin(2t)$.

Note: Let $f(t) = \sin t$. Then the graph of $y = f(t)$ is transformed to the graph of $y = 3f(2t)$. The point with coordinates (t, y) is mapped to the point with coordinates $\left(\frac{t}{2}, 3y\right)$.

t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$y = 3\sin(2t)$	0	3	0	−3	0



We make the following observations about the graph of $y = 3\sin(2t)$:

- amplitude is 3
- period is π

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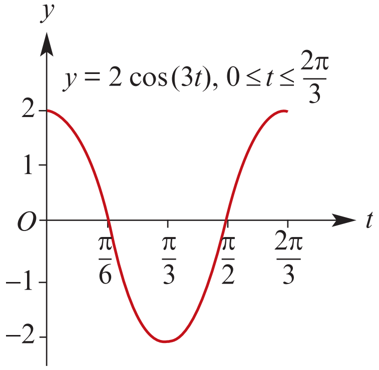
Graph of $y = 2\cos(3t)$ The image of the graph of $y = \cos t$ under a dilation of factor 2 from the t-axis and a dilation of factor $\frac{1}{3}$ from the y-axis is $y = 2\cos(3t)$.

t	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
$y = 2\cos(3t)$	2	0	-2	0	2



We make the following observations about the graph of $y = 2\cos(3t)$:

- amplitude is 2
- period is $\frac{2\pi}{3}$

Amplitude and period Comparing these results with those for $y = \sin t$ and $y = \cos t$, the following general rules can be stated for a and n positive:

Function	Amplitude	Period
$y = a\sin(nt)$	a	$\frac{2\pi}{n}$
$y = a\cos(nt)$	a	$\frac{2\pi}{n}$

Example 10

For each of the following functions with domain $\mathbb{R}$, state the amplitude and period:

a $f(t) = 2\sin(3t)$

b $f(t) = -\frac{1}{2}\sin\left(\frac{t}{2}\right)$

c $f(t) = 4\cos(3\pi t)$

Solution

a Amplitude is 2

Period is $\frac{2\pi}{3}$

b Amplitude is $\frac{1}{2}$

Period is $2\pi \div \frac{1}{2} = 4\pi$

c Amplitude is 4

Period is $\frac{2\pi}{3\pi} = \frac{2}{3}$

Graphs of $y = a\sin(nt)$ and $y = a\cos(nt)$

For a and n positive numbers, the graphs of $y = a\sin(nt)$ and $y = a\cos(nt)$ are obtained from the graphs of $y = \sin t$ and $y = \cos t$, respectively, by a dilation of factor a from the t-axis and a dilation of factor $\frac{1}{n}$ from the y-axis.

The point with coordinates (t, y) is mapped to the point with coordinates $\left(\frac{t}{n}, ay\right)$.

The following are important properties of both of the functions $f(t) = a\sin(nt)$ and $g(t) = a\cos(nt)$:

- The period is $\frac{2\pi}{n}$.
- The amplitude is a.
- The maximal domain is $\mathbb{R}$.
- The range is $[-a, a]$.

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Example 11

For each of the following, give a sequence of transformations which takes the graph of $y = \sin x$ to the graph of $y = g(x)$, and state the amplitude and period of $g(x)$:

a $g(x) = 3\sin(2x)$ **b** $g(x) = 4\sin\left(\frac{x}{2}\right)$

Solution

a The graph of $y = 3\sin(2x)$ is obtained from the graph of $y = \sin x$ by a dilation of factor 3 from the x-axis and a dilation of factor $\frac{1}{2}$ from the y-axis.

The function $g(x) = 3\sin(2x)$ has amplitude 3 and period $\frac{2\pi}{2} = \pi$.

b The graph of $y = 4\sin\left(\frac{x}{2}\right)$ is obtained from the graph of $y = \sin x$ by a dilation of factor 4 from the x-axis and a dilation of factor 2 from the y-axis.

The function $g(x) = 4\sin\left(\frac{x}{2}\right)$ has amplitude 4 and period $2\pi \div \frac{1}{2} = 4\pi$.

Example 12

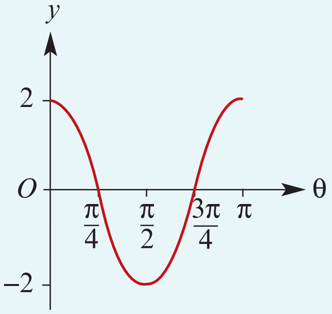
Sketch the graph of each of the following functions:

a $y = 2\cos(2\theta)$ **b** $y = \frac{1}{2}\sin\left(\frac{x}{2}\right)$

In each case, show one complete cycle.

Solution

a



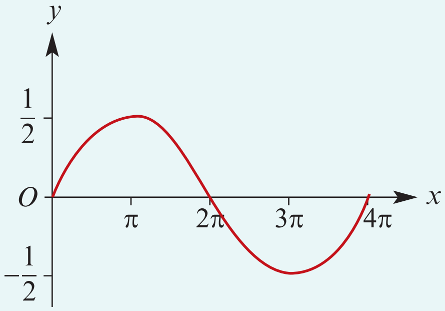
Explanation

The amplitude is 2.

The period is $\frac{2\pi}{2} = \pi$.

The graph of $y = 2\cos(2\theta)$ is obtained from the graph of $y = \cos\theta$ by a dilation of factor 2 from the θ-axis and a dilation of factor $\frac{1}{2}$ from the y-axis.

b



The amplitude is $\frac{1}{2}$.

The period is $2\pi \div \frac{1}{2} = 4\pi$.

The graph of $y = \frac{1}{2}\sin(\frac{x}{2})$ is obtained from the graph of $y = \sin x$ by a dilation of factor $\frac{1}{2}$ from the x-axis and a dilation of factor 2 from the y-axis.

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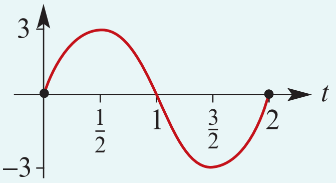
Example 13

Sketch the graph of $f\colon [0,2] \to \mathbb{R}$, $f(t) = 3\sin(\pi t)$.

Solution



Explanation

The amplitude is 3.

The period is $2\pi \div \pi = 2$.

The graph of $f(t) = 3\sin(\pi t)$ is obtained from the graph of $y = \sin t$ by a dilation of factor 3 from the t-axis and a dilation of factor $\frac{1}{\pi}$ from the y-axis.

Transformations: reflections

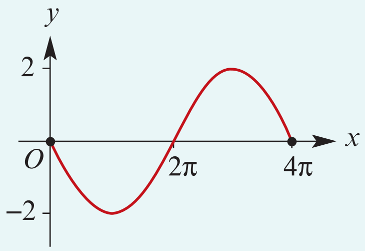
Example 14

Sketch the following graphs for $x \in [0, 4\pi]$:

a $f(x) = -2\sin\left(\frac{x}{2}\right)$ **b** $y = -\cos(2x)$

Solution

a

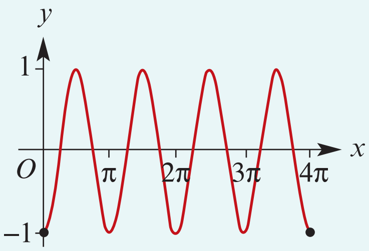


Explanation

The graph of $f(x) = -2\sin(\frac{x}{2})$ is obtained from the graph of $y = 2\sin(\frac{x}{2})$ by a reflection in the x-axis.

The amplitude is 2 and the period is 4π.

b



The graph of $y = -\cos(2x)$ is obtained from the graph of $y = \cos(2x)$ by a reflection in the x-axis.

The amplitude is 1 and the period is π.

Note: Recall that sin is an odd function and cos is an even function (i.e. $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$). When reflected in the y-axis, the graph of $y = \sin x$ transforms onto the graph of $y = -\sin x$, and the graph of $y = \cos x$ transforms onto itself.

Section summary

For positive numbers a and n, the graphs of $y = a\sin(nt)$, $y = -a\sin(nt)$, $y = a\cos(nt)$ and $y = -a\cos(nt)$ all have the following properties:

- The period is $\frac{2\pi}{n}$.
- The amplitude is a.
- The maximal domain is $\mathbb{R}$.
- The range is $[-a, a]$.

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Exercise 6D

Example 10

1 Write down **i** the period and **ii** the amplitude of each of the following:

a $3\sin\theta$ **b** $5\sin(3\theta)$ **c** $\frac{1}{2}\cos(2\theta)$ **d** $2\sin\left(\frac{1}{3}\theta\right)$

e $3\cos(4\theta)$ **f** $\frac{1}{2}\sin\theta$ **g** $3\cos\left(\frac{1}{2}\theta\right)$ **h** $2\sin\left(\frac{2\theta}{3}\right)$

Example 11

2 For each of the following, give a sequence of transformations which takes the graph of $y = \sin x$ to the graph of $y = g(x)$, and state the amplitude and period of $g(x)$:

a $g(x) = 4\sin(3x)$ **b** $g(x) = 5\sin\left(\frac{x}{3}\right)$

c $g(x) = 6\sin\left(\frac{x}{2}\right)$ **d** $g(x) = 4\sin(5x)$

3 For each of the following, give a sequence of transformations which takes the graph of $y = \cos x$ to the graph of $y = g(x)$, and state the amplitude and period of $g(x)$:

a $g(x) = 2\cos(3x)$ **b** $g(x) = 3\cos\left(\frac{x}{4}\right)$

c $g(x) = 6\cos\left(\frac{x}{5}\right)$ **d** $g(x) = 3\cos(7x)$

Example 12

4 Sketch the graph of each of the following, showing one complete cycle. State the amplitude and period.

a $y = 2\sin(3\theta)$ **b** $y = 2\cos(2\theta)$ **c** $y = 3\sin\left(\frac{1}{3}\theta\right)$

d $y = \frac{1}{3}\cos(2\theta)$ **e** $y = 3\sin(4\theta)$ **f** $y = 4\cos\left(\frac{1}{4}\theta\right)$

Example 13

5 Sketch the graph of $f\colon [0, 1] \to \mathbb{R}$, $f(t) = 3\sin(2\pi t)$.

6 Sketch the graph of $f\colon [0, 1] \to \mathbb{R}$, $f(t) = 3\sin\left(\frac{\pi t}{2}\right)$.

7 Sketch the graph of $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 5\cos(3x)$ for $0 \le x \le \pi$.

8 Sketch the graph of $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{2}\sin(2x)$ for $-\pi \le x \le 2\pi$.

9 Sketch the graph of $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 2\cos\left(\frac{3x}{2}\right)$ for $0 \le x \le 2\pi$.

Example 14

10 Sketch the graph of $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = -3\cos\left(\frac{x}{2}\right)$ for $0 \le x \le 4\pi$.

11 Find the equation of the image of the graph of $y = \sin x$ under a dilation of factor 2 from the x-axis followed by a dilation of factor 3 from the y-axis.

12 Find the equation of the image of the graph of $y = \cos x$ under a dilation of factor $\frac{1}{2}$ from the x-axis followed by a dilation of factor 3 from the y-axis.

13 Find the equation of the image of the graph of $y = \sin x$ under a dilation of factor $\frac{1}{2}$ from the x-axis followed by a dilation of factor 2 from the y-axis.

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6E Solution of trigonometric equations

In this section we revise methods for solving equations of the form $a\sin(nt) = b$ and $a\cos(nt) = b$.

Solving equations of the form $\sin t = b$ and $\cos t = b$

First we look at the techniques for solving equations of the form $\sin t = b$ and $\cos t = b$. These same techniques will be applied to solve more complicated trigonometric equations later in this section.

Example 15

Find all solutions of the equation $\sin\theta = \frac{1}{2}$ for $\theta \in [0, 4\pi]$.

Solution

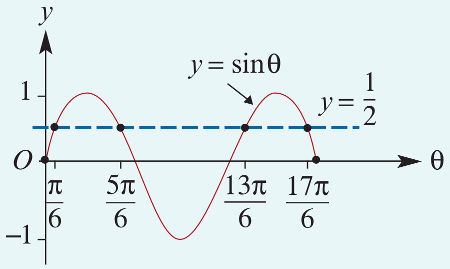
The solution for $\theta \in \left[0, \frac{\pi}{2}\right]$ is $\theta = \frac{\pi}{6}$.

The second solution is $\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

The third solution is $\theta = 2\pi + \frac{\pi}{6} = \frac{13\pi}{6}$.

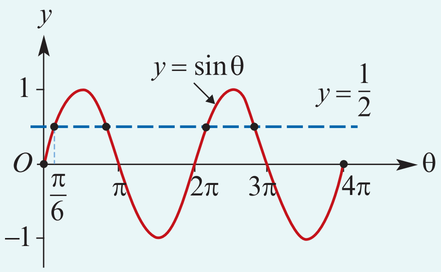
The fourth solution is $\theta = 2\pi + \frac{5\pi}{6} = \frac{17\pi}{6}$.

These four solutions are shown on the graph below.



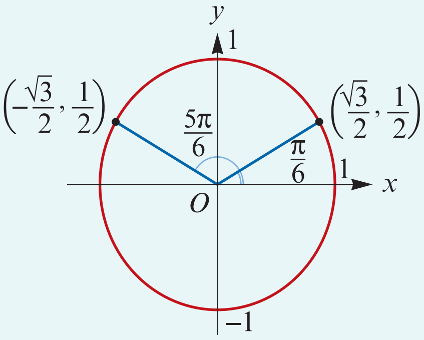
Explanation

By sketching a graph, we can see that there are four solutions in the interval $[0, 4\pi]$.



The first solution can be obtained from a knowledge of exact values or by using $\sin^{-1}$ on your calculator.

The second solution is obtained using symmetry. The sine function is positive in the 2nd quadrant and $\sin(\pi - \theta) = \sin\theta$.



y
1
$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
5π/6
π/6
1
x
O
−1

Further solutions are found by adding 2π, since $\sin\theta = \sin(2\pi + \theta)$.

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Example 16

Find two values of x:

a $\sin x = -0.3$ with $0 \le x \le 2\pi$

b $\cos x^\circ = -0.7$ with $0^\circ \le x^\circ \le 360^\circ$

Solution

a Consider $\sin\alpha = 0.3$ with $\alpha \in \left[0, \dfrac{\pi}{2}\right]$.

The solution is $\alpha = 0.30469\ldots$

The value of $\sin x$ is negative for $P(x)$ in the 3rd and 4th quadrants.

3rd quadrant:

$$\begin{aligned} x &= \pi + 0.30469\ldots \\ &= 3.446 \qquad \text{(to 3 d.p.)} \end{aligned}$$

4th quadrant:

$$\begin{aligned} x &= 2\pi - 0.30469\ldots \\ &= 5.978 \qquad \text{(to 3 d.p.)} \end{aligned}$$

The solutions of $\sin x = -0.3$ in $[0, 2\pi]$ are $x = 3.446$ and $x = 5.978$.

b Consider $\cos\alpha^\circ = 0.7$ with $\alpha^\circ \in [0^\circ, 90^\circ]$.

The solution is $\alpha^\circ = 45.57^\circ$.

The value of $\cos x^\circ$ is negative for $P(x^\circ)$ in the 2nd and 3rd quadrants.

2nd quadrant:

$$\begin{aligned} x^\circ &= 180^\circ - 45.57^\circ \\ &= 134.43^\circ \end{aligned}$$

3rd quadrant:

$$\begin{aligned} x^\circ &= 180^\circ + 45.57^\circ \\ &= 225.57^\circ \end{aligned}$$

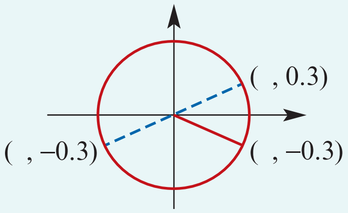
The solutions of $\cos x^\circ = -0.7$ in $[0^\circ, 360^\circ]$ are $x^\circ = 134.43^\circ$ and $x^\circ = 225.57^\circ$.

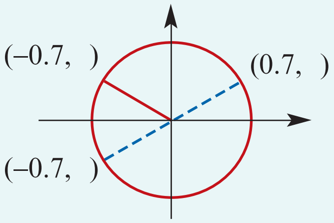
Explanation

First consider the corresponding equation for the 1st quadrant. Use your calculator to find the solution for α.

Decide which quadrants will contain a solution for x.

Find the solutions using the symmetry relationships (or the graph of $y = \sin x$).





Example 17

Find all the values of θ° between 0° and 360° for which:

a $\cos\theta° = \frac{\sqrt{3}}{2}$ **b** $\sin\theta° = -\frac{1}{2}$ **c** $\cos\theta° - \frac{1}{\sqrt{2}} = 0$

Solution

a $\cos\theta° = \frac{\sqrt{3}}{2}$

$\theta° = 30°$ or $\theta° = 360° - 30°$

$\theta° = 30°$ or $\theta° = 330°$

b $\sin\theta° = -\frac{1}{2}$

$\theta° = 180° + 30°$ or $\theta° = 360° - 30°$

$\theta° = 210°$ or $\theta° = 330°$

c $\cos\theta° - \frac{1}{\sqrt{2}} = 0$

$\therefore \quad \cos\theta° = \frac{1}{\sqrt{2}}$

$\theta° = 45°$ or $\theta° = 360° - 45°$

$\theta° = 45°$ or $\theta° = 315°$

Explanation

cos θ° is positive, and so $P(\theta°)$ lies in the 1st or 4th quadrant.

$\cos(360° - \theta°) = \cos\theta°$

sin θ° is negative, and so $P(\theta°)$ lies in the 3rd or 4th quadrant.

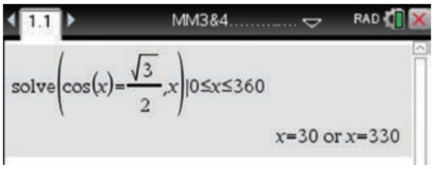
$\sin(180° + \theta°) = -\sin\theta°$

$\sin(360° - \theta°) = -\sin\theta°$

cos θ° is positive, and so $P(\theta°)$ lies in the 1st or 4th quadrant.

Using the TI-Nspire

For Example 17a, make sure the calculator is in degree mode and complete as shown.

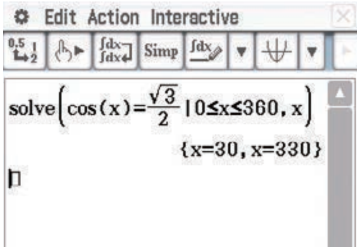


$\text{solve}\left(\cos(x)=\frac{\sqrt{3}}{2},x\right)|0\le x\le 360$

$x=30$ or $x=330$

Using the Casio ClassPad

- Ensure your calculator is in degree mode.
- Use the Math1 and Math3 keyboards to enter $\cos(x) = \frac{\sqrt{3}}{2} \mid 0 \le x \le 360$
- Highlight the equation and domain. Then select **Interactive** > **Equation/Inequality** > **solve** and ensure the variable is set to x.



$\text{solve}\left(\cos(x)=\frac{\sqrt{3}}{2} \mid 0\le x\le 360, x\right)$

$\{x=30, x=330\}$

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Solving equations of the form $a\sin(nt) = b$ and $a\cos(nt) = b$

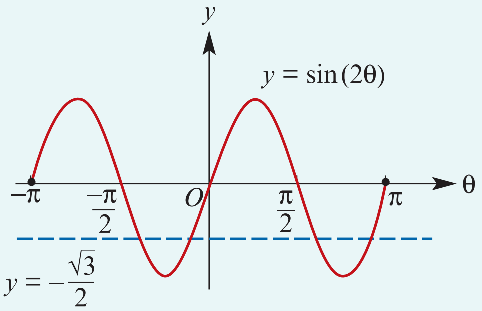
The techniques introduced above can be applied in a more general situation. This is achieved by a simple substitution, as shown in the following example.

Example 18

Solve the equation $\sin(2\theta) = -\dfrac{\sqrt{3}}{2}$ for $\theta \in [-\pi, \pi]$.

Solution

It is clear from the graph that there are four solutions.



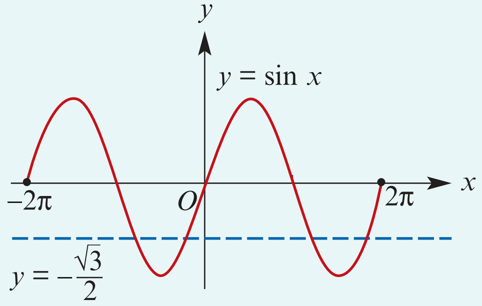
To solve the equation, let $x = 2\theta$.

Note: If $\theta \in [-\pi, \pi]$, then we have $x = 2\theta \in [-2\pi, 2\pi]$.

Now consider the equation

$$\sin x = -\frac{\sqrt{3}}{2} \quad \text{for } x \in [-2\pi, 2\pi]$$

The 1st quadrant solution to the equation $\sin\alpha = \dfrac{\sqrt{3}}{2}$ is $\alpha = \dfrac{\pi}{3}$.



Using symmetry, the solutions to $\sin x = -\dfrac{\sqrt{3}}{2}$ for $x \in [0, 2\pi]$ are

$x = \pi + \dfrac{\pi}{3}$ and $x = 2\pi - \dfrac{\pi}{3}$

i.e. $x = \dfrac{4\pi}{3}$ and $x = \dfrac{5\pi}{3}$

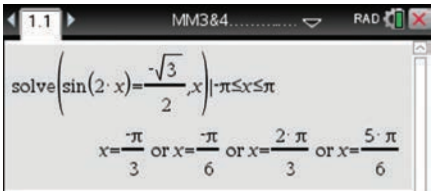
The other two solutions (obtained by subtracting 2π) are $x = \dfrac{4\pi}{3} - 2\pi$ and $x = \dfrac{5\pi}{3} - 2\pi$.

$\therefore$ The required solutions for x are $-\dfrac{2\pi}{3}$, $-\dfrac{\pi}{3}$, $\dfrac{4\pi}{3}$ and $\dfrac{5\pi}{3}$.

$\therefore$ The required solutions for θ are $-\dfrac{\pi}{3}$, $-\dfrac{\pi}{6}$, $\dfrac{2\pi}{3}$ and $\dfrac{5\pi}{6}$.

Using the TI-Nspire

Ensure that the calculator is in radian mode and complete as shown.



$\text{solve}\left(\sin(2\cdot x)=\dfrac{-\sqrt{3}}{2},x\right)|-\pi\leq x\leq\pi$

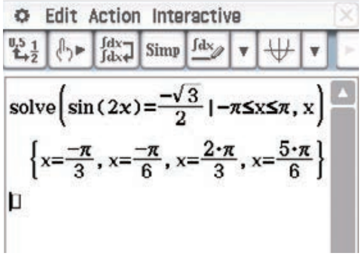
$x=\dfrac{-\pi}{3}$ or $x=\dfrac{-\pi}{6}$ or $x=\dfrac{2\cdot\pi}{3}$ or $x=\dfrac{5\cdot\pi}{6}$

Using the Casio ClassPad

- Ensure your calculator is in radian mode.
- Use the Math1 and Math3 keyboards to enter

$$\sin(2x) = \frac{-\sqrt{3}}{2} \,\Big|\, -\pi \le x \le \pi$$

- Highlight the equation and domain. Then select **Interactive** > **Equation/Inequality** > **solve** and ensure the variable is set to x.



$\text{solve}\left(\sin(2x)=\frac{-\sqrt{3}}{2} \,|\, -\pi \le x \le \pi, x\right)$

$\left\{x=\frac{-\pi}{3}, x=\frac{-\pi}{6}, x=\frac{2\cdot\pi}{3}, x=\frac{5\cdot\pi}{6}\right\}$

Section summary

- For solving equations of the form $\sin t = b$ and $\cos t = b$:
  - First find the solutions in the interval $[0, 2\pi]$. This can be done using your knowledge of exact values and symmetry properties, or with the aid of a calculator.
  - Further solutions can be found by adding and subtracting multiples of 2π.
- For solving equations of the form $a\sin(nt) = b$ and $a\cos(nt) = b$:
  - First substitute $x = nt$. Work out the interval in which solutions for x are required. Then proceed as in the case above to solve for x.
  - Once the solutions for x are found, the solutions for t can be found.

  For example: To solve $\sin(3t) = \frac{1}{2}$ for $t \in [0, 2\pi]$, first let $x = 3t$. The equation becomes $\sin x = \frac{1}{2}$ and the required solutions for x are in the interval $[0, 6\pi]$.

Exercise 6E

Skillsheet

Example 15

1 Solve each of the following for $x \in [0, 4\pi]$:

a $\sin x = \frac{1}{\sqrt{2}}$ **b** $\cos x = \frac{\sqrt{3}}{2}$ **c** $\sin x = \frac{-\sqrt{3}}{2}$

d $\cos x = \frac{1}{\sqrt{2}}$ **e** $\sin x = 1$ **f** $\cos x = -1$

2 Solve each of the following for $x \in [-\pi, \pi]$:

a $\sin x = -\frac{1}{2}$ **b** $\cos x = \frac{\sqrt{3}}{2}$ **c** $\cos x = \frac{-\sqrt{3}}{2}$

3 Solve each of the following for $x \in [0, 2\pi]$:

a $\sqrt{2}\sin x - 1 = 0$ **b** $\sqrt{2}\cos x + 1 = 0$ **c** $2\cos x + \sqrt{3} = 0$

d $2\sin x + 1 = 0$ **e** $1 - \sqrt{2}\cos x = 0$ **f** $4\cos x + 2 = 0$

Example 16

4 Find all values of x between 0 and 2π for which:

a $\sin x = 0.6$ **b** $\cos x = 0.8$ **c** $\sin x = -0.45$ **d** $\cos x = -0.2$

5 Find all values of $\theta°$ between 0° and 360° for which:

a $\sin\theta° = 0.3$ **b** $\cos\theta° = 0.4$ **c** $\sin\theta° = -0.8$ **d** $\cos\theta° = -0.5$

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Example 17

6 Without using a calculator, find all the values of θ between 0 and 360 for each of the following:

a $\cos \theta^\circ = \frac{1}{2}$ **b** $\sin \theta^\circ = \frac{\sqrt{3}}{2}$ **c** $\sin \theta^\circ = -\frac{1}{\sqrt{2}}$

d $2\cos \theta^\circ + 1 = 0$ **e** $2\sin \theta^\circ = \sqrt{3}$ **f** $2\cos \theta^\circ = -\sqrt{3}$

Example 18

7 Solve the following equations for $\theta \in [0, 2\pi]$:

a $\sin(2\theta) = -\frac{1}{2}$ **b** $\cos(2\theta) = \frac{\sqrt{3}}{2}$ **c** $\sin(2\theta) = \frac{1}{2}$

d $\sin(3\theta) = -\frac{1}{\sqrt{2}}$ **e** $\cos(2\theta) = -\frac{\sqrt{3}}{2}$ **f** $\sin(2\theta) = -\frac{1}{\sqrt{2}}$

8 Find, without using a calculator, all the values of x between 0 and 2π for each of the following:

a $2\cos(3x) + \sqrt{3} = 0$ **b** $4\sin(2x) - 2 = 0$ **c** $\sqrt{2}\cos(3x) - 1 = 0$

d $10\sin(3x) - 5 = 0$ **e** $2\sin(2x) = \sqrt{2}$ **f** $4\cos(3x) = -2\sqrt{3}$

g $-2\sin(3x) = \sqrt{2}$ **h** $4\cos(2x) = -2$ **i** $-2\cos(2x) = \sqrt{2}$

9 Solve the following equations for $\theta \in [0, 2\pi]$:

a $\sin(2\theta) = -0.8$ **b** $\sin(2\theta) = -0.6$ **c** $\cos(2\theta) = 0.4$ **d** $\cos(3\theta) = 0.6$

6F Sketch graphs of $y = a \sin n(t \pm \varepsilon)$ and $y = a \cos n(t \pm \varepsilon)$

In this section, we consider translations of graphs of functions of the form $f(t) = a\sin(nt)$ and $g(t) = a\cos(nt)$ in the direction of the t-axis.

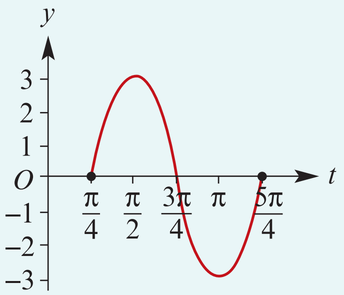
Example 19

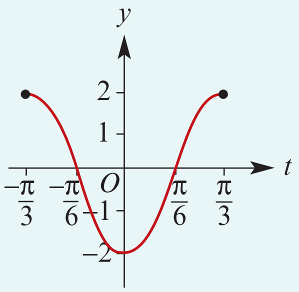
On separate axes, draw the graphs of the following functions. Use a calculator to help establish the shape. Set the window appropriately by noting the range and period.

a $y = 3\sin 2\left(t - \frac{\pi}{4}\right)$, $\frac{\pi}{4} \le t \le \frac{5\pi}{4}$ **b** $y = 2\cos 3\left(t + \frac{\pi}{3}\right)$, $-\frac{\pi}{3} \le t \le \frac{\pi}{3}$

Solution

a The range is $[-3, 3]$ and the period is π. **b** The range is $[-2, 2]$ and the period is $\frac{2\pi}{3}$.





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Observations from the example

a The graph of $y = 3\sin 2\left(t - \frac{\pi}{4}\right)$ is the same shape as $y = 3\sin(2t)$, but is translated $\frac{\pi}{4}$ units in the positive direction of the t-axis.

b The graph of $y = 2\cos 3\left(t + \frac{\pi}{3}\right)$ is the same shape as $y = 2\cos(3t)$, but is translated $\frac{\pi}{3}$ units in the negative direction of the t-axis.

The effect of $\pm\varepsilon$ is to translate the graph parallel to the t-axis. (Here $\pm\varepsilon$ is called the phase.)

Note: The techniques of Chapter 3 can be used to find the sequence of transformations. The graph of $y = \sin t$ is transformed to the graph of $y = 3\sin 2\left(t - \frac{\pi}{4}\right)$.

- Rearrange the second equation as $\frac{y'}{3} = \sin 2\left(t' - \frac{\pi}{4}\right)$.
- We can choose to write $y = \frac{y'}{3}$ and $t = 2\left(t' - \frac{\pi}{4}\right)$. Hence $y' = 3y$ and $t' = \frac{t}{2} + \frac{\pi}{4}$.
- The transformation is a dilation of factor 3 from the t-axis, followed by a dilation of factor $\frac{1}{2}$ from the y-axis, and then by a translation of $\frac{\pi}{4}$ units in the positive direction of the t-axis.

Example 20

For the function $f\colon [0, 2\pi] \to \mathbb{R}$, $f(x) = \sin\left(x - \frac{\pi}{3}\right)$:

a find $f(0)$ and $f(2\pi)$

b sketch the graph of f.

Solution

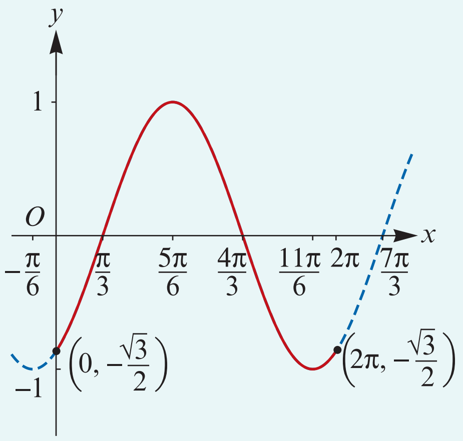
a $f(0) = \sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$ and $f(2\pi) = \sin\left(2\pi - \frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

b The graph of $y = \sin\left(x - \frac{\pi}{3}\right)$ is the graph of $y = \sin x$ translated $\frac{\pi}{3}$ units to the right.

The period of f is 2π and the amplitude is 1.

The endpoints are $\left(0, -\frac{\sqrt{3}}{2}\right)$ and $\left(2\pi, -\frac{\sqrt{3}}{2}\right)$.

There are many ways to proceed from here.



Method 1

- Start at $x = \frac{\pi}{3}$ and start to draw one cycle stopping at $x = 2\pi$.
- Each 'loop' of the graph is of length π and each 'half loop' is of length $\frac{\pi}{2}$.
- The 'half loop' going back from $x = \frac{\pi}{3}$ would end at $\frac{\pi}{3} - \frac{\pi}{2} = -\frac{\pi}{6}$.

Method 2

Find the x-axis intercepts by solving the equation $\sin\left(x - \frac{\pi}{3}\right) = 0$. Use symmetry to find the coordinates of the maximum and minimum points.

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Section summary

The graphs of $y = a\sin n(t \pm \varepsilon)$ and $y = a\cos n(t \pm \varepsilon)$ are translations of the graphs of $y = a\sin(nt)$ and $y = a\cos(nt)$ respectively.

The graphs are translated $\mp\varepsilon$ units parallel to the t-axis, where $\pm\varepsilon$ is called the phase.

Exercise 6F

Example 19

1 Sketch the graph of each of the following, showing one complete cycle. State the period and amplitude, and the greatest and least values of y.

a $y = 3\sin\left(\theta - \frac{\pi}{2}\right)$ **b** $y = \sin 2(\theta + \pi)$ **c** $y = 2\sin 3\left(\theta + \frac{\pi}{4}\right)$

d $y = \sqrt{3}\sin 2\left(\theta - \frac{\pi}{2}\right)$ **e** $y = 3\sin(2x - \pi)$ **f** $y = 2\cos 3\left(\theta + \frac{\pi}{4}\right)$

g $y = \sqrt{2}\sin 2\left(\theta - \frac{\pi}{3}\right)$ **h** $y = -3\sin\left(2x + \frac{\pi}{3}\right)$ **i** $y = -3\cos 2\left(\theta + \frac{\pi}{2}\right)$

Example 20

2 For the function $f\colon [0, 2\pi] \to \mathbb{R}$, $f(x) = \cos\left(x - \frac{\pi}{3}\right)$:

a find $f(0)$ and $f(2\pi)$

b sketch the graph of f.

3 For the function $f\colon [0, 2\pi] \to \mathbb{R}$, $f(x) = \sin 2\left(x - \frac{\pi}{3}\right)$:

a find $f(0)$ and $f(2\pi)$

b sketch the graph of f.

4 For the function $f\colon [-\pi, \pi] \to \mathbb{R}$, $f(x) = \sin 3\left(x + \frac{\pi}{4}\right)$:

a find $f(-\pi)$ and $f(\pi)$

b sketch the graph of f.

5 Find the equation of the image of $y = \sin x$ for each of the following transformations:

a dilation of factor 2 from the y-axis followed by dilation of factor 3 from the x-axis

b dilation of factor $\frac{1}{2}$ from the y-axis followed by dilation of factor 3 from the x-axis

c dilation of factor 3 from the y-axis followed by dilation of factor 2 from the x-axis

d dilation of factor $\frac{1}{2}$ from the y-axis followed by translation of $\frac{\pi}{3}$ units in the positive direction of the x-axis

e dilation of factor 2 from the y-axis followed by translation of $\frac{\pi}{3}$ units in the negative direction of the x-axis.

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6G Sketch graphs of $y = a \sin n(t \pm \varepsilon) \pm b$ and $y = a \cos n(t \pm \varepsilon) \pm b$

In general, the effect of ±b is to translate the graph ±b units parallel to the y-axis.

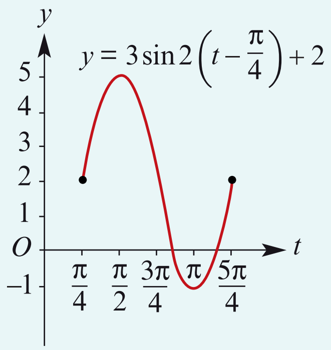
Example 21

Sketch each of the following graphs. Use a calculator to help establish the shape.

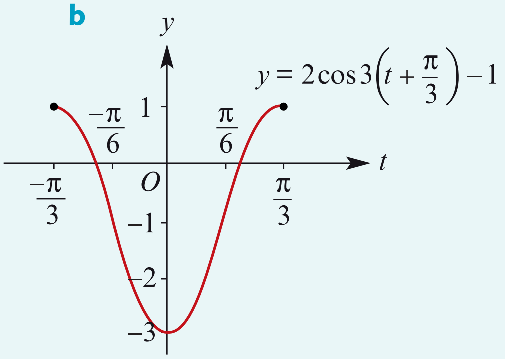
a $y = 3 \sin 2\left(t - \frac{\pi}{4}\right) + 2, \quad \frac{\pi}{4} \le t \le \frac{5\pi}{4}$ **b** $y = 2 \cos 3\left(t + \frac{\pi}{3}\right) - 1, \quad -\frac{\pi}{3} \le t \le \frac{\pi}{3}$

Solution

a



b



Finding axis intercepts

Example 22

Sketch the graph of each of the following for $x \in [0, 2\pi]$. Clearly indicate axis intercepts.

a $y = \sqrt{2}\sin(x) + 1$ **b** $y = 2\cos(2x) - 1$ **c** $y = 2\sin 2\left(x - \frac{\pi}{3}\right) - \sqrt{3}$

Solution

a To determine the x-axis intercepts, solve the equation $\sqrt{2}\sin(x) + 1 = 0$.

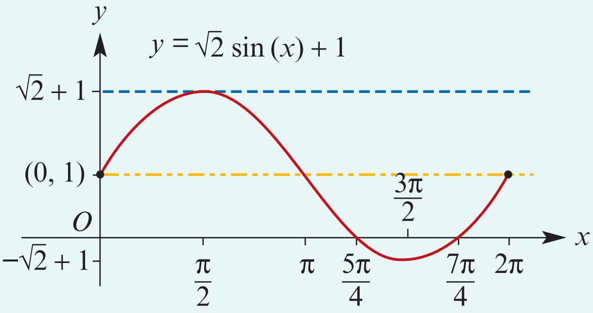
$$\sqrt{2}\sin(x) + 1 = 0$$

$$\therefore \quad \sin x = -\frac{1}{\sqrt{2}}$$

$$\therefore x = \pi + \frac{\pi}{4} \text{ or } 2\pi - \frac{\pi}{4}$$

$$\therefore x = \frac{5\pi}{4} \text{ or } \frac{7\pi}{4}$$

The x-axis intercepts are $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$.



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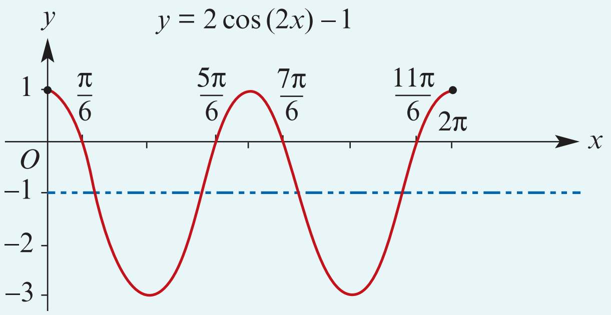
b $2\cos(2x) - 1 = 0$

$$\therefore \quad \cos(2x) = \frac{1}{2}$$

$$\therefore 2x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3} \text{ or } \frac{11\pi}{3}$$

$$\therefore \quad x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6} \text{ or } \frac{11\pi}{6}$$

The x-axis intercepts are $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$ and $\frac{11\pi}{6}$.



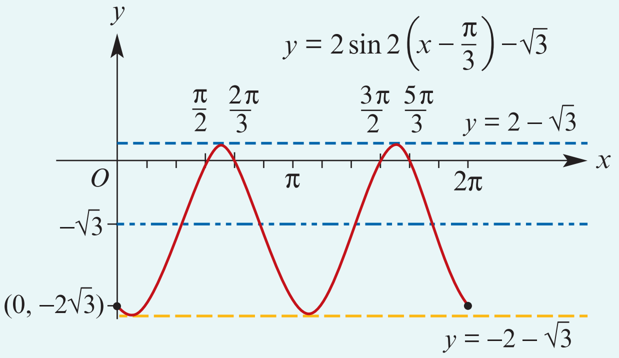
c $\sin 2\left(x - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$$\therefore 2\left(x - \frac{\pi}{3}\right) = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3} \text{ or } \frac{8\pi}{3}$$

$$\therefore \quad x - \frac{\pi}{3} = \frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6} \text{ or } \frac{4\pi}{3}$$

$$\therefore \quad x = \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{2} \text{ or } \frac{5\pi}{3}$$

The x-axis intercepts are $\frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{2}$ and $\frac{5\pi}{3}$.



Section summary

The graphs of $y = a \sin n(t \pm \varepsilon) \pm b$ and $y = a \cos n(t \pm \varepsilon) \pm b$ are translations of the graphs of $y = a \sin(nt)$ and $y = a \cos(nt)$ respectively.

The graphs are translated $\mp\varepsilon$ units parallel to the t-axis, where $\pm\varepsilon$ is called the phase. They are also translated $\pm b$ units parallel to the y-axis.

Exercise 6G

1 Sketch each of the following graphs. Use a calculator to help establish the shape. Label the endpoints with their coordinates.

Example 21

a $y = 2\sin 2\left(t - \frac{\pi}{3}\right) + 2, \quad \frac{\pi}{3} \le t \le \frac{4\pi}{3}$ **b** $y = 2\cos 3\left(t + \frac{\pi}{4}\right) - 1, \quad -\frac{\pi}{4} \le t \le \frac{5\pi}{12}$

Example 22

Skillsheet

2 Sketch the graph of each of the following for $x \in [0, 2\pi]$. List the x-axis intercepts of each graph for this interval.

a $y = 2\sin(x) + 1$ **b** $y = 2\sin(2x) - \sqrt{3}$ **c** $y = \sqrt{2}\cos(x) + 1$

d $y = 2\sin(2x) - 2$ **e** $y = \sqrt{2}\sin\left(x - \frac{\pi}{4}\right) + 1$

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3 Sketch the graph of each of the following for $x \in [-\pi, 2\pi]$:

a $y = 2\sin(3x) - 2$ **b** $y = 2\cos 3\left(x - \frac{\pi}{4}\right)$ **c** $y = 2\sin(2x) - 3$

d $y = 2\cos(2x) + 1$ **e** $y = 2\cos 2\left(x - \frac{\pi}{3}\right) - 1$ **f** $y = 2\sin 2\left(x + \frac{\pi}{6}\right) + 1$

4 Sketch the graph of each of the following for $x \in [-\pi, \pi]$:

a $y = 2\sin 2\left(x + \frac{\pi}{3}\right) + 1$ **b** $y = -2\sin 2\left(x + \frac{\pi}{6}\right) + 1$ **c** $y = 2\cos 2\left(x + \frac{\pi}{4}\right) + \sqrt{3}$

5 Sketch the graph of each of the following, showing one complete cycle. State the period, amplitude and range in each case.

a $y = 2\sin\left(\theta - \frac{\pi}{3}\right)$ **b** $y = \sin 2(\theta - \pi)$ **c** $y = 3\sin 2\left(\theta + \frac{\pi}{4}\right)$

d $y = \sqrt{3}\sin 3\left(\theta - \frac{\pi}{2}\right)$ **e** $y = 2\sin(3x) + 1$ **f** $y = 3\cos 2\left(x + \frac{\pi}{2}\right) - 1$

g $y = \sqrt{2}\sin 2\left(\theta - \frac{\pi}{6}\right) + 2$ **h** $y = 3 - 4\sin(2x)$ **i** $y = 2 - 3\cos 2\left(\theta - \frac{\pi}{2}\right)$

6 Find the equation of the image of the graph of $y = \cos x$ under:

a a dilation of factor $\frac{1}{2}$ from the x-axis, followed by a dilation of factor 3 from the y-axis, followed by a translation of $\frac{\pi}{4}$ units in the positive direction of the x-axis

b a dilation of factor 2 from the x-axis, followed by a translation of $\frac{\pi}{4}$ units in the positive direction of the x-axis

c a dilation of factor $\frac{1}{3}$ from the x-axis, followed by a reflection in the x-axis, then followed by a translation of $\frac{\pi}{3}$ units in the positive direction of the x-axis.

7 Give a sequence of transformations that takes the graph of $y = \sin x$ to the graph of:

a $y = -3\sin(2x)$ **b** $y = -3\sin 2\left(x - \frac{\pi}{3}\right)$

c $y = 3\sin 2\left(x - \frac{\pi}{3}\right) + 2$ **d** $y = 5 - 2\sin 2\left(x - \frac{\pi}{3}\right)$

8 Sketch the graph of each of the following for $x \in [0, 2\pi]$. List the x-axis intercepts of each graph for this interval.

a $y = 2\cos x + 1$ **b** $y = 2\cos(2x) - \sqrt{3}$ **c** $y = \sqrt{2}\cos x - 1$

d $y = 2\cos x - 2$ **e** $y = \sqrt{2}\cos\left(x - \frac{\pi}{4}\right) + 1$

9 Sketch the graph of each of the following for $x \in [-\pi, \pi]$:

a $y = 2\sin\left(x - \frac{\pi}{4}\right) + 1$ **b** $y = 1 - 2\sin x$ **c** $y = 2\cos 3\left(x - \frac{\pi}{4}\right)$

d $y = 2\cos\left(3x - \frac{\pi}{4}\right)$ **e** $y = 1 - \cos(2x)$ **f** $y = -1 - \sin x$

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6H Addition of ordinates for circular functions

Sums of trigonometric functions play an important role in mathematics and have many applications, such as audio compression. We recall the following from Chapter 1:

Key points to consider when sketching $y = (f + g)(x)$

- When $f(x) = 0$, $(f + g)(x) = g(x)$.
- When $g(x) = 0$, $(f + g)(x) = f(x)$.
- If $f(x)$ and $g(x)$ are positive, then $(f + g)(x) > g(x)$ and $(f + g)(x) > f(x)$.
- If $f(x)$ and $g(x)$ are negative, then $(f + g)(x) < g(x)$ and $(f + g)(x) < f(x)$.
- If $f(x)$ is positive and $g(x)$ is negative, then $g(x) < (f + g)(x) < f(x)$.
- Look for values of x for which $f(x) + g(x) = 0$.

Example 23

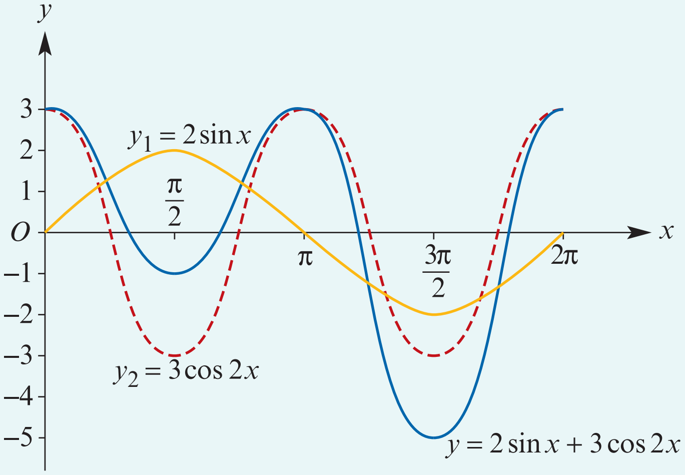
Using the same scale and axes, sketch the graphs of $y_1 = 2\sin x$ and $y_2 = 3\cos(2x)$ for $0 \le x \le 2\pi$. Use addition of ordinates to sketch the graph of $y = 2\sin x + 3\cos(2x)$.

Solution

The graphs of $y_1 = 2\sin x$ and $y_2 = 3\cos(2x)$ are shown. To obtain points on the graph of $y = 2\sin x + 3\cos(2x)$, the process of addition of ordinates is used.

Let $y = y_1 + y_2$ where $y_1 = 2\sin x$ and $y_2 = 3\cos(2x)$. A table of values is shown on the right.

x	y_1	y_2	y
0	0	3	3
$\frac{\pi}{4}$	$\sqrt{2}$	0	$\sqrt{2}$
$\frac{\pi}{2}$	2	−3	−1
π	0	3	3
$\frac{3\pi}{2}$	−2	−3	−5



Exercise 6H

Example 23

1 Use addition of ordinates to sketch the graph of each of the following for $\theta \in [-\pi, \pi]$:

a $y = \sin\theta + 2\cos\theta$

b $y = 2\cos(2\theta) + 3\sin(2\theta)$

c $y = \frac{1}{2}\cos(2\theta) - \sin\theta$

d $y = 3\cos\theta + \sin(2\theta)$

e $y = 2\sin\theta - 4\cos\theta$

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6I Determining rules for graphs of circular functions

In previous chapters, we introduced procedures for finding the rule for a graph known to come from a polynomial, exponential or logarithmic function. In this section, we find rules for graphs of functions known to be of the form $f(t) = A\sin(nt + \varepsilon) + b$.

Example 24

A function has rule $f(t) = A\sin(nt)$. The amplitude is 6; the period is 10. Find A and n and sketch the graph of $y = f(t)$ for $0 \le t \le 10$.

Solution

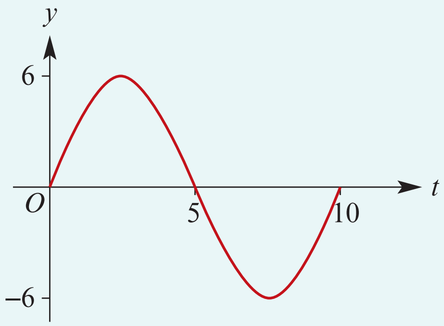
$$\text{Period} = \frac{2\pi}{n} = 10$$

$$\therefore \quad n = \frac{\pi}{5}$$

The amplitude is 6 and therefore $A = 6$.

The function has rule

$$f(t) = 6\sin\left(\frac{\pi t}{5}\right)$$

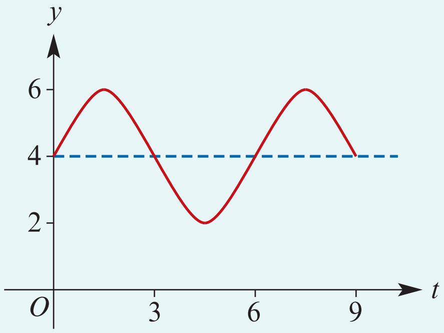


Example 25

The graph shown is that of a function with rule

$$y = A\sin(nt) + b$$

Find A, n and b.



Solution

The amplitude is 2 and so $A = 2$.

The period is 6. Therefore $\dfrac{2\pi}{n} = 6$ and so $n = \dfrac{\pi}{3}$.

The 'centreline' has equation $y = 4$ and so $b = 4$.

Hence the rule is

$$y = 2\sin\left(\frac{\pi t}{3}\right) + 4$$

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Example 26

A function with rule $y = A\sin(nt) + b$ has range $[-2, 4]$ and period 3. Find A, n and b.

Solution

The amplitude $A = \frac{1}{2}(4 - (-2)) = 3$. The 'centreline' has equation $y = 1$ and so $b = 1$.

The period is 3. Therefore $\dfrac{2\pi}{n} = 3$, which implies $n = \dfrac{2\pi}{3}$.

Hence the rule is $y = 3\sin\left(\dfrac{2\pi}{3}t\right) + 1$.

Example 27

A function with rule $y = A\sin(nt + \varepsilon)$ has the following properties:

- range $= [-2, 2]$
- period $= 6$
- when $t = 4$, $y = 0$.

Find values for A, n and ε.

Solution

Since the range is $[-2, 2]$, the amplitude $A = 2$.

Since the period is 6, we have $\dfrac{2\pi}{n} = 6$, which implies $n = \dfrac{\pi}{3}$.

Hence $y = 2\sin\left(\dfrac{\pi}{3}t + \varepsilon\right)$. When $t = 4$, $y = 0$ and so

$$2\sin\left(\frac{4\pi}{3} + \varepsilon\right) = 0$$

$$\sin\left(\frac{4\pi}{3} + \varepsilon\right) = 0$$

Therefore $\dfrac{4\pi}{3} + \varepsilon = 0$ or $\pm\pi$ or $\pm 2\pi$ or $\dots$

We choose the simplest solution, which is $\varepsilon = -\dfrac{4\pi}{3}$.

The rule $y = 2\sin\left(\dfrac{\pi}{3}t - \dfrac{4\pi}{3}\right)$ satisfies the three properties.

Exercise 6I

Example 24

1 **a** A function has rule $f(t) = A\sin(nt)$. The amplitude is 4; the period is 6. Find A and n and sketch the graph of $y = f(t)$ for $0 \le t \le 6$.

b A function has rule $f(t) = A\sin(nt)$. The amplitude is 2; the period is 7. Find A and n and sketch the graph of $y = f(t)$ for $0 \le t \le 7$.

c A function has rule $f(t) = A\cos(nt)$. The amplitude is 3; the period is 5. Find A and n and sketch the graph of $y = f(t)$ for $0 \le t \le 5$.

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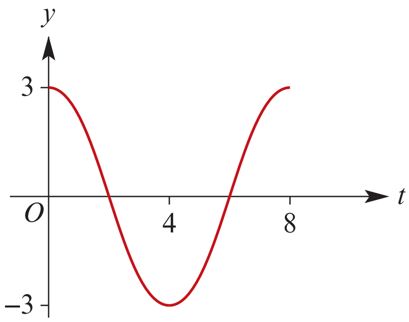
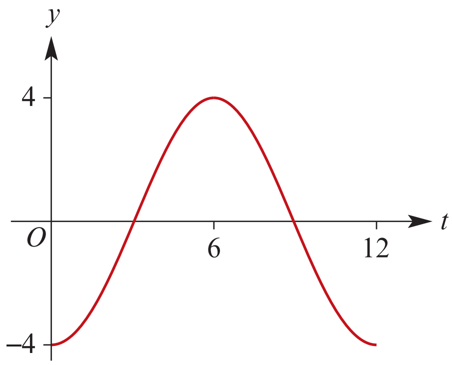
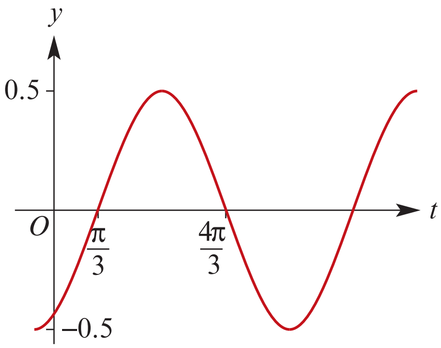
Example 25 **2** The graph shown has rule of the form $y = A\cos(nt)$.
Find the values of A and n.



3 The graph shown has rule of the form $y = A\cos(nt)$.
Find the values of A and n.



4 The graph shown has rule of the form $y = A\sin(t + \varepsilon)$.
Find possible values for A and ε.



Example 26 **5** A function with rule $y = A\sin(nt) + b$ has range $[2, 8]$ and period $\frac{2\pi}{3}$. Find the values of A, n and b.

Example 27 **6** A function with rule $y = A\sin(nt + \varepsilon)$ has the following three properties:

- range $= [-4, 4]$
- period $= 8$
- when $t = 2$, $y = 0$.

Find values for A, n and ε.

7 A function with rule $y = A\sin(nt + \varepsilon)$ has range $[-2, 2]$ and period 6, and when $t = 1$, $y = 1$. Find possible values for A, n and ε.

8 A function with rule $y = A\sin(nt + \varepsilon) + d$ has range $[-2, 6]$ and period 8, and when $t = 2$, $y = 2$. Find possible values for A, n, d and ε.

9 A function with rule $y = A\sin(nt + \varepsilon) + d$ has range $[0, 4]$ and period 6, and when $t = 1$, $y = 3$. Find possible values for A, n, d and ε.

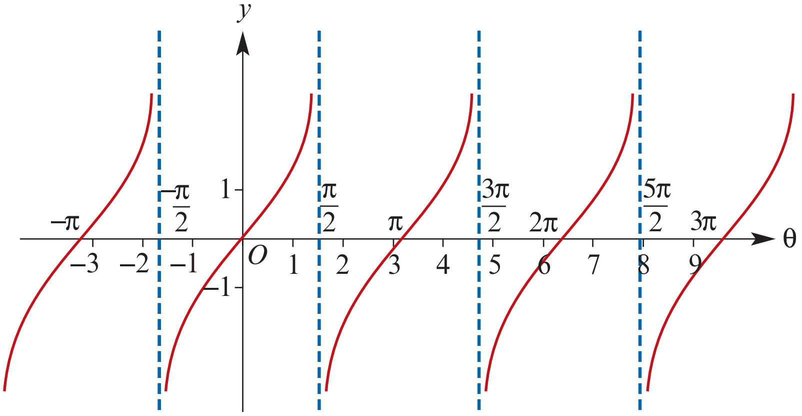
6J The tangent function

The tangent function is given by

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad \text{for } \cos\theta \neq 0$$

A table of values for $y = \tan\theta$ is given below:

θ	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π	$\frac{9\pi}{4}$	$\frac{5\pi}{2}$	$\frac{11\pi}{4}$	3π
y	0	1	ud	−1	0	1	ud	−1	0	1	ud	−1	0	1	ud	−1	0



Note: There are vertical asymptotes at $\theta = \frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$ and $\frac{5\pi}{2}$.

Observations from the graph of $y = \tan\theta$

- The graph repeats itself every π units, i.e. the period of tan is π.
- The range of tan is $\mathbb{R}$.
- The vertical asymptotes have equations $\theta = \frac{(2k+1)\pi}{2}$ where $k \in \mathbb{Z}$.
- The axis intercepts are at $\theta = k\pi$ where $k \in \mathbb{Z}$.

Graph of $y = a\tan(nt)$

For a and n positive numbers, the graph of $y = a\tan(nt)$ is obtained from the graph of $y = \tan t$ by a dilation of factor a from the t-axis and a dilation of factor $\frac{1}{n}$ from the y-axis.

The following are important properties of $f(t) = a\tan(nt)$:

- The period is $\frac{\pi}{n}$.
- The range is $\mathbb{R}$.
- The vertical asymptotes have equations $t = \frac{(2k+1)\pi}{2n}$ where $k \in \mathbb{Z}$.
- The axis intercepts are at $t = \frac{k\pi}{n}$ where $k \in \mathbb{Z}$.

Example 28

Sketch the graph of each of the following for $x \in [-\pi, \pi]$:

a $y = 3\tan(2x)$

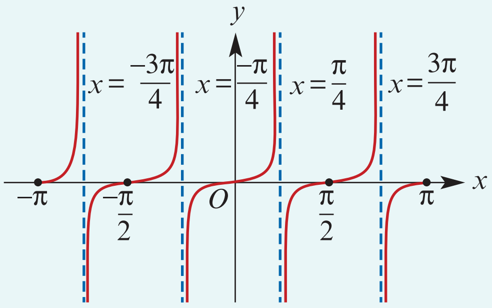
b $y = -2\tan(3x)$

Solution

a Period $= \dfrac{\pi}{n} = \dfrac{\pi}{2}$

Asymptotes: $x = \dfrac{(2k+1)\pi}{4}, \ k \in \mathbb{Z}$

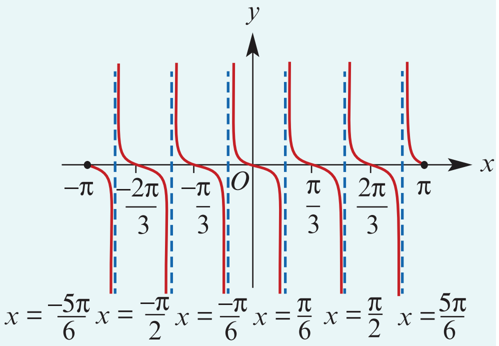
Axis intercepts: $x = \dfrac{k\pi}{2}, \ k \in \mathbb{Z}$



b Period $= \dfrac{\pi}{n} = \dfrac{\pi}{3}$

Asymptotes: $x = \dfrac{(2k+1)\pi}{6}, \ k \in \mathbb{Z}$

Axis intercepts: $x = \dfrac{k\pi}{3}, \ k \in \mathbb{Z}$



Example 29

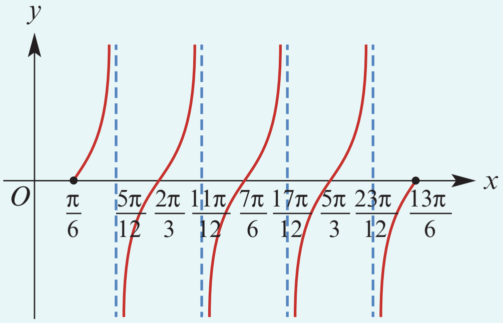
Sketch the graph of $y = 3\tan\left(2x - \dfrac{\pi}{3}\right)$ for $\dfrac{\pi}{6} \le x \le \dfrac{13\pi}{6}$.

Solution

Consider $y = 3\tan 2\left(x - \dfrac{\pi}{6}\right)$.

The graph is the image of $y = \tan x$ under:

- a dilation of factor 3 from the x-axis
- a dilation of factor $\frac{1}{2}$ from the y-axis
- a translation of $\dfrac{\pi}{6}$ units in the positive direction of the x-axis.



Period $= \dfrac{\pi}{2}$

Asymptotes: $x = \dfrac{(2k+1)\pi}{4} + \dfrac{\pi}{6} = \dfrac{(6k+5)\pi}{12}, k \in \mathbb{Z}$

Axis intercepts: $x = \dfrac{k\pi}{2} + \dfrac{\pi}{6} = \dfrac{(3k+1)\pi}{6}, k \in \mathbb{Z}$

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Solution of equations involving the tangent function

We now consider the solution of equations involving the tangent function. We will then apply this to finding the x-axis intercepts for graphs of the tangent function which have been translated parallel to the y-axis.

We recall the following exact values:

$$\tan 0 = 0, \qquad \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}, \qquad \tan\left(\frac{\pi}{4}\right) = 1, \qquad \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$$

and the symmetry properties:

- $\tan(\pi + \theta) = \tan\theta$
- $\tan(-\theta) = -\tan\theta$

Example 30

Solve the equation $3\tan(2x) = \sqrt{3}$ for $x \in (0, 2\pi)$.

Solution

$$3\tan(2x) = \sqrt{3}$$

$$\tan(2x) = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$$

$$\therefore \quad 2x = \frac{\pi}{6} \text{ or } \frac{7\pi}{6} \text{ or } \frac{13\pi}{6} \text{ or } \frac{19\pi}{6}$$

$$x = \frac{\pi}{12} \text{ or } \frac{7\pi}{12} \text{ or } \frac{13\pi}{12} \text{ or } \frac{19\pi}{12}$$

Explanation

Since we want solutions for x in $(0, 2\pi)$, we find solutions for $2x$ in $(0, 4\pi)$.

Once we have found one solution for $2x$, we can obtain all other solutions by adding and subtracting multiples of π.

Example 31

Solve the equation $\tan\left(\frac{1}{2}\left(x - \frac{\pi}{4}\right)\right) = -1$ for $x \in [-2\pi, 2\pi]$.

Solution

$$\tan\left(\frac{1}{2}\left(x - \frac{\pi}{4}\right)\right) = -1$$

implies

$$\frac{1}{2}\left(x - \frac{\pi}{4}\right) = \frac{-\pi}{4} \text{ or } \frac{3\pi}{4}$$

$$x - \frac{\pi}{4} = \frac{-\pi}{2} \text{ or } \frac{3\pi}{2}$$

$$\therefore \quad x = \frac{-\pi}{4} \text{ or } \frac{7\pi}{4}$$

Explanation

Note that

$$x \in [-2\pi, 2\pi] \iff x - \frac{\pi}{4} \in \left[-\frac{9\pi}{4}, \frac{7\pi}{4}\right]$$

$$\iff \frac{1}{2}\left(x - \frac{\pi}{4}\right) \in \left[-\frac{9\pi}{8}, \frac{7\pi}{8}\right]$$

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Example 32

Sketch the graph of $y = 3\tan\left(2x - \frac{\pi}{3}\right) + \sqrt{3}$ for $\frac{\pi}{6} \le x \le \frac{13\pi}{6}$.

Solution

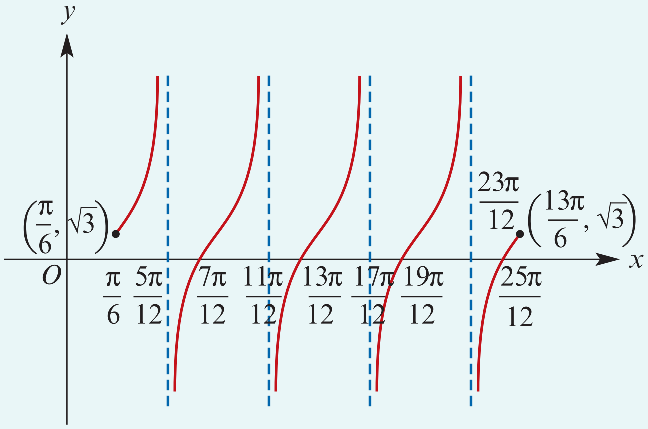
First write the equation as

$$y = 3\tan 2\left(x - \frac{\pi}{6}\right) + \sqrt{3}$$

The graph is the image of $y = \tan x$ under:

- a dilation of factor 3 from the x-axis
- a dilation of factor $\frac{1}{2}$ from the y-axis
- a translation of $\frac{\pi}{6}$ units in the positive direction of the x-axis
- a translation of $\sqrt{3}$ units in the positive direction of the y-axis.

The graph can be obtained from the graph in Example 29 by a translation $\sqrt{3}$ units in the positive direction of the y-axis.



To find the x-axis intercepts, solve the equation:

$$3\tan\left(2x - \frac{\pi}{3}\right) + \sqrt{3} = 0 \quad \text{for} \quad \frac{\pi}{6} \le x \le \frac{13\pi}{6}$$

$$\therefore \quad \tan\left(2x - \frac{\pi}{3}\right) = -\frac{\sqrt{3}}{3} = \frac{-1}{\sqrt{3}}$$

$$\therefore \quad 2x - \frac{\pi}{3} = \frac{5\pi}{6} \text{ or } \frac{11\pi}{6} \text{ or } \frac{17\pi}{6} \text{ or } \frac{23\pi}{6}$$

$$2x = \frac{7\pi}{6} \text{ or } \frac{13\pi}{6} \text{ or } \frac{19\pi}{6} \text{ or } \frac{25\pi}{6}$$

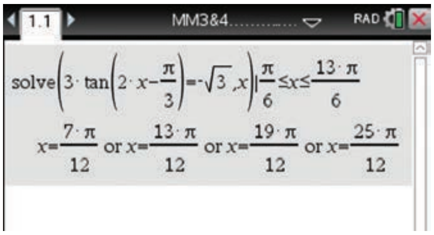
$$x = \frac{7\pi}{12} \text{ or } \frac{13\pi}{12} \text{ or } \frac{19\pi}{12} \text{ or } \frac{25\pi}{12}$$

Hence the x-axis intercepts are $\frac{7\pi}{12}, \frac{13\pi}{12}, \frac{19\pi}{12}$ and $\frac{25\pi}{12}$.

Using the TI-Nspire

To find the x-axis intercepts, enter:

$$\text{solve}\left(3\tan\left(2x-\frac{\pi}{3}\right)=-\sqrt{3},\ x\right)\ \Big|\ \frac{\pi}{6}\le x\le\frac{13\pi}{6}$$

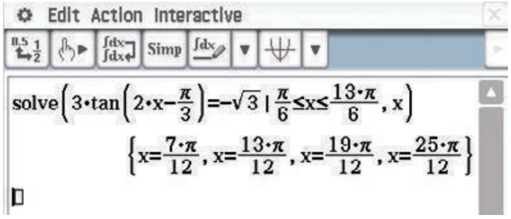


$\text{solve}\left(3\cdot\tan\left(2\cdot x-\frac{\pi}{3}\right)=-\sqrt{3},x\right)\Big|\frac{\pi}{6}\le x\le\frac{13\cdot\pi}{6}$

$x=\frac{7\cdot\pi}{12}$ or $x=\frac{13\cdot\pi}{12}$ or $x=\frac{19\cdot\pi}{12}$ or $x=\frac{25\cdot\pi}{12}$

Using the Casio ClassPad

- Make sure that the calculator is in radian mode.
- To find the x-axis intercepts, enter
  $3\tan\left(2x-\frac{\pi}{3}\right)=-\sqrt{3}\ \Big|\ \frac{\pi}{6}\le x\le\frac{13\pi}{6}$
- Highlight and then select **Interactive** > **Equation/Inequality** > **solve**.



$\text{solve}\left(3\cdot\tan\left(2\cdot x-\frac{\pi}{3}\right)=-\sqrt{3}\ \Big|\ \frac{\pi}{6}\le x\le\frac{13\cdot\pi}{6},x\right)$

$\left\{x=\frac{7\cdot\pi}{12},x=\frac{13\cdot\pi}{12},x=\frac{19\cdot\pi}{12},x=\frac{25\cdot\pi}{12}\right\}$

Solution of equations of the form $\sin(nx) = k\cos(nx)$

We can find the coordinates of the points of intersection of certain sine and cosine graphs by using the following observation:

If $\sin(nx) = k\cos(nx)$, then $\tan(nx) = k$.

This is obtained by dividing both sides of the equation $\sin(nx) = k\cos(nx)$ by $\cos(nx)$, for $\cos(nx) \neq 0$.

Example 33

On the same set of axes, sketch the graphs of $y = \sin x$ and $y = \cos x$ for $x \in [0, 2\pi]$ and find the coordinates of the points of intersection.

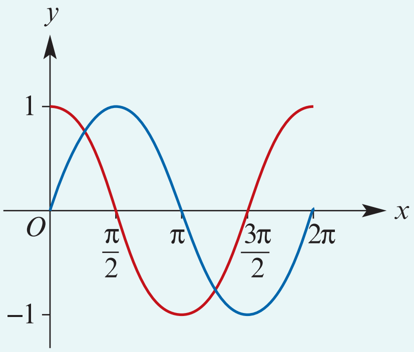
Solution

$\sin x = \cos x$ implies $\tan x = 1$

$$\therefore\quad x = \frac{\pi}{4} \text{ or } \frac{5\pi}{4}$$

The coordinates of the points of intersection are

$$\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right) \text{ and } \left(\frac{5\pi}{4}, -\frac{1}{\sqrt{2}}\right)$$



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Example 34

Solve the equation $\sin(2x) = \cos(2x)$ for $x \in [0, 2\pi]$.

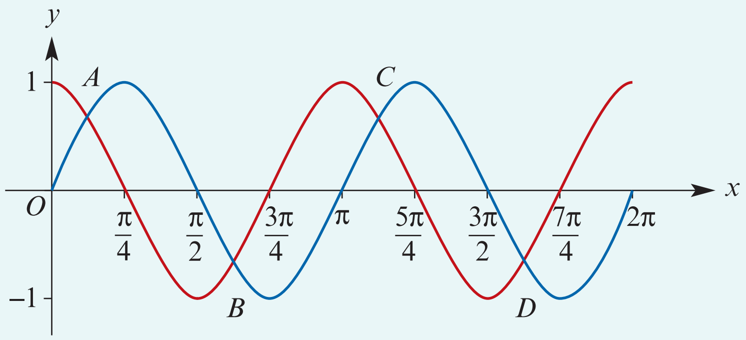
Solution

$\sin(2x) = \cos(2x)$ implies $\tan(2x) = 1$

$$\therefore \quad 2x = \frac{\pi}{4} \text{ or } \frac{5\pi}{4} \text{ or } \frac{9\pi}{4} \text{ or } \frac{13\pi}{4}$$

$$\therefore \quad x = \frac{\pi}{8} \text{ or } \frac{5\pi}{8} \text{ or } \frac{9\pi}{8} \text{ or } \frac{13\pi}{8}$$

This can be shown graphically.



The points of intersection A, B, C and D occur when $x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}$ and $\frac{13\pi}{8}$ respectively.

Section summary

- The tangent function is given by $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ for $\cos\theta \neq 0$.
  - The period is π.
  - The vertical asymptotes have equations $\theta = \dfrac{(2k+1)\pi}{2}$ where $k \in \mathbb{Z}$.
  - The axis intercepts are at $\theta = k\pi$ where $k \in \mathbb{Z}$.
- Useful symmetry properties:
  - $\tan(\pi + \theta) = \tan\theta$
  - $\tan(-\theta) = -\tan\theta$

Exercise 6J

1 State the period for each of the following:

a $\tan(3\theta)$ **b** $\tan\left(\dfrac{\theta}{2}\right)$ **c** $\tan\left(\dfrac{3\theta}{2}\right)$ **d** $\tan(\pi\theta)$ **e** $\tan\left(\dfrac{\pi\theta}{2}\right)$

Example 28

2 Sketch the graph of each of the following for $x \in (0, 2\pi)$:

a $y = \tan(2x)$ **b** $y = 2\tan(3x)$ **c** $y = -2\tan(2x)$

Example 29

3 Sketch the graph of each of the following for $x \in (0, 2\pi)$:

a $y = 2\tan\left(x + \dfrac{\pi}{4}\right)$ **b** $y = 2\tan 3\left(x + \dfrac{\pi}{2}\right)$ **c** $y = 3\tan 2\left(x - \dfrac{\pi}{4}\right)$

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Example 30 **4** **a** Solve the equation $\tan(2x) = 1$ for $x \in (0, 2\pi)$.

b Solve the equation $\tan(2x) = -1$ for $x \in (-\pi, \pi)$.

c Solve the equation $\tan(2x) = -\sqrt{3}$ for $x \in (-\pi, \pi)$.

d Solve the equation $\tan(2x) = \sqrt{3}$ for $x \in (-\pi, \pi)$.

e Solve the equation $\tan(2x) = \dfrac{1}{\sqrt{3}}$ for $x \in (-\pi, \pi)$.

Example 31 **5** Solve the equation $\tan 2\left(x - \dfrac{\pi}{3}\right) = 1$ for $x \in [0, 2\pi]$.

6 Solve the equation $\tan\left(x - \dfrac{\pi}{4}\right) = \sqrt{3}$ for $x \in [0, 2\pi]$.

Example 32 **7** Sketch the graph of each of the following for $x \in (0, 2\pi)$:

a $y = 3\tan x + 1$ **b** $y = 2\tan\left(x + \dfrac{\pi}{2}\right) + 1$ **c** $y = 3\tan 2\left(x - \dfrac{\pi}{4}\right) - 2$

8 Sketch the graph of $y = -2\tan(\pi\theta)$ for $-2 \le \theta \le 2$.

9 Sketch the graph of $y = \tan(-2\theta)$ for $\dfrac{-\pi}{2} \le \theta \le \dfrac{\pi}{2}$.

Example 33 **10** **a** On the same set of axes, sketch the graphs of $y = \cos(2x)$ and $y = -\sin(2x)$ for $x \in [-\pi, \pi]$.

b Find the coordinates of the points of intersection.

c On the same set of axes, sketch the graph of $y = \cos(2x) - \sin(2x)$.

Example 34 **11** Solve each of the following equations for $x \in [0, 2\pi]$:

a $\sqrt{3}\sin x = \cos x$ **b** $\sin(4x) = \cos(4x)$ **c** $\sqrt{3}\sin(2x) = \cos(2x)$

d $-\sqrt{3}\sin(2x) = \cos(2x)$ **e** $\sin(3x) = -\cos(3x)$ **f** $\sin x = 0.5\cos x$

g $\sin x = 2\cos x$ **h** $\sin(2x) = -\cos(2x)$ **i** $\cos(3x) = \sqrt{3}\sin(3x)$

j $\sin(3x) = \sqrt{3}\cos(3x)$

12 **a** On the same set of axes, sketch the graphs of $y = \cos x$ and $y = \sqrt{3}\sin x$ for $x \in [0, 2\pi]$.

b Find the coordinates of the points of intersection.

c On the same set of axes, sketch the graph of $y = \cos x + \sqrt{3}\sin x$.

13 Solve each of the following equations for $0 \le x \le 2\pi$:

a $\tan\left(2x - \dfrac{\pi}{4}\right) = \sqrt{3}$ **b** $3\tan(2x) = -\sqrt{3}$ **c** $\tan\left(3x - \dfrac{\pi}{6}\right) = -1$

14 A function with rule $y = A\tan(nt)$ has the following properties:

- the asymptotes have equations $t = \dfrac{(2k+1)\pi}{6}$ where $k \in \mathbb{Z}$
- when $t = \dfrac{\pi}{12}$, $y = 5$.

Find values for A and n.

15 A function with rule $y = A\tan(nt)$ has period 2 and, when $t = \frac{1}{2}$, $y = 6$. Find values for A and n.

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6K General solution of trigonometric equations

We have seen how to solve equations involving circular functions over a restricted domain. We now consider the general solutions of such equations over the maximal domain for each function.

By convention:

- $\cos^{-1}$ has range $[0, \pi]$
- $\sin^{-1}$ has range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
- $\tan^{-1}$ has range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

For example:

$$\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}, \qquad \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}, \qquad \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}, \qquad \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

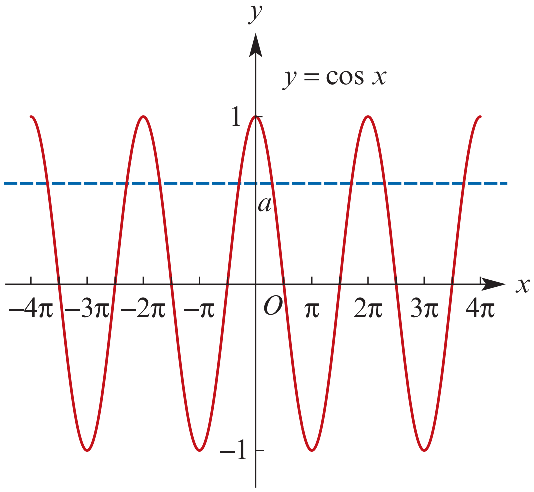
If an equation involving a circular function has one or more solutions in one 'cycle', then it will have corresponding solutions in each 'cycle' of its domain, i.e. there will be infinitely many solutions.

For example, consider the equation

$$\cos x = a$$

for some fixed $a \in [-1, 1]$. The solution in the interval $[0, \pi]$ is given by

$$x = \cos^{-1}(a)$$



By the symmetry properties of the cosine function, the other solutions are given by

$$-\cos^{-1}(a),\ \pm 2\pi + \cos^{-1}(a),\ \pm 2\pi - \cos^{-1}(a),\ \pm 4\pi + \cos^{-1}(a),\ \pm 4\pi - \cos^{-1}(a),\ \dots$$

In general, we have the following:

- For $a \in [-1, 1]$, the general solution of the equation $\cos x = a$ is

  $$x = 2n\pi \pm \cos^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

- For $a \in \mathbb{R}$, the general solution of the equation $\tan x = a$ is

  $$x = n\pi + \tan^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

- For $a \in [-1, 1]$, the general solution of the equation $\sin x = a$ is

  $$x = 2n\pi + \sin^{-1}(a) \quad \text{or} \quad x = (2n+1)\pi - \sin^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

Note: An alternative and more concise way to express the general solution of $\sin x = a$ is $x = n\pi + (-1)^n \sin^{-1}(a)$, where $n \in \mathbb{Z}$.

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Example 35

Find the general solution of each of the following equations:

a $\cos x = 0.5$ **b** $\sqrt{3}\tan(3x) = 1$ **c** $2\sin x = \sqrt{2}$

Solution

a $\cos x = 0.5$

$$\begin{aligned} x &= 2n\pi \pm \cos^{-1}(0.5) \\ &= 2n\pi \pm \frac{\pi}{3} \\ &= \frac{(6n \pm 1)\pi}{3}, \quad n \in \mathbb{Z} \end{aligned}$$

b $\tan(3x) = \dfrac{1}{\sqrt{3}}$

$$\begin{aligned} 3x &= n\pi + \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \\ &= n\pi + \frac{\pi}{6} \\ &= \frac{(6n+1)\pi}{6} \\ x &= \frac{(6n+1)\pi}{18}, \quad n \in \mathbb{Z} \end{aligned}$$

c $\sin x = \dfrac{\sqrt{2}}{2} = \dfrac{1}{\sqrt{2}}$

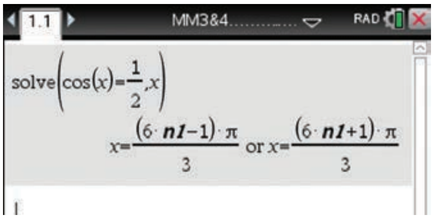
$$\begin{aligned} x &= 2n\pi + \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) \\ &= 2n\pi + \frac{\pi}{4} \\ &= \frac{(8n+1)\pi}{4}, \quad n \in \mathbb{Z} \end{aligned}$$

or

$$\begin{aligned} x &= (2n+1)\pi - \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) \\ &= (2n+1)\pi - \frac{\pi}{4} \\ &= \frac{(8n+3)\pi}{4}, \quad n \in \mathbb{Z} \end{aligned}$$

Using the TI-Nspire

- Make sure the calculator is in radian mode.
- Use **solve** from the **Algebra** menu and complete as shown. Note the use of $\frac{1}{2}$ rather than 0.5 to ensure that the answer is exact.



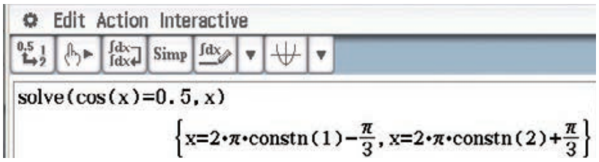
$\text{solve}\left(\cos(x)=\frac{1}{2},x\right)$

$x=\frac{(6\cdot n1-1)\cdot\pi}{3}$ or $x=\frac{(6\cdot n1+1)\cdot\pi}{3}$

Using the Casio ClassPad

- Check that the calculator is in radian mode.
- In Main √α, enter and highlight the equation $\cos(x) = 0.5$.
- Select **Interactive** > **Equation/Inequality** > **solve**. Then tap EXE.
- To view the entire solution, rotate the screen by selecting Rotate.

Note: Replace constn(1) and constn(2) with n in the written answer.



solve(cos(x)=0.5,x)

$\left\{x=2\cdot\pi\cdot\text{constn}(1)-\frac{\pi}{3},\ x=2\cdot\pi\cdot\text{constn}(2)+\frac{\pi}{3}\right\}$

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Example 36

Find the first three positive solutions of each of the following equations:

a $\cos x = 0.5$ **b** $\sqrt{3}\tan(3x) = 1$ **c** $2\sin x = \sqrt{2}$

Solution

a The general solution (from Example 35a) is given by $x = \dfrac{(6n \pm 1)\pi}{3}$, $n \in \mathbb{Z}$.

When $n = 0$, $x = \pm\dfrac{\pi}{3}$, and when $n = 1$, $x = \dfrac{5\pi}{3}$ or $x = \dfrac{7\pi}{3}$.

Thus the first three positive solutions of $\cos x = 0.5$ are $x = \dfrac{\pi}{3}, \dfrac{5\pi}{3}, \dfrac{7\pi}{3}$.

b The general solution (from Example 35b) is given by $x = \dfrac{(6n + 1)\pi}{18}$, $n \in \mathbb{Z}$.

When $n = 0$, $x = \dfrac{\pi}{18}$, and when $n = 1$, $x = \dfrac{7\pi}{18}$, and when $n = 2$, $x = \dfrac{13\pi}{18}$.

Thus the first three positive solutions of $\sqrt{3}\tan(3x) = 1$ are $x = \dfrac{\pi}{18}, \dfrac{7\pi}{18}, \dfrac{13\pi}{18}$.

c The general solution (from Example 35c) is $x = \dfrac{(8n + 1)\pi}{4}$ or $x = \dfrac{(8n + 3)\pi}{4}$, $n \in \mathbb{Z}$.

When $n = 0$, $x = \dfrac{\pi}{4}$ or $x = \dfrac{3\pi}{4}$, and when $n = 1$, $x = \dfrac{9\pi}{4}$ or $x = \dfrac{11\pi}{4}$.

Thus the first three positive solutions of $2\sin x = \sqrt{2}$ are $x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{9\pi}{4}$.

Example 37

Find the general solution for each of the following:

a $\sin\left(x - \dfrac{\pi}{3}\right) = \dfrac{\sqrt{3}}{2}$ **b** $\tan\left(2x - \dfrac{\pi}{3}\right) = 1$

Solution

a $\sin\left(x - \dfrac{\pi}{3}\right) = \dfrac{\sqrt{3}}{2}$

$$x - \frac{\pi}{3} = n\pi + (-1)^n \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$\therefore \quad x = n\pi + (-1)^n\left(\frac{\pi}{3}\right) + \frac{\pi}{3}, \quad n \in \mathbb{Z}$$

The solutions are $x = \dfrac{(3n + 2)\pi}{3}$ for n even and $x = n\pi$ for n odd.

b $\tan\left(2x - \dfrac{\pi}{3}\right) = 1$

$$2x - \frac{\pi}{3} = n\pi + \frac{\pi}{4}$$

$$2x = n\pi + \frac{\pi}{4} + \frac{\pi}{3}$$

$$\therefore \quad x = \frac{1}{2}\left(n\pi + \frac{7\pi}{12}\right)$$

$$= \frac{(12n + 7)\pi}{24}, \quad n \in \mathbb{Z}$$

Section summary

- For $a \in [-1, 1]$, the general solution of the equation $\cos x = a$ is

$$x = 2n\pi \pm \cos^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

- For $a \in \mathbb{R}$, the general solution of the equation $\tan x = a$ is

$$x = n\pi + \tan^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

- For $a \in [-1, 1]$, the general solution of the equation $\sin x = a$ is

$$x = 2n\pi + \sin^{-1}(a) \quad \text{or} \quad x = (2n+1)\pi - \sin^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

Exercise 6K

1 Evaluate each of the following for:

i $n = 1$ **ii** $n = 2$ **iii** $n = -2$

a $2n\pi \pm \cos^{-1}(1)$ **b** $2n\pi \pm \cos^{-1}\left(-\frac{1}{2}\right)$

Example 35

2 Find the general solution of each of the following equations:

a $\cos x = \frac{\sqrt{3}}{2}$ **b** $2\sin(3x) = \sqrt{3}$ **c** $\sqrt{3}\tan x = 3$

Example 36

3 Find the first two positive solutions of each of the following equations:

a $\sin x = 0.5$ **b** $2\cos(2x) = \sqrt{3}$ **c** $\sqrt{3}\tan(2x) = -3$

4 Given that a trigonometric equation has general solution $x = n\pi + (-1)^n \sin^{-1}\left(\frac{1}{2}\right)$, where $n \in \mathbb{Z}$, find the solutions of the equation in the interval $[-2\pi, 2\pi]$.

5 Given that a trigonometric equation has general solution $x = 2n\pi \pm \cos^{-1}\left(\frac{1}{2}\right)$, where $n \in \mathbb{Z}$, find the solutions of the equation in the interval $[-\pi, 2\pi]$.

Example 37

6 Find the general solution for each of the following:

a $\cos 2\left(x + \frac{\pi}{3}\right) = \frac{1}{2}$ **b** $2\tan 2\left(x + \frac{\pi}{4}\right) = 2\sqrt{3}$ **c** $2\sin\left(x + \frac{\pi}{3}\right) = -1$

7 Find the general solution of $2\cos\left(2x + \frac{\pi}{4}\right) = \sqrt{2}$ and hence find all the solutions for x in the interval $(-2\pi, 2\pi)$.

8 Find the general solution of $\sqrt{3}\tan\left(\frac{\pi}{6} - 3x\right) - 1 = 0$ and hence find all the solutions for x in the interval $[-\pi, 0]$.

9 Find the general solution of $2\sin(4\pi x) + \sqrt{3} = 0$ and hence find all the solutions for x in the interval $[-1, 1]$.

6L Applications of circular functions

A **sinusoidal function** has a rule of the form $y = a\sin(nt + \varepsilon) + b$ or, equivalently, of the form $y = a\cos(nt + \varepsilon) + b$. Such functions can be used to model periodic motion.

Example 38

A wheel is mounted on a wall and rotates such that the distance, d cm, of a particular point P on the wheel from the ground is given by the rule

$$d = 100 - 60\cos\left(\frac{4\pi}{3}t\right)$$

where t is the time in seconds.

a How far is the point P above the ground when $t = 0$?

b How long does it take for the wheel to rotate once?

c Find the maximum and minimum distances of the point P above the ground.

d Sketch the graph of d against t.

e In the first rotation, find the intervals of time when the point P is less than 70 cm above the ground.

Solution

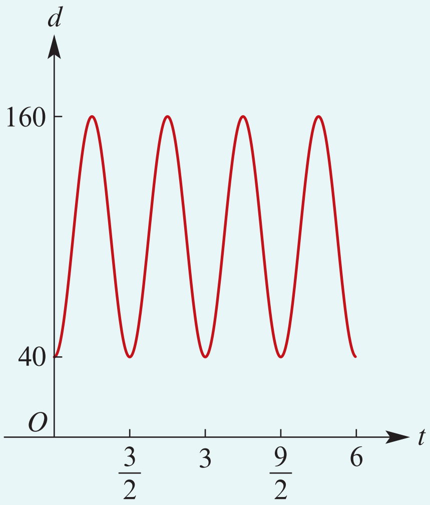
a When $t = 0$, $d = 100 - 60 \times 1 = 40$. The point is 40 cm above the ground.

b The period is $2\pi \div \frac{4\pi}{3} = \frac{3}{2}$. The wheel takes $\frac{3}{2}$ seconds to rotate once.

c The minimum occurs when $\cos\left(\frac{4\pi}{3}t\right) = 1$, which gives $d = 100 - 60 = 40$. Hence the minimum distance is 40 cm.

The maximum occurs when $\cos\left(\frac{4\pi}{3}t\right) = -1$, which gives $d = 100 + 60 = 160$. Hence the maximum distance is 160 cm.

d



e

$$100 - 60\cos\left(\frac{4\pi}{3}t\right) = 70$$

$$-60\cos\left(\frac{4\pi}{3}t\right) = -30$$

$$\cos\left(\frac{4\pi}{3}t\right) = \frac{1}{2}$$

$$\therefore \quad \frac{4\pi}{3}t = \frac{\pi}{3} \text{ or } \frac{5\pi}{3}$$

$$t = \frac{1}{4} \text{ or } \frac{5}{4}$$

From the graph, the distance is less than 70 cm for $t \in \left(0, \frac{1}{4}\right)$ and for $t \in \left(\frac{5}{4}, \frac{3}{2}\right)$.

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Example 39

It is suggested that the height, $h(t)$ metres, of the tide above mean sea level on 1 January at Warnung is given approximately by the rule $h(t) = 4\sin\left(\frac{\pi}{6}t\right)$, where t is the number of hours after midnight.

a Draw the graph of $y = h(t)$ for $0 \le t \le 24$.

b When was high tide?

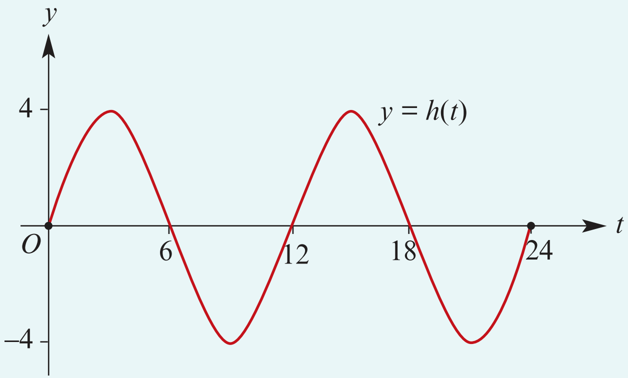
c What was the height of the high tide?

d What was the height of the tide at 8 a.m.?

e A boat can only cross the harbour bar when the tide is at least 1 metre above mean sea level. When could the boat cross the harbour bar on 1 January?

Solution

a



Note: Period $= 2\pi \div \frac{\pi}{6} = 12$

b High tide occurs when $h(t) = 4$:

$$4\sin\left(\frac{\pi}{6}t\right) = 4$$

$$\sin\left(\frac{\pi}{6}t\right) = 1$$

$$\frac{\pi}{6}t = \frac{\pi}{2}, \frac{5\pi}{2}$$

$$\therefore \quad t = 3, 15$$

i.e. high tide occurs at 03:00 and 15:00 (3 p.m.).

c The high tide has height 4 metres above the mean height.

d $h(8) = 4\sin\left(\frac{8\pi}{6}\right) = 4\sin\left(\frac{4\pi}{3}\right) = 4 \times \frac{-\sqrt{3}}{2} = -2\sqrt{3}$

At 8 a.m. the water is $2\sqrt{3}$ metres below the mean height.

e We first consider $4\sin\left(\frac{\pi}{6}t\right) = 1$:

$$\sin\left(\frac{\pi}{6}t\right) = \frac{1}{4}$$

$$\therefore \quad \frac{\pi}{6}t = 0.2526, 2.889, 6.5358, 9.172$$

$$\therefore \quad t = 0.4824, 5.5176, 12.4824, 17.5173$$

i.e. the water is at height 1 metre at 00:29, 05:31, 12:29, 17:31.
Thus the boat can pass across the harbour bar between 00:29 and 05:31, and between 12:29 and 17:31.

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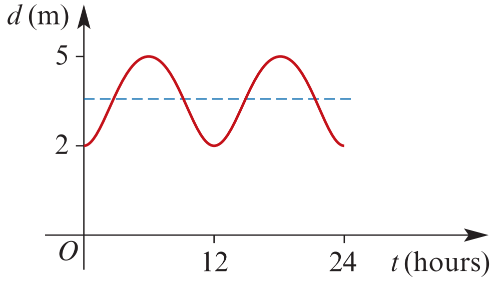
Exercise 6L

Example 38 **1** The graph shows the distance, $d(t)$, of the tip of the hour hand of a large clock from the ceiling at time t hours.



- **a** The function d is sinusoidal. Find:
  - **i** the amplitude
  - **ii** the period
  - **iii** the rule for $d(t)$
  - **iv** the length of the hour hand.
- **b** At what times is the distance less than 3.5 metres from the ceiling?

Example 39 **2** The water level on a beach wall is given by

$$d(t) = 6 + 4\cos\left(\frac{\pi}{6}t - \frac{\pi}{3}\right)$$

where t is the number of hours after midnight and d is the depth of the water in metres.

- **a** Sketch the graph of $d(t)$ for $0 \le t \le 24$.
- **b** What is the earliest time of day at which the water is at its highest?
- **c** When is the water 2 m up the wall?

3 In a tidal river, the time between high tides is 12 hours. The average depth of water at a point in the river is 5 m; at high tide the depth is 8 m. Assume that the depth of water, $h(t)$ m, at this point is given by

$$h(t) = A\sin(nt + \varepsilon) + b$$

where t is the number of hours after noon. At noon there is a high tide.

- **a** Find the values of A, n, b and ε.
- **b** At what times is the depth of the water 6 m?
- **c** Sketch the graph of $y = h(t)$ for $0 \le t \le 24$.

4 A particle moves along a straight line. Its position, x metres, relative to a fixed point O on the line is given by $x = 3 + 2\sin(3t)$, where t is the time in seconds.

- **a** Find its greatest distance from O.
- **b** Find its least distance from O.
- **c** Find the times at which it is 5 m from O for $0 \le t \le 5$.
- **d** Find the times at which it is 3 m from O for $0 \le t \le 3$.
- **e** Describe the motion of the particle.

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5 The temperature, A°C, inside a house at t hours after 4 a.m. is given by the rule $A = 21 - 3\cos\left(\frac{\pi t}{12}\right)$, for $0 \le t \le 24$. The temperature, B°C, outside the house at the same time is given by $B = 22 - 5\cos\left(\frac{\pi t}{12}\right)$, for $0 \le t \le 24$.

- **a** Find the temperature inside the house at 8 a.m.
- **b** Write down an expression for $D = A - B$, the difference between the inside and outside temperatures.
- **c** Sketch the graph of D for $0 \le t \le 24$.
- **d** Determine when the inside temperature is less than the outside temperature.

6 Passengers on a ferris wheel access their seats from a platform 5 m above the ground. As each seat is filled, the ferris wheel moves around so that the next seat can be filled. Once all seats are filled, the ride begins and lasts for 6 minutes. The height, h m, of Isobel's seat above the ground t seconds after the ride has begun is given by $h = 15\sin(10t - 45)° + 16.5$.

- **a** Use a calculator to sketch the graph of h against t for the first 2 minutes of the ride.
- **b** How far above the ground is Isobel's seat at the commencement of the ride?
- **c** After how many seconds does Isobel's seat pass the access platform?
- **d** How many times will her seat pass the access platform in the first 2 minutes?
- **e** How many times will her seat pass the access platform during the entire ride?

Due to a malfunction, the ferris wheel stops abruptly 1 minute 40 seconds into the ride.

- **f** How far above the ground is Isobel stranded?
- **g** If Isobel's brother Hamish had a seat 1.5 m above the ground at the commencement of the ride, how far above the ground is Hamish stranded?

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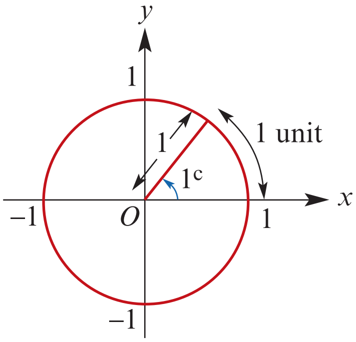
Chapter summary

- **Definition of a radian**

  One radian (written 1^c) is the angle formed at the centre of the unit circle by an arc of length 1 unit.

  $$1^c = \frac{180^\circ}{\pi} \qquad 1^\circ = \frac{\pi^c}{180}$$



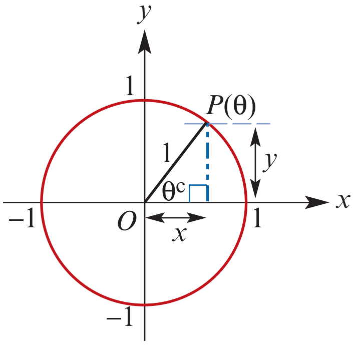
- **Sine and cosine functions**

  x-coordinate of $P(\theta)$ on unit circle:

  $$x = \cos\theta, \qquad \theta \in \mathbb{R}$$

  y-coordinate of $P(\theta)$ on unit circle:

  $$y = \sin\theta, \qquad \theta \in \mathbb{R}$$



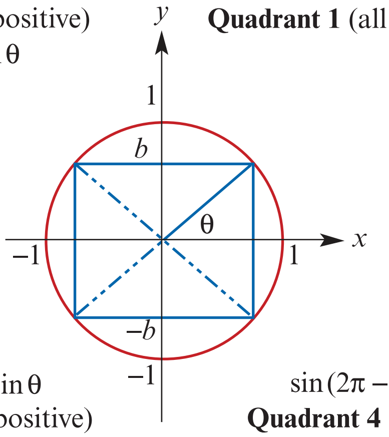
- **Tangent function**

  $$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad \text{for } \cos\theta \neq 0$$

- **Symmetry properties of circular functions**

Quadrant 2 (sin is positive)
$\sin(\pi - \theta) = b = \sin\theta$

Quadrant 1 (all functions are positive)
e.g. $\sin\theta = b$



$\sin(\pi + \theta) = -b = -\sin\theta$
Quadrant 3 (tan is positive)

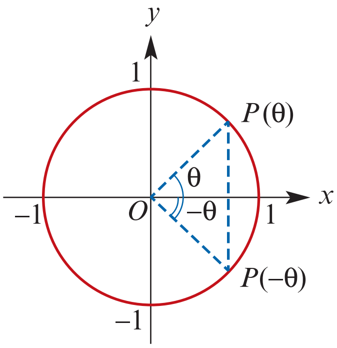
$\sin(2\pi - \theta) = -b = -\sin\theta$
Quadrant 4 (cos is positive)

Negative angles:

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

$$\tan(-\theta) = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$$



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Complementary angles:

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta, \qquad \sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta, \qquad \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$$

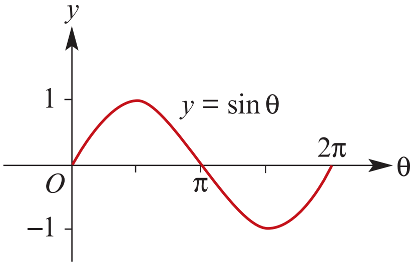
- Pythagorean identity

$$\cos^2\theta + \sin^2\theta = 1$$

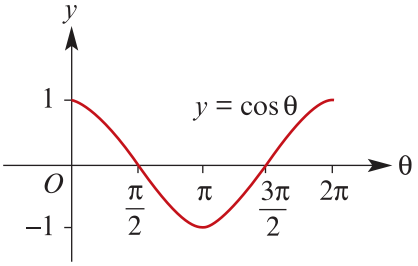
- Exact values of circular functions

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	undefined

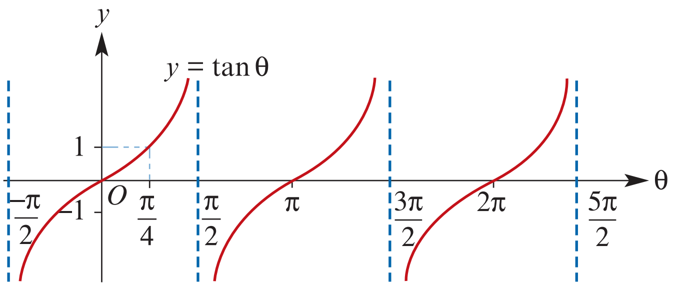
- Graphs of circular functions



Amplitude = 1
Period = 2π



Amplitude = 1
Period = 2π



Amplitude is undefined
Period = π

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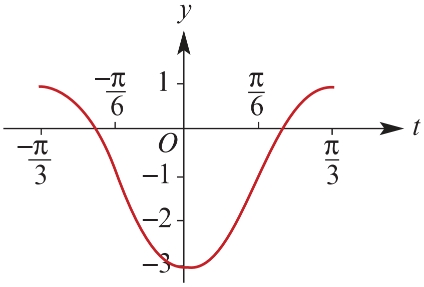
- Transformations of sine and cosine graphs:
  $y = a\sin n(t \pm \varepsilon) \pm b$ and $y = a\cos n(t \pm \varepsilon) \pm b$



  e.g. $y = 2\cos 3\left(t + \frac{\pi}{3}\right) - 1$

  - Amplitude, $a = 2$
  - Period $= \frac{2\pi}{n} = \frac{2\pi}{3}$
  - The graph is the same shape as $y = 2\cos(3t)$ but is translated $\frac{\pi}{3}$ units in the negative direction of the t-axis and 1 unit in the negative direction of the y-axis.

- Transformations of the graph of $y = \tan t$
  e.g. $y = a\tan n(t + \varepsilon) + b$, where $n > 0$
  - Period $= \frac{\pi}{n}$
  - Asymptotes: $t = \frac{(2k+1)\pi}{2n} - \varepsilon$, where $k \in \mathbb{Z}$

- Solution of trigonometric equations
  e.g. Solve $\cos x^\circ = -0.7$ for $x \in [0, 360]$.
  First look at the 1st quadrant: If $\cos \alpha^\circ = 0.7$, then $\alpha = 45.6$.
  Since $\cos x^\circ$ is negative for $P(x^\circ)$ in the 2nd and 3rd quadrants, the solutions are

  $$x = 180 - 45.6 = 134.4 \quad \text{and} \quad x = 180 + 45.6 = 225.6$$

- General solution of trigonometric equations
  - For $a \in [-1, 1]$, the general solution of the equation $\cos x = a$ is
    $$x = 2n\pi \pm \cos^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$
  - For $a \in \mathbb{R}$, the general solution of the equation $\tan x = a$ is
    $$x = n\pi + \tan^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$
  - For $a \in [-1, 1]$, the general solution of the equation $\sin x = a$ is
    $$x = 2n\pi + \sin^{-1}(a) \quad \text{or} \quad x = (2n+1)\pi - \sin^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

Technology-free questions

1 Solve each of the following equations for $x \in [-\pi, 2\pi]$:

a $\sin x = \frac{1}{2}$ **b** $2\cos x = -1$ **c** $2\cos x = \sqrt{3}$ **d** $\sqrt{2}\sin x + 1 = 0$

e $4\sin x + 2 = 0$ **f** $\sin(2x) + 1 = 0$ **g** $\cos(2x) = \frac{-1}{\sqrt{2}}$ **h** $2\sin(3x) - 1 = 0$

2 Sketch the graph of each of the following, showing one cycle. Clearly label axis intercepts.

a $f(x) = \sin(3x)$ **b** $f(x) = 2\sin(2x) - 1$ **c** $g(x) = 2\sin(2x) + 1$

d $f(x) = 2\sin\left(x - \frac{\pi}{4}\right)$ **e** $f(x) = 2\sin\left(\frac{\pi x}{3}\right)$ **f** $h(x) = 2\cos\left(\frac{\pi x}{4}\right)$

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3 Solve each of the following equations for $x \in [0, 360]$:

a $\sin x^\circ = 0.5$ **b** $\cos(2x)^\circ = 0$ **c** $2\sin x^\circ = -\sqrt{3}$

d $\sin(2x + 60)^\circ = \dfrac{-\sqrt{3}}{2}$ **e** $2\sin(\frac{1}{2}x)^\circ = \sqrt{3}$

4 Sketch the graph of each of the following, showing one cycle. Clearly label axis intercepts.

a $y = 2\sin\left(x + \dfrac{\pi}{3}\right) + 2$ **b** $y = -2\sin\left(x + \dfrac{\pi}{3}\right) + 1$ **c** $y = 2\sin\left(x - \dfrac{\pi}{4}\right) + \sqrt{3}$

d $y = -3\sin x$ **e** $y = \sin\left(x - \dfrac{\pi}{6}\right) + 3$ **f** $y = 2\sin\left(x - \dfrac{\pi}{2}\right) + 1$

5 Sketch, on the same set of axes, the curves $y = \cos x$ and $y = \sin(2x)$ for the interval $0 \le x \le 2\pi$, labelling each curve carefully. State the number of solutions in this interval for each of the following equations:

a $\sin(2x) = 0.6$ **b** $\sin(2x) = \cos x$ **c** $\sin(2x) - \cos x = 1$

6 Sketch on separate axes for $0^\circ \le x^\circ \le 360^\circ$:

a $y = 3\cos x^\circ$ **b** $y = \cos(2x)^\circ$ **c** $y = \cos(x - 30)^\circ$

7 Solve each of the following for $x \in [-\pi, \pi]$:

a $\tan x = \sqrt{3}$ **b** $\tan x = -1$

c $\tan(2x) = -1$ **d** $\tan(2x) + \sqrt{3} = 0$

8 Solve the equation $\sin x = \sqrt{3}\cos x$ for $x \in [-\pi, \pi]$.

9 The graphs of $y = a\cos x$ and $y = \sin x$, where a is a real constant, have a point of intersection at $x = \dfrac{\pi}{6}$.

a Find the value of a.

b If $x \in [-\pi, \pi]$, find the x-coordinate(s) of the other point(s) of intersection of the two graphs.

10 Find the general solution for each of the following:

a $\sin(2x) = -1$ **b** $\cos(3x) = 1$ **c** $\tan x = -1$

Multiple-choice questions

1 The period of the graph of $y = 3\sin\left(\dfrac{1}{2}x - \pi\right) + 4$ is

A π **B** 3 **C** 4π **D** $\pi + 4$ **E** 2π

2 The range of the graph of $y = f(x)$, where $f(x) = 5\cos\left(2x - \dfrac{\pi}{3}\right) - 7$, is

A $[-12, -2]$ **B** $[-7, 7]$ **C** $(-2, 5)$ **D** $[-2, 5]$ **E** $[-2, 12]$

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3 The equation of the image of the graph of $y = \sin x$ under a transformation of a dilation of factor $\frac{1}{2}$ from the y-axis followed by a translation of $\dfrac{\pi}{4}$ units in the positive direction of the x-axis is

A $y = \sin\left(\dfrac{1}{2}x + \dfrac{\pi}{4}\right)$ **B** $y = \sin\left(\dfrac{1}{2}x - \dfrac{\pi}{4}\right)$ **C** $y = 2\sin\left(x - \dfrac{\pi}{4}\right)$

D $y = \sin\left(2x - \dfrac{\pi}{4}\right)$ **E** $y = \sin 2\left(x - \dfrac{\pi}{4}\right)$

4 The function $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = a\sin(bx) + c$, where a, b and c are positive constants, has period

A a **B** b **C** $\dfrac{2\pi}{a}$ **D** $\dfrac{2\pi}{b}$ **E** $\dfrac{b}{2\pi}$

5 The equation $3\sin x - 1 = b$, where b is a positive real number, has one solution in the interval $(0, 2\pi)$. The value of b is

A 2 **B** 0.2 **C** 3 **D** 5 **E** 6

6 The range of the function $f\colon \left[0, \dfrac{2\pi}{9}\right) \to \mathbb{R}$, $f(x) = \cos(3x) - 1$ is

A $(-1, 0)$ **B** $(-2, 0]$ **C** $\left(-\dfrac{3}{2}, 0\right]$ **D** $\left[-\dfrac{3}{2}, 0\right)$ **E** $[-2, 0)$

7 Let $f(x) = p\cos(5x) + q$ where $p > 0$. Then $f(x) \le 0$ for all values of x if

A $q \ge 0$ **B** $-p \le q \le p$ **C** $p \le -q$ **D** $p \ge q$ **E** $-q \le p$

8 The vertical distance of a point on a wheel from the ground as it rotates is given by $D(t) = 3 - 3\sin(6\pi t)$, where t is the time in seconds. The time in seconds for a full rotation of the wheel is

A $\dfrac{1}{6\pi}$ **B** $\dfrac{1}{3}$ **C** 6π **D** $\dfrac{1}{3\pi}$ **E** 3

9 Let $f\colon \left[0, \dfrac{\pi}{2}\right] \to \mathbb{R}$ where $f(x) = \cos(3x) - 2$. The graph of f is transformed by a reflection in the x-axis followed by a dilation of factor 3 from the y-axis. The resulting graph is defined by

A $g\colon \left[0, \dfrac{\pi}{2}\right] \to \mathbb{R}$, $g(x) = 6 - 3\cos(3x)$ **B** $g\colon \left[0, \dfrac{\pi}{2}\right] \to \mathbb{R}$, $g(x) = 3\cos x - 6$

C $g\colon \left[0, \dfrac{3\pi}{2}\right] \to \mathbb{R}$, $g(x) = 2 - \cos x$ **D** $g\colon \left[0, \dfrac{\pi}{2}\right] \to \mathbb{R}$, $g(x) = \cos(-x) - 2$

E $g\colon \left[0, \dfrac{3\pi}{2}\right] \to \mathbb{R}$, $g(x) = \cos(9x) + 1$

10 The equation of the image of $y = \cos x$ under a transformation of a dilation of factor 2 from the x-axis, followed by a translation of $\dfrac{\pi}{4}$ units in the positive direction of the x-axis is

A $y = \cos\left(\dfrac{1}{2}x + \dfrac{\pi}{4}\right)$ **B** $y = \cos\left(\dfrac{1}{2}x - \dfrac{\pi}{4}\right)$ **C** $y = 2\cos\left(x + \dfrac{\pi}{4}\right)$

D $y = 2\sin\left(x - \dfrac{\pi}{4}\right)$ **E** $y = 2\cos\left(x - \dfrac{\pi}{4}\right)$

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11 A sequence of transformations which takes the graph of $y = \cos x$ to the graph of $y = -2\cos\left(\frac{x}{3}\right)$ is

A a dilation of factor $\frac{1}{3}$ from the x-axis, followed by dilation of factor $\frac{1}{2}$ from the y-axis, followed by a reflection in the x-axis

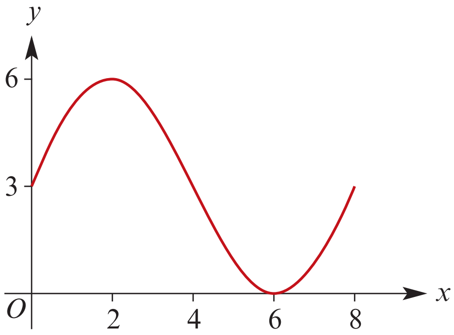
B a dilation of factor $\frac{1}{2}$ from the x-axis, followed by dilation of factor 3 from the y-axis, followed by a reflection in the y-axis

C a dilation of factor 2 from the x-axis, followed by dilation of factor 3 from the y-axis, followed by a reflection in the x-axis

D a dilation of factor 3 from the x-axis, followed by dilation of factor 2 from the y-axis, followed by a reflection in the x-axis

E a dilation of factor 2 from the x-axis, followed by dilation of factor $\frac{1}{3}$ from the y-axis, followed by a reflection in the x-axis

12 Which of the following is likely to be the rule for the graph of the circular function shown?



A $y = 3 + 3\cos\left(\frac{\pi x}{4}\right)$

B $y = 3 + 3\sin\left(\frac{\pi x}{4}\right)$

C $y = 3 + 3\sin(4\pi x)$

D $y = 3 + 3\cos\left(\frac{x}{4}\right)$

E $y = 3 + 3\sin\left(\frac{x}{4}\right)$

Extended-response questions

1 In a tidal river, the time between high tide and low tide is 6 hours. The average depth of water at a point in the river is 4 metres; at high tide the depth is 5 metres.

a Sketch the graph of the depth of water at the point for the time interval from 0 to 24 hours if the relationship between time and depth is sinusoidal and there is high tide at noon.

b If a boat requires a depth of 4 metres of water in order to sail, at what time before noon can it enter the point in the river and by what time must it leave if it is not to be stranded?

c If a boat requires a depth of 3.5 metres of water in order to sail, at what time before noon can it enter the point in the river and by what time must it leave if it is not to be stranded?

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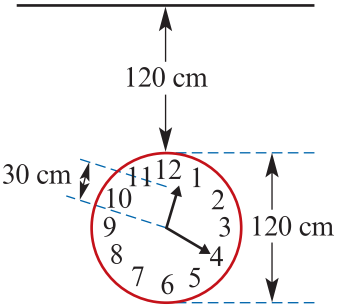
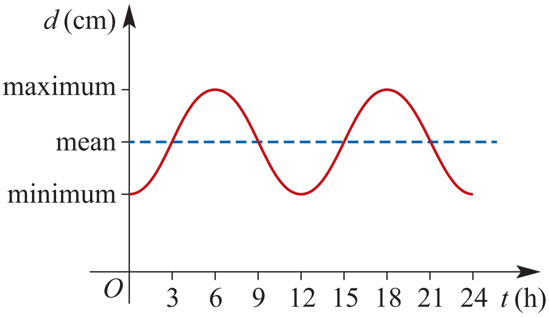
2 A clock hangs 120 cm below a ceiling. The diameter of the clock is 120 cm, and the length of the hour hand is 30 cm. The graph shows the distance from the ceiling to the tip of the hour hand over a 24-hour period.





a What are the values for the maximum, minimum and mean distance?

b An equation that determines this curve is of the form

$$y = A\sin(nt + \varepsilon) + b$$

Find suitable values of A, n, ε and b.

c Find the distance from the ceiling to the tip of the hour hand at:

- **i** 2 a.m.
- **ii** 11 p.m.

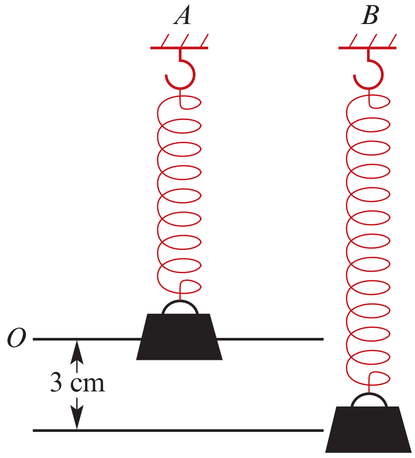
d Find the times in the morning at which the tip of the hour hand is 200 cm below the ceiling.

3 A weight is suspended from a spring as shown. The weight is pulled down 3 cm from O and released. The vertical displacement from O at time t is described by a function of the form

$$y = a\cos(nt)$$

where y cm is the vertical displacement at time t seconds. The following data were recorded.

t	0	0.5	1
y	−3	3	−3



It was also noted that the centre of the weight went no further than 3 cm from the centre O.

a Find the values of a and n.

b Sketch the graph of y against t.

c Find when the centre of the weight is first:

- **i** 1.5 cm above O
- **ii** 1.5 cm below O.

d When does the weight first reach a point 1 cm below O?

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4 The manager of a reservoir and its catchment area has noted that the inflow of water into the reservoir is very predictable and in fact models the inflow using a function with rule of the form

$$y = a\sin(nt + \varepsilon) + b$$

The following observations were made:

- The average inflow is 100 000 m^3/day.
- The minimum daily inflow is 80 000 m^3/day.
- The maximum daily inflow is 120 000 m^3/day, and this occurs on 1 May ($t = 121$) each year.

a Find the values of a, b and n and the smallest possible positive value for ε.

b Sketch the graph of y against t.

c Find the times of year when the inflow per day is:

  i 90 000 m^3/day

  ii 110 000 m^3/day

d Find the inflow rate on 1 June.

5 The number of hours of daylight at a point on the Antarctic Circle is given approximately by $d = 12 + 12\cos\left(\frac{1}{6}\pi\left(t + \frac{1}{3}\right)\right)$, where t is the number of months that have elapsed since 1 January.

a **i** Find d on 21 June ($t \approx 5.7$).

  ii Find d on 21 March ($t \approx 2.7$).

b When will there be 5 hours of daylight?

6 The depth, $D(t)$ m, of water at the entrance to a harbour at t hours after midnight on a particular day is given by $D(t) = 10 + 3\sin\left(\frac{\pi t}{6}\right)$, $0 \le t \le 24$.

a Sketch the graph of $y = D(t)$ for $0 \le t \le 24$.

b Find the values of t for which $D(t) \ge 8.5$.

c Boats that need a depth of w m are permitted to enter the harbour only if the depth of the water at the entrance is at least w m for a continuous period of 1 hour. Find, correct to one decimal place, the largest value of w that satisfies this condition.

7 The depth of water at the entrance to a harbour t hours after high tide is D m, where $D = p + q\cos(rt)^\circ$ for suitable constants p, q, r. At high tide the depth is 7 m; at low tide, 6 hours later, the depth is 3 m.

a Show that $r = 30$ and find the values of p and q.

b Sketch the graph of D against t for $0 \le t \le 12$.

c Find how soon after low tide a ship that requires a depth of at least 4 m of water will be able to enter the harbour.

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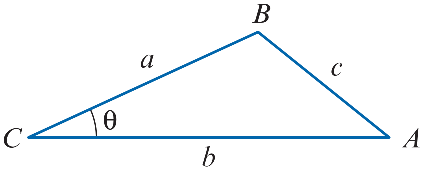
8 The area of a triangle is given by

$$A = \frac{1}{2}ab\sin\theta$$

and the perimeter is given by

$$P = a + b + \sqrt{a^2 + b^2 - 2ab\cos\theta}$$



a For $a = b = 10$ and $\theta = \frac{\pi}{3}$, find:

i the area of the triangle

ii the perimeter of the triangle.

b For $a = b = 10$, find the value(s) of θ for which $A = P$. (Give value(s) correct to two decimal places.)

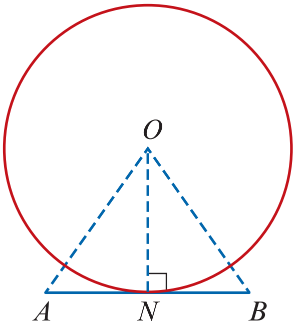
c Show graphically that, if $a = b = 6$, then $P > A$ for all θ.

d Assume $\theta = \frac{\pi}{2}$. If $a = 6$, find the value of b such that $A = P$.

e For $a = 10$ and $b = 6$, find the value(s) of θ for which $A = P$.

f If $a = b$ and $\theta = \frac{\pi}{3}$, find the value of a such that $A = P$.

9 AB is one side of a regular n-sided polygon that circumscribes a circle (i.e. each edge of the polygon is tangent to the circle). The circle has radius 1.



a Show that the area of triangle OAB is $\tan\left(\frac{\pi}{n}\right)$.

b Show that the area, A, of the polygon is given by $A = n\tan\left(\frac{\pi}{n}\right)$.

c Use a calculator to help sketch the graph of $A(x) = x\tan\left(\frac{\pi}{x}\right)$ for $x \geq 3$. Label the horizontal asymptote.

d What is the difference in area of the polygon and the circle when:

i $n = 3$ **ii** $n = 4$ **iii** $n = 12$ **iv** $n = 50$?

e State the area of an n-sided polygon that circumscribes a circle of radius r cm.

f **i** Find a formula for the area of an n-sided regular polygon that can be inscribed in a circle of radius 1.

ii Sketch the graph of this function for $x \geq 3$.

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7

Further functions

Objectives

- To graph **power functions** with rational non-integer index.
- To review and extend our study of all the functions of Mathematical Methods by revisiting:
  - sums, differences and products of functions
  - addition of ordinates
  - one-to-one functions, strictly increasing functions, strictly decreasing functions, odd functions and even functions
  - compositions of functions
  - inverse functions
  - transformations of functions.
- To use **functional equations** to describe properties of functions.
- To use parameters to describe **families of functions**.

In Chapter 1, we introduced:

- operations on functions, including addition, multiplication, composition and inverse
- properties of functions, including one-to-one, strictly increasing, strictly decreasing, odd and even.

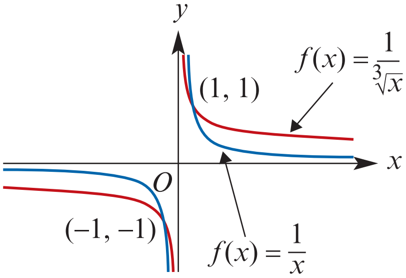
In this chapter, we revisit these concepts with all the functions of Mathematical Methods at our disposal: power functions, polynomial functions, exponential and logarithmic functions, and circular functions.

7A More power functions

In Chapter 1 we looked at power functions of the form $f(x) = x^n$, $f(x) = x^{-n}$ and $f(x) = x^{\frac{1}{n}}$, where n is a positive integer. In Chapter 3 we looked at transformations of these functions. Here we briefly consider some other power functions to complete our collection.

▶ The function $f(x) = x^{-\frac{1}{n}}$ where n is a positive integer

We can write $x^{-\frac{1}{n}} = \dfrac{1}{x^{\frac{1}{n}}} = \dfrac{1}{\sqrt[n]{x}}$.

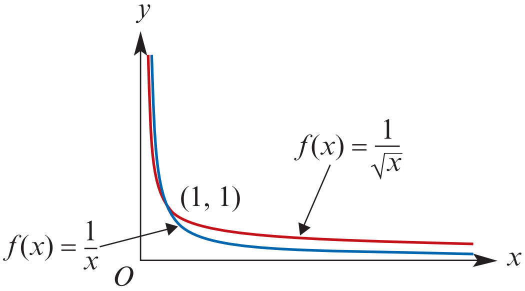


For the function f with rule $f(x) = \dfrac{1}{\sqrt[n]{x}}$:

- the maximal domain is $\mathbb{R} \setminus \{0\}$ if n is odd
- the maximal domain is $\mathbb{R}^+$ if n is even.

The first diagram shows the graphs of $y = \dfrac{1}{\sqrt[3]{x}}$ and $y = \dfrac{1}{x}$ for $x \in \mathbb{R} \setminus \{0\}$.

The second diagram shows the graphs of $y = \dfrac{1}{\sqrt{x}}$ and $y = \dfrac{1}{x}$ for $x \in \mathbb{R}^+$.



Each graph has a horizontal asymptote with equation $y = 0$ and a vertical asymptote with equation $x = 0$.

If n is an odd positive integer, then $f(x) = x^{-\frac{1}{n}}$ is an odd function, since $f(-x) = -f(x)$.

Example 1

For each of the following, use your calculator to help sketch the graph of $y = f(x)$ for the maximal domain. State this maximal domain and the range corresponding to this domain. Also state whether the function is odd, even or neither.

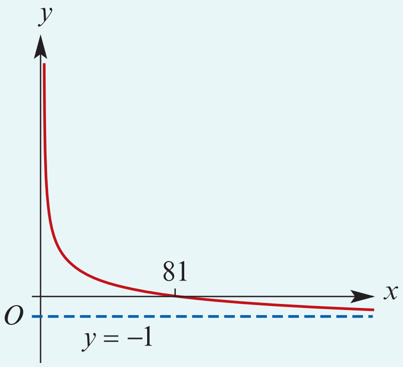
a $f(x) = 3x^{-\frac{1}{4}} - 1$ **b** $f(x) = 6x^{-\frac{1}{5}} + 1$

Solution

a The maximal domain is $\mathbb{R}^+$.

The range is $(-1, \infty)$.

The function is neither odd nor even.



To find the x-axis intercept:

$$\begin{aligned} 3x^{-\frac{1}{4}} - 1 &= 0 \\ x^{-\frac{1}{4}} &= \frac{1}{3} \\ x^{\frac{1}{4}} &= 3 \\ \therefore \quad x &= 3^4 = 81 \end{aligned}$$

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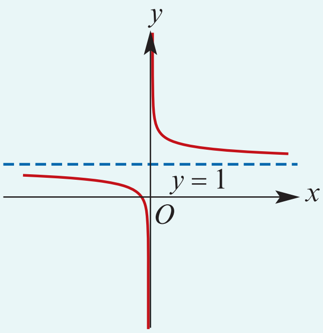
b $f(x) = 6x^{-\frac{1}{5}} + 1$

The maximal domain is $\mathbb{R} \setminus \{0\}$.
The range is $\mathbb{R} \setminus \{1\}$.
The function is neither odd nor even.

The line $x = 0$ is a vertical asymptote.
The line $y = 1$ is a horizontal asymptote.



The function $f(x) = x^{\frac{p}{q}}$ where p and q are positive integers

The special case where $p = 1$ has been considered in Chapter 1.

The expression $x^{\frac{p}{q}}$, where p and q are positive integers, can be defined as

$$x^{\frac{p}{q}} = \left(x^{\frac{1}{q}}\right)^p = (\sqrt[q]{x})^p$$

To employ this definition we will always first write the fractional power in simplest form.

For the function f with rule $f(x) = x^{\frac{p}{q}}$:

- the maximal domain is $\mathbb{R}$ if q is odd
- the maximal domain is $\mathbb{R}^+ \cup \{0\}$ if q is even.

Here are some examples of evaluating such functions:

$$8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2 = 2^2 = 4 \qquad (-8)^{\frac{2}{3}} = \left((-8)^{\frac{1}{3}}\right)^2 = (-2)^2 = 4$$

$$10\,000^{\frac{3}{4}} = \left(10\,000^{\frac{1}{4}}\right)^3 = 10^3 = 1000 \qquad 0.0001^{\frac{3}{4}} = \left(0.0001^{\frac{1}{4}}\right)^3 = 0.1^3 = 0.001$$

An investigation of these graphs with your calculator is worthwhile. Not every case will be illustrated here.

Example 2

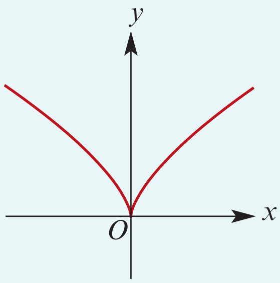
For each of the following, use your calculator to help sketch the graph of $y = f(x)$ for the maximal domain. State this maximal domain and the range corresponding to this domain. Also state whether the function is odd, even or neither.

a $f(x) = x^{\frac{2}{3}}$ **b** $f(x) = x^{\frac{3}{2}}$

Solution

a $f(x) = x^{\frac{2}{3}}$

The maximal domain is $\mathbb{R}$.
The range is $[0, \infty)$.
The function $f(x) = x^{\frac{2}{3}}$ is even, since $f(-x) = f(x)$.



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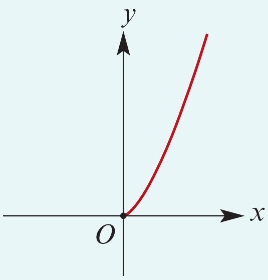
b $f(x) = x^{\frac{3}{2}}$

The maximal domain is $[0, \infty)$.

The range is $[0, \infty)$.

The function $f(x) = x^{\frac{3}{2}}$ is neither odd nor even.



Section summary

The power function $f(x) = x^{-\frac{1}{n}}$ where n is a positive integer

- The rule can also be written as $f(x) = \dfrac{1}{\sqrt[n]{x}}$.
- The maximal domain of f is:
  - $\mathbb{R} \setminus \{0\}$ if n is odd
  - $\mathbb{R}^+$ if n is even.
- If n is odd, then f is an odd function, since $f(-x) = -f(x)$.

The power function $f(x) = x^{\frac{p}{q}}$ where p and q are positive integers

- The expression $x^{\frac{p}{q}}$ is defined as $(\sqrt[q]{x})^p$, provided the fraction $\frac{p}{q}$ is in simplest form.
- The maximal domain of f is:
  - $\mathbb{R}$ if q is odd
  - $\mathbb{R}^+ \cup \{0\}$ if q is even.

Exercise 7A

Example 1

1 For each of the following, use your calculator to help sketch the graph of $y = f(x)$ for the maximal domain. State this maximal domain and the corresponding range. Also state whether the function is odd, even or neither.

a $f(x) = 2x^{-\frac{1}{4}} - 1$ **b** $f(x) = 3x^{-\frac{1}{5}} - 1$

2 Evaluate each of the following:

a $32^{\frac{2}{5}}$ **b** $(-32)^{\frac{2}{5}}$ **c** $32^{\frac{3}{5}}$

d $(-32)^{\frac{3}{5}}$ **e** $(-8)^{\frac{5}{3}}$ **f** $(-27)^{\frac{4}{3}}$

Example 2

3 For each of the following, use your calculator to help sketch the graph of $y = f(x)$ for the maximal domain. State this maximal domain and the corresponding range. Also state whether the function is odd, even or neither.

a $f(x) = x^{\frac{3}{5}}$ **b** $f(x) = x^{\frac{3}{4}}$

4 For each of the following rules of functions:

i state the maximal domain of the function, the corresponding range and the equations of any asymptotes

ii sketch the graph using your calculator for assistance.

a $f(x) = \dfrac{1}{\sqrt{x}}$ **b** $f(x) = x^{\frac{5}{3}}$ **c** $f(x) = -x^{\frac{3}{5}}$

d $f(x) = \dfrac{8}{\sqrt[3]{x}}$ **e** $f(x) = \dfrac{4}{\sqrt[5]{x}}$ **f** $f(x) = x^{\frac{7}{5}}$

5 **a** Find $\left\{x : x^{\frac{3}{2}} > x^2\right\}$. **b** Find $\left\{x : x^{\frac{3}{2}} < x^{-2}\right\}$.

6 For each of the following, state whether the function is odd, even or neither:

a $f(x) = \dfrac{1}{x}$ **b** $f(x) = \dfrac{1}{x^2}$ **c** $f(x) = \sqrt[3]{x}$

d $f(x) = \dfrac{1}{\sqrt[3]{x}}$ **e** $f(x) = x^{\frac{2}{3}}$ **f** $f(x) = x^{\frac{5}{7}}$

7B Composite and inverse functions

In the previous chapters, we have considered compositions and inverses for different families of functions. In this section, we revisit these two concepts using all the functions of Mathematical Methods.

We recall the following from Chapter 1.

Composition of functions The composition of g with f is written $g \circ f$ (read 'composition of f followed by g') and the rule for the composite function is $g \circ f(x) = g(f(x))$.

If $\operatorname{ran} f \subseteq \operatorname{dom} g$, then the composition $g \circ f$ is defined and $\operatorname{dom}(g \circ f) = \operatorname{dom} f$.

Inverse functions If f is a one-to-one function, then a new function f^{-1}, called the inverse of f, may be defined by

$$f^{-1}(x) = y \text{ if } f(y) = x, \quad \text{for } x \in \operatorname{ran} f \text{ and } y \in \operatorname{dom} f$$

Example 3

Express each of the following as the composition of two functions:

a $h(x) = e^{x^2}$ **b** $h(x) = \sin(x^2)$ **c** $h(x) = (x^2 - 2)^n, \quad n \in \mathbb{N}$

Solution

a $h(x) = e^{x^2}$
Choose $f(x) = x^2$
and $g(x) = e^x$.
Then $h(x) = g \circ f(x)$.

b $h(x) = \sin(x^2)$
Choose $f(x) = x^2$
and $g(x) = \sin x$.
Then $h(x) = g \circ f(x)$.

c $h(x) = (x^2 - 2)^n, \quad n \in \mathbb{N}$
Choose $f(x) = x^2 - 2$
and $g(x) = x^n$.
Then $h(x) = g \circ f(x)$.

Note: These are not the only possible answers, but the 'natural' choices have been made.

Example 4

Let $f(x) = e^{2x}$ and let $g(x) = \dfrac{1}{\sqrt{x}}$ for $x \in \mathbb{R}^+$. Find:

a f^{-1}

b g^{-1}

c $f \circ g$

d $g \circ f$

e $(f \circ g)^{-1}$

f $(g \circ f)^{-1}$

Solution

a $f^{-1}(x) = \dfrac{1}{2}\log_e x, \quad x \in \mathbb{R}^+$

b $g^{-1}(x) = \dfrac{1}{x^2}, \quad x \in \mathbb{R}^+$

c $f \circ g(x) = f\left(\dfrac{1}{\sqrt{x}}\right) = e^{\frac{2}{\sqrt{x}}}, \quad x \in \mathbb{R}^+$

d $g \circ f(x) = g(e^{2x}) = \dfrac{1}{e^x}, \quad x \in \mathbb{R}$

e For $(f \circ g)^{-1}$, let $x = e^{\frac{2}{\sqrt{y}}}$. Then

$$\log_e x = \frac{2}{\sqrt{y}}$$

$$\therefore \quad y = \left(\frac{2}{\log_e x}\right)^2$$

$$(f \circ g)^{-1}(x) = \left(\frac{2}{\log_e x}\right)^2, \quad x \in (1, \infty)$$

f For $(g \circ f)^{-1}$, let $x = \dfrac{1}{e^y}$. Then

$$e^y = \frac{1}{x}$$

$$\therefore \quad y = \log_e\left(\frac{1}{x}\right) = -\log_e x$$

$$(g \circ f)^{-1}(x) = -\log_e x, \quad x \in \mathbb{R}^+$$

Strictly increasing and strictly decreasing functions

We introduced strictly increasing and strictly decreasing functions in Chapter 1.

- A function f is **strictly increasing** if $a > b$ implies $f(a) > f(b)$, for all $a, b \in \operatorname{dom} f$.
- A function f is **strictly decreasing** if $a > b$ implies $f(a) < f(b)$, for all $a, b \in \operatorname{dom} f$.

In Section 1G we noted that, if f is a strictly increasing function, then it is one-to-one and so it has an inverse function.

Recall that a function f is **one-to-one** if $a \neq b$ implies $f(a) \neq f(b)$, for all $a, b \in \operatorname{dom} f$.

If f is strictly increasing, then it is a one-to-one function.

If f is strictly decreasing, then it is a one-to-one function.

Proof We prove only the first of the two statements.

Assume f is strictly increasing and let $a, b \in \operatorname{dom} f$ with $a \neq b$. Then $a > b$ or $b > a$. Therefore $f(a) > f(b)$ or $f(b) > f(a)$, since f is strictly increasing. In both cases, we have $f(a) \neq f(b)$. Hence f is a one-to-one function.

Note: If a function f is continuous and one-to-one on an interval, then f is either strictly increasing or strictly decreasing on this interval. The proof of this result is beyond the requirements of this course.

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If f is strictly increasing, then f^{-1} is also strictly increasing.

If f is strictly decreasing, then f^{-1} is also strictly decreasing.

Proof We prove only the first of the two statements.

Assume f is strictly increasing and let $a, b \in \operatorname{dom} f^{-1}$ with $a > b$.

If $f^{-1}(a) = f^{-1}(b)$, then $f(f^{-1}(a)) = f(f^{-1}(b))$ and so $a = b$, which is not the case.

If $f^{-1}(a) < f^{-1}(b)$, then $f(f^{-1}(a)) < f(f^{-1}(b))$ and so $a < b$, which is not the case.

Thus, we must have $f^{-1}(a) > f^{-1}(b)$, and hence f^{-1} is strictly increasing.

These results help us to understand the graphs of strictly increasing and strictly decreasing functions and their inverses.

Example 5

For each function f, find the inverse function f^{-1}, and state whether f and f^{-1} are strictly increasing, strictly decreasing or neither:

a $f\colon \mathbb{R}^+ \cup \{0\} \to \mathbb{R},\ f(x) = x^{\frac{2}{3}}$ **b** $f\colon \mathbb{R}^- \cup \{0\} \to \mathbb{R},\ f(x) = x^{\frac{2}{3}}$

Solution

a Write $x = y^{\frac{2}{3}}$ and solve for y:

$$x = \left(y^{\frac{1}{3}}\right)^2$$
$$y^{\frac{1}{3}} = \pm\sqrt{x}$$
$$y = (\pm\sqrt{x})^3 = \pm x^{\frac{3}{2}}$$

The domain of f is $\mathbb{R}^+ \cup \{0\}$;
the range of f is $\mathbb{R}^+ \cup \{0\}$.

Hence $f^{-1}\colon \mathbb{R}^+ \cup \{0\} \to \mathbb{R},\ f^{-1}(x) = x^{\frac{3}{2}}$

Both f and f^{-1} are strictly increasing.

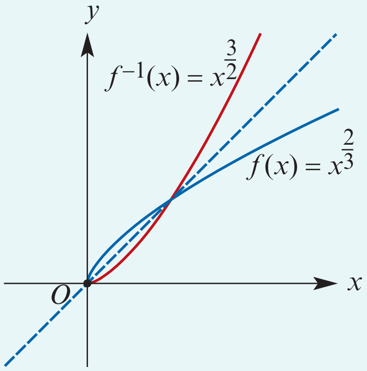
b Write $x = y^{\frac{2}{3}}$. Then $y = \pm x^{\frac{3}{2}}$.

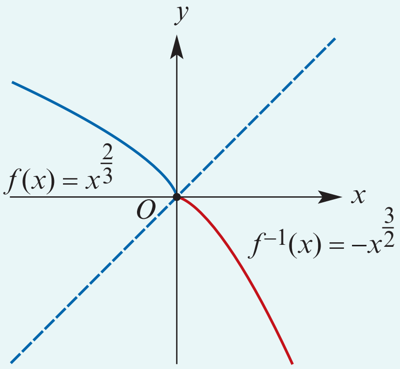
The domain of f is $\mathbb{R}^- \cup \{0\}$;
the range of f is $\mathbb{R}^+ \cup \{0\}$.

Hence $f^{-1}\colon \mathbb{R}^+ \cup \{0\} \to \mathbb{R},\ f^{-1}(x) = -x^{\frac{3}{2}}$

Both f and f^{-1} are strictly decreasing.

Explanation





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Exercise 7B

Skillsheet
Example 3

1 Express each of the following as the composition of two functions:

a $h(x) = e^{x^3}$ **b** $h(x) = \sin(2x^2)$ **c** $h(x) = (x^2 - 2x)^n$ where $n \in \mathbb{N}$

d $h(x) = \cos(x^2)$ **e** $h(x) = \cos^2 x$ **f** $h(x) = (x^2 - 1)^4$

g $h(x) = \cos^2(2x)$ **h** $h(x) = (x^2 - 2x)^3 - 2(x^2 - 2x)$

Example 4

2 Let $f(x) = 4e^{3x}$ and let $g(x) = \dfrac{2}{\sqrt[3]{x}}$ for $x \in \mathbb{R} \setminus \{0\}$. Find:

a f^{-1} **b** g^{-1} **c** $f \circ g$ **d** $g \circ f$ **e** $(f \circ g)^{-1}$ **f** $(g \circ f)^{-1}$

Example 5

3 For each function f, find the inverse function f^{-1}, and state whether f and f^{-1} are strictly increasing, strictly decreasing or neither:

a $f\colon \mathbb{R}^+ \cup \{0\} \to \mathbb{R},\ f(x) = x^{\frac{2}{5}}$ **b** $f\colon \mathbb{R}^- \cup \{0\} \to \mathbb{R},\ f(x) = x^{\frac{2}{5}}$

c $f\colon \mathbb{R}^+ \to \mathbb{R},\ f(x) = x^{\frac{5}{2}}$

4 Let $g(x) = x^2$. For each of the following functions f:

i Find the rules $f \circ g(x)$ and $g \circ f(x)$.

ii Find the range of $y = f \circ g(x)$ and $y = g \circ f(x)$ (and state the maximal domain for each of the composite functions to exist).

a $f(x) = 3\sin(2x)$ **b** $f(x) = -2\cos(2x)$ **c** $f(x) = e^x$

d $f(x) = e^{2x} - 1$ **e** $f(x) = -2e^x - 1$ **f** $f(x) = \log_e(2x)$

g $f(x) = \log_e(x - 1)$ **h** $f(x) = -\log_e x$

5 Let $f(x) = 2x - \dfrac{\pi}{3}$ and $g(x) = \sin x$.

a Find $g \circ f$.

b Describe a sequence of transformations that takes the graph of $y = g(x)$ to the graph of $y = g \circ f(x)$.

6 Consider $f\colon (\frac{1}{3}, \infty) \to \mathbb{R},\ f(x) = 3x - 2$ and $g\colon (-1, \infty) \to \mathbb{R},\ g(x) = \log_e(x + 1)$.

a Find $g \circ f$.

b Describe a sequence of transformations that takes the graph of $y = g(x)$ to the graph of $y = g \circ f(x)$.

7 **a** Given that $[g(x)]^2 - 7g(x) + 12 = 0$, find possible rules for $g(x)$.

b Given that $[g(x)]^2 - 7xg(x) + 12x^2 = 0$, find possible rules for $g(x)$.

8 Given that $e^{g(x)} = 2x - 1$, find the rule for $g(x)$.

9 The functions f and g are defined by $f\colon \mathbb{R} \to \mathbb{R},\ f(x) = e^{4x}$ and $g\colon \mathbb{R}^+ \to \mathbb{R}$, $g(x) = 2\sqrt{x}$. Find each of the following:

a $g \circ f(x)$ **b** $(g \circ f)^{-1}(x)$ **c** $f \circ g^{-1}(x)$

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10 The functions f and g are defined by $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = e^{-2x}$ and $g\colon \mathbb{R} \to \mathbb{R}$, $g(x) = x^3 + 1$.

a Find the inverse function of each of these functions.

b Find the rules $f \circ g(x)$ and $g \circ f(x)$ and state the range of each of these composite functions.

11 The function f is defined by $f\colon (-1, \infty) \to \mathbb{R}$, $f(x) = \dfrac{1}{x+1}$.

a Find f^{-1}.

b Solve the equation $f(x) = f^{-1}(x)$ for x.

12 The functions f and g are defined by $f\colon (-1, \infty) \to \mathbb{R}$, $f(x) = \log_e(x + 1)$ and $g\colon (-1, \infty) \to \mathbb{R}$, $g(x) = x^2 + 2x$.

a Define f^{-1} and g^{-1}, giving their rules and domains.

b Find the rule for $f \circ g(x)$.

13 The functions f and g are defined by $f\colon (0, \infty) \to \mathbb{R}$, $f(x) = \log_e x$ and $g\colon (0, \infty) \to \mathbb{R}$, $g(x) = \dfrac{1}{x}$. Find $f \circ g(x)$ and simplify $f(x) + f \circ g(x)$.

14 The functions g and h are defined by $g\colon \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$, $g(x) = 5x^2 + 3$ and $h\colon [3, \infty) \to \mathbb{R}$, $h(x) = \sqrt{\dfrac{x-3}{5}}$. Find $h(g(x))$.

15 For $f(x) = (x - 4)(x - 6)$ and $g(x) = x^2 - 4$:

a Find $f(g(x))$ and $g(f(x))$.

b Solve the equation $g(f(x)) - f(g(x)) = 158$ for x.

16 For $f(x) = 4 - x^2$, solve the equation $f(f(x)) = 0$ for x.

17 For $f(x) = e^x - e^{-x}$, show that:

a $f(-x) = -f(x)$

b $[f(x)]^3 = f(3x) - 3f(x)$

18 The inverse function of the linear function $f(x) = ax + b$ is $f^{-1}(x) = 6x + 3$. Find the values of a and b.

19 Show that $f = f^{-1}$ for $f(x) = \dfrac{x+2}{x-1}$.

20 Let $g\colon \mathbb{R} \to \mathbb{R}$ such that $\log_e(g(x)) = ax + b$. Given that $g(0) = 1$ and $g(1) = e^6$, find a and b and hence find $g(x)$.

21 **a** Let $f\colon [0, \infty) \to \mathbb{R}$, $f(x) = \dfrac{e^x + e^{-x}}{2}$. Find f^{-1}.

b Let $g\colon \mathbb{R} \to \mathbb{R}$, $g(x) = \dfrac{e^x - e^{-x}}{2}$. Find g^{-1}.

c Is f a strictly increasing function on the stated interval?

d Is g a strictly increasing function on the stated interval?

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22 Let f and g be functions such that the composite $g \circ f$ is defined.

- **a** Prove that, if both f and g are strictly increasing, then $g \circ f$ is strictly increasing.
- **b** Prove that, if both f and g are strictly decreasing, then $g \circ f$ is strictly increasing.
- **c** What can be said about the composite $g \circ f$ if one of the two functions f and g is strictly increasing and the other is strictly decreasing?

7C Sums and products of functions and addition of ordinates

In Chapter 1 we saw that, for functions f and g, the new functions $f + g$ and fg can be defined by

$$(f+g)(x) = f(x) + g(x) \qquad (fg)(x) = f(x)\,g(x)$$

$$\operatorname{dom}(f+g) = \operatorname{dom} f \cap \operatorname{dom} g \qquad \operatorname{dom}(fg) = \operatorname{dom} f \cap \operatorname{dom} g$$

We also considered graphing by addition of ordinates. The new functions that have been defined in Chapters 4 to 6 may now be included.

Example 6

For $f(x) = \cos x$ and $g(x) = e^{-x}$:

- **a** Find the rules for $(f+g)(x)$ and $(fg)(x)$.
- **b** Evaluate $(f+g)(0)$ and $(fg)(0)$.

Solution

- **a** $(f+g)(x) = \cos x + e^{-x}$ and $(fg)(x) = e^{-x}\cos x$
- **b** $(f+g)(0) = 1 + 1 = 2$ and $(fg)(0) = 1 \times 1 = 1$

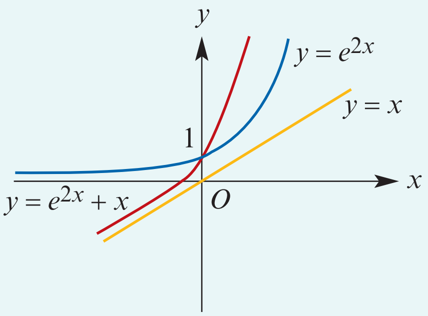
Example 7

For $f(x) = x$ and $g(x) = e^{2x}$, sketch the graph of $y = (f+g)(x)$.

Solution

Note that $(f+g)(0) = 0 + e^0 = 1$ and that $(f+g)(x) = 0$ implies $x + e^{2x} = 0$. This equation cannot be solved analytically, but a calculator can be used to find the approximate solution $x = -0.43$, correct to two decimal places.

Also note that, as $x \to -\infty$, $(f+g)(x) \to x$ from 'above'.



Section summary

- **Sum of functions** $(f+g)(x) = f(x) + g(x)$, where $\text{dom}(f+g) = \text{dom}\, f \cap \text{dom}\, g$
- **Difference of functions** $(f-g)(x) = f(x) - g(x)$, where $\text{dom}(f-g) = \text{dom}\, f \cap \text{dom}\, g$
- **Product of functions** $(f \cdot g)(x) = f(x) \cdot g(x)$, where $\text{dom}(f \cdot g) = \text{dom}\, f \cap \text{dom}\, g$
- **Addition of ordinates** This technique can be used to help sketch the graph of the sum of two functions. Key points to consider when sketching $y = (f+g)(x)$:
  - When $f(x) = 0$, $(f+g)(x) = g(x)$.
  - When $g(x) = 0$, $(f+g)(x) = f(x)$.
  - If $f(x)$ and $g(x)$ are positive, then $(f+g)(x) > g(x)$ and $(f+g)(x) > f(x)$.
  - If $f(x)$ and $g(x)$ are negative, then $(f+g)(x) < g(x)$ and $(f+g)(x) < f(x)$.
  - If $f(x)$ is positive and $g(x)$ is negative, then $g(x) < (f+g)(x) < f(x)$.
  - Look for values of x for which $f(x) + g(x) = 0$.

Exercise 7C

Example 6

1 Let $f(x) = e^{2x}$ and $g(x) = -2x$.

a **i** Find the rule for $f+g$. **ii** Find the rule for fg.

b **i** Evaluate $(f+g)(-\frac{1}{2})$. **ii** Evaluate $(fg)(-\frac{1}{2})$.

Example 7

2 Sketch the graphs of $f(x) = e^{-2x}$ and $g(x) = -2x$ on the one set of axes and hence, using addition of ordinates, sketch the graph of $y = e^{-2x} - 2x$.

3 Sketch the graphs of $f(x) = 2e^{2x}$ and $g(x) = x + 2$ on the one set of axes and hence, using addition of ordinates, sketch the graph of $y = 2e^{2x} + x + 2$.

4 Let $f(x) = \sin\left(\dfrac{\pi x}{2}\right)$ and $g(x) = -2x$.

a **i** Find the rule for $f+g$. **ii** Find the rule for fg.

b **i** Evaluate $(f+g)(1)$. **ii** Evaluate $(fg)(1)$.

5 Let $f(x) = \cos\left(\dfrac{\pi x}{2}\right)$ and $g(x) = e^{x}$.

a **i** Find the rule for $f+g$. **ii** Find the rule for fg.

b **i** Evaluate $(f+g)(0)$. **ii** Evaluate $(fg)(0)$.

6 Prove that any function f with domain $\mathbb{R}$ can be expressed as the sum of an even function and an odd function.

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7D Function notation and identities

Many of the properties which have been investigated for the functions introduced in the previous chapters may be expressed using function notation.

For example, the rules for logarithms

$$\log_e(x) + \log_e(y) = \log_e(xy) \qquad \log_e(x) - \log_e(y) = \log_e\left(\frac{x}{y}\right)$$

can be written in the following way if $f(x) = \log_e x$:

$$f(x) + f(y) = f(xy) \qquad f(x) - f(y) = f\left(\frac{x}{y}\right)$$

The rules for exponential functions

$$e^{x+y} = e^x \times e^y \qquad e^{x-y} = \frac{e^x}{e^y}$$

can be written in the following way if $f(x) = e^x$:

$$f(x+y) = f(x)f(y) \qquad f(x-y) = \frac{f(x)}{f(y)}$$

Example 8

a For the function with rule $f(x) = 2x$, show that $f(x+y) = f(x) + f(y)$ for all x and y.

b For the function with rule $f(x) = x + 2$, show that $f(x+y) \neq f(x) + f(y)$ for all x and y.

Solution

a $f(x+y) = 2(x+y) = 2x + 2y = f(x) + f(y)$

b $f(x+y) = (x+y) + 2 = (x+2) + (y+2) - 2 = f(x) + f(y) - 2$

If $f(x) + f(y) - 2 = f(x) + f(y)$, then $-2 = 0$, which is a contradiction.

Example 9

If $f(x) = \frac{1}{x}$, verify that $f(x) + f(y) = (x+y)\,f(xy)$ for all non-zero real numbers x and y.

Solution

$$f(x) + f(y) = \frac{1}{x} + \frac{1}{y} = \frac{y+x}{xy} = (x+y) \times \frac{1}{xy} = (x+y)\,f(xy)$$

Example 10

For the function $f(x) = \cos x$, give an example to show that $f(x+y) \neq f(x) + f(y)$ for some x and y.

Solution

For $x = 0$ and $y = \pi$:

$$f(0+\pi) = f(\pi) = -1$$

$$f(0) + f(\pi) = 1 + (-1) = 0$$

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Section summary

Functional equations can be useful for describing properties of functions. For example, the exponential function $f(x) = e^x$ satisfies $f(x + y) = f(x)\,f(y)$.

Note that, in general,

$$f(x + y) \neq f(x) + f(y)$$
$$f(xy) \neq f(x)\,f(y)$$

Exercise 7D

Example 8

1 **a** For the function with rule $f(x) = 2x$, show that $f(x - y) = f(x) - f(y)$ for all x and y.

b For the function with rule $f(x) = x - 3$, verify that $f(x - y) \neq f(x) - f(y)$ for all x and y.

2 For $f(x) = kx$, find an equivalent expression for $f(x - y)$ in terms of $f(x)$ and $f(y)$.

3 For $f(x) = 2x + 3$, show that $f(x + y)$ can be written in the form $f(x) + f(y) + a$ and give the value of a.

Example 9

4 If $f(x) = \dfrac{3}{x}$, show that $f(x) + f(y) = (x + y)\,f(xy)$ for all non-zero real numbers x and y.

5 A function g satisfies the property that $[g(x)]^2 = g(x)$. Find the possible values of $g(x)$.

6 A function g satisfies the property that $\dfrac{1}{g(x)} = g(x)$. Find the possible values of $g(x)$.

Example 10

7 For the function with rule $f(x) = x^3$, give an example to show that $f(x+y) \neq f(x)+f(y)$ for some x and y.

8 For the function $f\colon \mathbb{R} \to \mathbb{R}$ where $f(x) = \sin x$, give an example to show that $f(x + y) \neq f(x) + f(y)$ for some x and y.

9 For the function with rule $f(x) = \dfrac{1}{x^2}$, show that $f(x) + f(y) = (x^2 + y^2)\,f(xy)$.

10 **a** For $h(x) = x^2$, give an example to show that $h(x + y) \neq h(x) + h(y)$ for some x and y.

b Show that $h(x + y) = h(x) + h(y)$ implies $x = 0$ or $y = 0$.

11 For $g(x) = 2^{3x}$, show that $g(x + y) = g(x)\,g(y)$.

12 Show that the functions with rules of the form $f(x) = x^n$, where n is a natural number, satisfy the identities $f(xy) = f(x)\,f(y)$ and $f\left(\dfrac{x}{y}\right) = \dfrac{f(x)}{f(y)}$.

13 For the function with rule $f(x) = ax$ where $a \in \mathbb{R} \setminus \{0, 1\}$, give an example to show that $f(xy) \neq f(x)\,f(y)$ for some x and y.

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14 For the function $f\colon \mathbb{R} \setminus \{-1\} \to \mathbb{R},\ f(x) = \dfrac{1}{x+1}$, show that $f(f(x)) + f(x+1) = 1$ for all $x \in \mathbb{R} \setminus \{-1, -2\}$.

15 Give an example to show that in general $f \circ (g + h) \neq (f \circ g) + (f \circ h)$.

16 Prove that $(g + h) \circ f = (g \circ f) + (h \circ f)$.

17 Let $g\colon \mathbb{R}^+ \to \mathbb{R},\ g(x) = xe^x$ and $f\colon \mathbb{R}^+ \to \mathbb{R},\ f(x) = \log_e x$. Show that $f(g(x)) - f(x) = x$ for all $x > 0$ and show that $\dfrac{g(f(x))}{f(x)} = x$ for all $x > 1$.

7E Families of functions and solving literal equations

In Chapter 2 we used parameters to describe the solutions of simultaneous equations. In this section we use parameters to describe families of functions.

Here are some families of functions:

$$
\begin{array}{ll}
f\colon \mathbb{R} \to \mathbb{R},\ f(x) = mx & \text{where } m \in \mathbb{R} \\
f\colon \mathbb{R} \to \mathbb{R},\ f(x) = ax^3 & \text{where } a \in \mathbb{R} \setminus \{0\} \\
f\colon \mathbb{R} \to \mathbb{R},\ f(x) = mx + 2 & \text{where } m \in \mathbb{R}^+ \\
f\colon \mathbb{R} \to \mathbb{R},\ f(x) = ke^{mx} & \text{where } m \in \mathbb{R} \setminus \{0\} \text{ and } k \in \mathbb{R} \setminus \{0\}
\end{array}
$$

The use of parameters makes it possible to describe general properties.

What can be said in general about each of these families? The following example explores the family of functions of the form $f\colon \mathbb{R} \to \mathbb{R},\ f(x) = mx + 2$ where $m \in \mathbb{R}^+$.

Example 11

For $f\colon \mathbb{R} \to \mathbb{R},\ f(x) = mx + 2$ where $m \in \mathbb{R}^+$:

a Find the x-axis intercept of the graph of $y = f(x)$.

b For which values of m is the x-axis intercept less than -2?

c Find the inverse function of f.

d Find the equation of the line perpendicular to the graph of $y = f(x)$ at the point $(0, 2)$.

Solution

a When $y = 0$, we have $mx + 2 = 0$ and so $x = -\dfrac{2}{m}$. The x-axis intercept is $-\dfrac{2}{m}$.

b Consider $-\dfrac{2}{m} < -2$ and solve for m:

$$
\begin{aligned}
\frac{2}{m} &> 2 \\
2 &> 2m \qquad \text{Multiply both sides of the inequality by } m\ (m > 0). \\
\therefore\quad m &< 1
\end{aligned}
$$

Therefore the x-axis intercept is less than -2 for $0 < m < 1$.

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c Consider $x = my + 2$ and solve for y:

$$my = x - 2$$

$$\therefore \quad y = \frac{x-2}{m}$$

Therefore $f^{-1}(x) = \frac{1}{m}x - \frac{2}{m}$. The domain of f^{-1} is $\mathbb{R}$.

d The perpendicular line has gradient $-\frac{1}{m}$.

The equation is $y - 2 = -\frac{1}{m}(x - 0)$, which rearranges to $y = -\frac{1}{m}x + 2$.

Example 12

The graph of a quadratic function passes through the points $(1, 6)$ and $(2, 4)$. Find the coefficients of the quadratic rule in terms of c, the y-axis intercept of the graph.

Solution

Let $f(x) = ax^2 + bx + c$ be a function in this family. Then $f(1) = 6$ and $f(2) = 4$.

The following equations are obtained:

$$a + b + c = 6 \quad \text{and} \quad 4a + 2b + c = 4$$

Solving these gives

$$a = \frac{c-8}{2} \quad \text{and} \quad b = \frac{20-3c}{2}$$

The equation of the quadratic in terms of c is

$$y = \left(\frac{c-8}{2}\right)x^2 + \left(\frac{20-3c}{2}\right)x + c$$

The following example demonstrates how to solve literal equations involving exponential and logarithmic functions.

Example 13

Solve each of the following for x. All constants are positive reals.

a $ae^{bx} - c = 0$ **b** $\log_e(x - a) = b$ **c** $\log_e(cx - a) = 1$

Solution

a
$$\begin{aligned} ae^{bx} - c &= 0 \\ e^{bx} &= \frac{c}{a} \\ bx &= \log_e\left(\frac{c}{a}\right) \\ x &= \frac{1}{b}\log_e\left(\frac{c}{a}\right) \end{aligned}$$

b
$$\begin{aligned} \log_e(x - a) &= b \\ x - a &= e^b \\ x &= e^b + a \end{aligned}$$

c
$$\begin{aligned} \log_e(cx - a) &= 1 \\ cx - a &= e \\ cx &= a + e \\ x &= \frac{a+e}{c} \end{aligned}$$

Example 14

A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by the matrix equation

$$\begin{bmatrix} 5 & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

where k is a non-zero real number.

a Find x and y in terms of x' and y'.

b Find the image of the curve with equation $y = \frac{1}{x}$ under this transformation.

c Find the value of k if the image passes through the origin.

Solution

a From the matrix equation:

$$5x + 5 = x'$$
$$ky + 2 = y'$$

Therefore $x = \frac{x' - 5}{5}$ and $y = \frac{y' - 2}{k}$.

b The image has equation $\frac{y' - 2}{k} = \frac{5}{x' - 5}$, which can be written as $y = \frac{5k}{x - 5} + 2$.

c If the graph passes through the origin, then $0 = -k + 2$ and so $k = 2$.

Exercise 7E

Example 11

Skillsheet

1 Consider $f : \mathbb{R} \to \mathbb{R}$, $f(x) = mx - 4$ where $m \in \mathbb{R} \setminus \{0\}$.

- **a** Find the x-axis intercept of the graph of $y = f(x)$.
- **b** For which values of m is the x-axis intercept less than or equal to 1?
- **c** Find the inverse function of f.
- **d** Find the coordinates of the point of intersection of the graph of $y = f(x)$ with the graph of $y = x$.
- **e** Find the equation of the line perpendicular to the line $y = f(x)$ at the point $(0, -4)$.

2 Consider $f : \mathbb{R} \to \mathbb{R}$, $f(x) = -2x + c$ where $c \in \mathbb{R}$.

- **a** Find the x-axis intercept of the graph of $y = f(x)$.
- **b** For which values of c is the x-axis intercept less than or equal to 1?
- **c** Find the inverse function of f.
- **d** Find the coordinates of the point of intersection of the graph of $y = f(x)$ with the graph of $y = x$.
- **e** Find the equation of the line perpendicular to the line $y = f(x)$ at the point $(0, c)$.

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3 Consider the family of quadratics with rules of the form $y = x^2 - bx$, where b is a non-zero real number.

a Find the x-axis intercepts.

b Find the coordinates of the vertex of the parabola.

c **i** Find the coordinates of the points of intersection of the graph of $y = x^2 - bx$ with the line $y = -x$ in terms of b.

ii For what value(s) of b is there one intersection point?

iii For what value(s) of b are there two intersection points?

Example 12

4 The graph of a quadratic function passes through the points $(-1, 6)$ and $(1, 4)$. Find the coefficients of the quadratic rule in terms of c, the y-axis intercept of the graph.

5 **a** The graph of $f(x) = x^2$ is translated to the graph of $y = f(x + h)$. Find the possible values of h if $f(1 + h) = 8$.

b The graph of $f(x) = x^2$ is transformed to the graph of $y = f(ax)$. Find the possible values of a if the graph of $y = f(ax)$ passes through the point with coordinates $(1, 8)$.

c The quadratic with equation $y = ax^2 + bx$ has vertex with coordinates $(1, 8)$. Find the values of a and b.

6 Consider the family of functions with rules of the form $f(x) = \sqrt{2a - x}$, where a is a positive real number.

a State the maximal domain of f.

b Find the coordinates of the point of intersection of the graph of $y = f(x)$ with the graph of $y = x$.

c For what value of a does the line with equation $y = x$ intersect the graph of $y = f(x)$ at the point with coordinates $(1, 1)$?

d For what value of a does the line with equation $y = x$ intersect the graph of $y = f(x)$ at the point with coordinates $(2, 2)$?

e For what value of a does the line with equation $y = x$ intersect the graph of $y = f(x)$ at the point with coordinates (c, c), where c is a positive real number?

7 Consider the function with rule $f(x) = (x^2 - ax)^2$.

a State the coordinates of the x-axis intercepts.

b State the coordinates of the y-axis intercept.

c For $a > 0$, find the maximum value of the function in the interval $[0, a]$.

d Find the possible values of a for which the point $(-1, 16)$ lies on the graph of $y = f(x)$.

Example 13

8 Solve each of the following for x. All constants are positive reals.

a $-ae^{bx} + c = 0$

b $c \log_e(x + a) = b$

c $\log_e(cx - a) = 0$

d $e^{ax+b} = c$

9 Consider the family of functions with rules of the form $f(x) = c\log_e(x - a)$, where a and c are positive constants.

- **a** State the equation of the vertical asymptote.
- **b** State the coordinates of the x-axis intercept.
- **c** State the coordinates of the point where the graph crosses the line $y = 1$.
- **d** If the graph of the function crosses the line $y = 1$ when $x = 2$, find the value of c in terms of a.

10 Consider the family of functions with rules of the form $f(x) = e^{x-1} - b$, where $b > 0$.

- **a** State the equation of the horizontal asymptote.
- **b** State the coordinates of the x-axis intercept.
- **c** Give the values of b for which the x-axis intercept is:
  - **i** at the origin
  - **ii** a negative number.

11 The graph of a cubic function passes through the points $(-1, 6)$, $(1, -2)$ and $(2, 4)$. Find the coefficients of the cubic rule in terms of d, the y-axis intercept of the graph.

12 A quadratic function has rule $f(x) = \left(\frac{c-8}{2}\right)x^2 + \left(\frac{20-3c}{2}\right)x + c$. Find the values of c for which:

- **a** the graph of $y = f(x)$ touches the x-axis
- **b** the graph of $y = f(x)$ has two distinct x-axis intercepts.

13 The graph of a cubic function passes through the points $(-2, 8)$, $(1, 1)$ and $(3, 4)$. Find the coefficients of the quadratic rule in terms of d, the y-axis intercept of the graph.

Example 14 **14** A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by the matrix equation

$$\begin{bmatrix} -4 & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

where k is a non-zero real number.

- **a** Find x and y in terms of x' and y'.
- **b** Find the image of the curve with equation $y = \frac{1}{x}$ under this transformation.
- **c** Find the value of k if the image passes through the origin.

15 A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by the matrix equation

$$\begin{bmatrix} -4 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ -2 \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

where a is a non-zero real number.

- **a** Find x and y in terms of x' and y'.
- **b** Find the image of the curve with equation $y = 2^x$ under this transformation.
- **c** Find the value of a if the image passes through the origin.

Technology-free questions

Spreadsheet
AS
Nrich

1 For each of the following, use your calculator to help sketch the graph of $y = f(x)$ for the maximal domain. State this maximal domain and the corresponding range. Also state whether the function is odd, even or neither.

a $f(x) = 3x^{-\frac{1}{4}} + 1$ **b** $f(x) = 2x^{-\frac{1}{5}} - 2$

2 Evaluate each of the following:

a $243^{\frac{2}{5}}$ **b** $(-243)^{\frac{2}{5}}$ **c** $243^{\frac{3}{5}}$ **d** $(-243)^{\frac{2}{5}}$ **e** $(-27)^{\frac{5}{3}}$ **f** $(-125)^{\frac{4}{3}}$

3 Let $g(x) = x^2$. For each of the following functions f:

i Find the rules $f \circ g(x)$ and $g \circ f(x)$.

ii Find the range of $y = f \circ g(x)$ and $y = g \circ f(x)$ (and state the maximal domain for each of the composite functions to exist).

a $f(x) = 3\cos(2x)$ **b** $f(x) = \log_e(3x)$

c $f(x) = \log_e(2 - x)$ **d** $f(x) = -\log_e(2x)$

4 Express each of the following as the composition of two functions:

a $h(x) = \cos(x^2)$ **b** $h(x) = (x^2 - x)^n$ where $n \in \mathbb{N}$

c $h(x) = \log_e(\sin x)$ **d** $h(x) = -2\sin^2(2x)$

e $h(x) = (x^2 - 3x)^4 - 2(x^2 - 3x)^2$

5 Let $f(x) = 2\cos\left(\dfrac{\pi x}{2}\right)$ and $g(x) = e^{-x}$.

a **i** Find the rule for $f + g$. **ii** Find the rule for fg.

b **i** Evaluate $(f + g)(0)$. **ii** Evaluate $(fg)(0)$.

6 Let $f\colon [a, \infty) \to \mathbb{R}$ where $f(x) = -(3x - 2)^2 + 3$.

a Find the smallest value of a such that f is one-to-one.

b With this value of a, state the range of f.

c Sketch the graph of f.

d Find f^{-1} and state the domain and range of f^{-1}.

e Sketch the graphs of f and f^{-1} on the one set of axes.

7 Consider the family of functions with rules of the form $f(x) = c\log_e(x - a)$, where a and c are positive constants.

a State the equation of the vertical asymptote.

b State the coordinates of the x-axis intercept.

c State the coordinates of the point where the graph crosses the line $y = c$.

d Find the inverse function f^{-1} of f.

e State the range of f^{-1}.

f If $f^{-1}(1) = 2$ and $f^{-1}(2) = 4$, find the exact values of a and c.

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8 The inverse function of the linear function $f(x) = ax + b$ is $f^{-1}(x) = 4x - 6$. Find the values of a and b.

9 Find the inverse function of each of the following functions:

a $f(x) = 3x^{\frac{1}{3}} + 1$ **b** $f(x) = 4x^{\frac{1}{3}} - 2$

c $f(x) = (3x - 2)^3 + 4$ **d** $f(x) = -2x^3 + 3$

Multiple-choice questions

1 The graph of the function with rule $h(x) = \dfrac{x^4 + 2}{x^2}$ can be drawn by adding the ordinates of the graphs of two functions f and g. The rules for f and g could be

A $f(x) = x^4, \quad g(x) = \dfrac{2}{x^2}$ **B** $f(x) = x^2, \quad g(x) = \dfrac{2}{x^2}$

C $f(x) = x^4 + 2, \quad g(x) = x^2$ **D** $f(x) = x^4 + 2, \quad g(x) = \dfrac{2}{x^2}$

E $f(x) = x^2, \quad g(x) = 2$

2 Which one of the following functions is not a one-to-one function?

A $f : \mathbb{R}^+ \to \mathbb{R}, f(x) = \dfrac{1}{x^2}$ **B** $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3$

C $f : \mathbb{R} \to \mathbb{R}, f(x) = 10^x$ **D** $f : \mathbb{R}^+ \to \mathbb{R}, f(x) = \log_{10} x$

E $f : \mathbb{R} \to \mathbb{R}, f(x) = \cos x$

3 For the function with rule $f(x) = e^x$, which one of the following is *not* correct for all positive real x and y?

A $f(x + y) = f(x)f(y)$ **B** $f^{-1}(xy) = f^{-1}(x) + f^{-1}(y)$

C $f^{-1}(x^y) = yf^{-1}(x)$ **D** $f^{-1}(1) = 0$ **E** $f^{-1}(x) = \dfrac{1}{f(x)}$

4 If $f(x) = \cos x$ and $g(x) = 3x^2$, then $g\left(f\left(\dfrac{\pi}{3}\right)\right)$ is equal to

A $\cos\left(\dfrac{\pi^2}{9}\right)$ **B** $\dfrac{1}{\sqrt{2}}$ **C** 1 **D** $\dfrac{3}{4}$ **E** $\dfrac{4}{3}$

5 Which of the following is not an even function?

A $f : \mathbb{R} \to \mathbb{R}, f(x) = 4x^2$ **B** $f : \mathbb{R} \to \mathbb{R}, f(x) = \cos^2 x$

C $f : \mathbb{R} \to \mathbb{R}, f(x) = \cos x$ **D** $f : \mathbb{R} \to \mathbb{R}, f(x) = 4x^2 - 3$

E $f : \mathbb{R} \to \mathbb{R}, f(x) = (x - 2)^2$

6 It is known that the graph of the function with rule $y = 2ax + \cos(2x)$ has an x-axis intercept when $x = \pi$. The value of a is

A 2 **B** $\dfrac{1}{2\pi}$ **C** 2π **D** -2π **E** $\dfrac{-1}{2\pi}$

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7 Let $g(x) = \log_e(x - 5)$ for $x > 5$. If $2[g(x)] = g(f(x))$, then $f(x)$ is equal to

A $5x - 8$ **B** $x^2 - 10x + 30$ **C** $5x^2$
D $(2x - 10)^2$ **E** $2x - 2$

8 If the equation $f(3x) = 3f(x)$ is true for all real values of x, then the rule for f could be

A x^2 **B** $3x + 3$ **C** $4x$ **D** $\log_e(x + 3)$ **E** $x - 5$

9 The function $g\colon [-a, a] \to \mathbb{R}$, $g(x) = 3\sin(2x)$ has an inverse function. The maximum possible value of a is

A 3 **B** $\dfrac{\pi}{6}$ **C** $\dfrac{\pi}{3}$ **D** $\dfrac{\pi}{4}$ **E** $\dfrac{\pi}{2}$

10 If $f\colon (-\infty, 3) \to \mathbb{R}$, $f(x) = 4\log_e(3 - x)$ and $g\colon [2, \infty) \to \mathbb{R}$, $g(x) = 4\sqrt{x - 2}$, then the maximal domain of the function $f + g$ is

A $\mathbb{R}$ **B** $[-2, 3)$ **C** $[2, 3)$ **D** $[-3, 3)$ **E** $(2, 3]$

11 A transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ that maps the curve with equation $y = \sin x$ onto the curve $y = -4\sin\left(3x - \dfrac{\pi}{3}\right) + 2$ is given by

A $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{9} & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{\pi}{3} \\ 2 \end{bmatrix}$

B $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{\pi}{9} \\ 2 \end{bmatrix}$

C $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{\pi}{3} \\ 2 \end{bmatrix}$

D $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{\pi}{3} \\ -2 \end{bmatrix}$

E $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{\pi}{3} \\ 2 \end{bmatrix}$

12 A transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ that maps the curve with equation $y = 2\sin\left(2x - \dfrac{\pi}{4}\right) - 3$ onto the curve $y = \sin x$ is given by

A $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{\pi}{4} \\ 3 \end{bmatrix}$

B $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -\frac{\pi}{4} \\ -3 \end{bmatrix}$

C $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{\pi}{4} \\ 3 \end{bmatrix}$

D $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -\frac{\pi}{4} \\ \frac{3}{2} \end{bmatrix}$

E $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{\pi}{4} \\ 3 \end{bmatrix}$

13 Let $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$. Which one of the following is not true?

A $f(xy) = f(x)f(y)$ **B** $f(-x) = -f(x)$ **C** $x > y$ implies $f(x) > f(y)$
D $f(2x) = 8f(x)$ **E** $f(x - y) = f(x) - f(y)$

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14 A function f has the following properties for all real values of x:

$$f(2\pi - x) = -f(x) \quad \text{and} \quad f(\pi - x) = f(x)$$

A possible rule for f is

A $f(x) = \cos x$ **B** $f(x) = \sin x$ **C** $f(x) = \tan x$

D $f(x) = 3\cos(2x)$ **E** $f(x) = 1 - \sin x$

Extended-response questions

1 Consider $f\colon \mathbb{R}^+ \to \mathbb{R}$, $f(x) = e^{-x}$ and $g\colon (-\infty, 1) \to \mathbb{R}$, $g(x) = \dfrac{1}{x-1}$.

a State the ranges of f and g.

b Find f^{-1} and g^{-1}.

c **i** Find $g \circ f$. **ii** Sketch the graph of $y = g \circ f(x)$.

d **i** Find $(g \circ f)^{-1}$. **ii** Sketch the graph of $y = (g \circ f)^{-1}(x)$.

2 **a** For $f\colon [5, \infty) \to \mathbb{R}$, $f(x) = \sqrt{x-3}$:

- **i** Sketch the graph of $y = f(x)$ for $x \in [5, \infty)$.
- **ii** State the range.
- **iii** Find f^{-1}.

b For $h\colon [4, \infty) \to \mathbb{R}$, $h(x) = \sqrt{x-p}$ with inverse function h^{-1} that has domain $[1, \infty)$:

- **i** Find p.
- **ii** Find the rule for h^{-1}.
- **iii** Sketch the graphs of $y = h(x)$ and $y = h^{-1}(x)$ on the one set of axes.

3 Let $f\colon (0, \pi) \to \mathbb{R}$ with $f(x) = \sin x$ and $g\colon [1, \infty) \to \mathbb{R}$ with $g(x) = \dfrac{1}{x}$.

a Find the range of f.

b Find the range of g.

c Give a reason why $f \circ g$ is defined and find $f \circ g(x)$.

d State, with reason, whether $g \circ f$ is defined.

e Find g^{-1}, giving its domain and range.

f Give a reason why $g^{-1} \circ f$ is defined and find $g^{-1} \circ f(x)$. Also state the domain and range of this function.

4 Let $f\colon [-a, \infty) \to \mathbb{R}$, $f(x) = k\log_e(x + a) + c$ where k, a and c are positive constants.

- The graph of f has a vertical asymptote $x = -2$.
- The graph has y-axis intercept 2.
- There is a point on the graph with coordinates $(d, 12)$.

a State the value of a. **b** Find the value of c in terms of k.

c Find k in terms of d. **d** If $d = 2e - 2$, find the value of k.

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8

Revision of Chapters 1–7

8A Technology-free questions

1 State the maximal domain and range of each of the following:

a $f(x) = \dfrac{1}{x} + 2$ **b** $f(x) = 3 - 2\sqrt{3x - 2}$ **c** $f(x) = \dfrac{4}{(x-2)^2} + 3$

d $h(x) = 4 - \dfrac{3}{x - 2}$ **e** $f(x) = \sqrt{x - 2} - 5$

2 Find the inverse of the function with the rule $f(x) = \sqrt{x - 2} + 4$ and sketch both functions on the one set of axes.

3 Find the inverse of the function with the rule $f(x) = \dfrac{x - 2}{x + 1}$.

4 Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2e^{3x} - 1$.

a Find the rule and domain of f^{-1}.

b Sketch the graphs of f and f^{-1} on the one set of axes.

c Sketch the graph of $y = f(f^{-1}(x))$ for its maximal domain.

d Sketch the graph of $y = f^{-1}(f(x))$ for its maximal domain.

e Find $y = f(f^{-1}(2x))$.

5 Simplify $2\log_{10} 5 + 3\log_{10} 2 - \log_{10} 20$.

6 Find x in terms of a if $3\log_a x = 3 + \log_a 12$.

7 Solve $2 \times 2^{-x} = 1024$.

8 Solve the equation $4e^{2x} = 9$ for x.

9 **a** The graph of the function f with rule $f(x) = 2\log_e(x + 2)$ intersects the axes at the points $(a, 0)$ and $(0, b)$. Find the exact values of a and b.

b Hence sketch the graph of $y = f(x)$.

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10 Solve the equation $2^{4x} - 5 \times 2^{2x} + 4 = 0$ for x.

11 Solve the equation $\sin\left(\frac{3x}{2}\right) = \frac{1}{2}$ for $x \in [-\pi, \pi]$.

12 **a** State the range and period of the function $h\colon \mathbb{R} \to \mathbb{R}$, $h(x) = 5 - 3\cos\left(\frac{\pi x}{3}\right)$.

b Solve the equation $\cos\left(2x + \frac{\pi}{6}\right) = \frac{1}{2}$ for $x \in [0, \pi]$.

13 Consider the simultaneous equations

$$mx + y = 2$$
$$2x + (m-1)y = -4$$

Find the values of m such that the system of equations has:

a a unique solution **b** no solution **c** infinitely many solutions.

14 If a graph has rule $y = \frac{a}{x^2} + b$ and passes through the points $(1, -1)$ and $(-2, \frac{1}{2})$, find the values of a and b.

15 Find the value(s) of m for which the equation $x^2 + mx + 2 = 0$ has:

a one solution **b** two solutions **c** no solution.

16 Two points A and B have coordinates $(a, -2)$ and $(3, 1)$.

a Find the value(s) of a if:

- **i** the midpoint of AB is $(0, -\frac{1}{2})$
- **ii** the length of AB is $\sqrt{13}$
- **iii** the gradient of AB is $\frac{1}{2}$.

b Find the equation of the line passing through A and B if $a = -2$, and find the angle the line makes with the positive direction of the x-axis.

17 Let $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 2x^3$.

a State whether the function f is even, odd or neither.

b Find the inverse function f^{-1}.

c Find:

i $f^{-1}(16)$ **ii** $f^{-1}(-2)$ **iii** $\{x : f(x) = f^{-1}(x)\}$

18 Let $f(x) = 2 - x$ and $g(x) = \sqrt{2x - 3}$. Find:

a $f(-2)$ **b** $g(4)$ **c** $f(2a)$ **d** $g(a-1)$

e $\{x : f(x) = 10\}$ **f** $\{x : g(x) = 10\}$ **g** $\{x : f(2x) > 0\}$

19 Let $f(x) = 4x - 3$ and $g(x) = x^2 + 2x$.

a Find:

i $f \circ g$ **ii** $g \circ f$ **iii** $g \circ f^{-1}$

b Find a transformation that takes the graph of $y = g(x)$ to the graph of $y = g(f(x))$.

c Find a transformation that takes the graph of $y = x^2$ to the graph of $y = g(x)$.

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20 Solve $\cos x = \frac{\sqrt{3}}{2}$, giving the general solution.

21 A function has rule $y = Ae^{kt}$. Given that $y = 4$ when $t = 1$ and that $y = 10$ when $t = 2$, find the values of A and k.

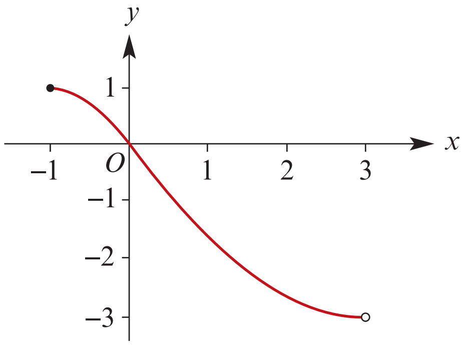
22 Solve each of the following inequalities for x:

a $2x^3 - 3x^2 - 11x + 6 \geq 0$

b $-x^3 + 4x^2 - 4x > 0$

8B Multiple-choice questions

1 The domain of the function whose graph is shown is



A $(-3, 1]$
B $(-1, 3]$
C $[1, 3]$
D $[-1, 3)$
E $(-1, 3)$

2 Which of the following sets of ordered pairs does not represent a function where y is the value of the function?

A $\{(x, y) : x = 2y^2,\ x \geq 0\}$
B $\left\{(x, y) : y = \frac{1}{x},\ x \in \mathbb{R} \setminus \{0\}\right\}$
C $\{(x, y) : y = 2x^3 + 3,\ x \in \mathbb{R}\}$
D $\{(x, y) : y = 3x^2 + 7,\ x \in \mathbb{R}\}$
E $\{(x, y) : y = e^x - 1,\ x \in \mathbb{R}\}$

3 The implied (largest possible) domain for the function with the rule $y = \frac{1}{\sqrt{2 - x}}$ is

A $\mathbb{R} \setminus \{2\}$ **B** $(-\infty, 2)$ **C** $(2, \infty)$ **D** $(-\infty, 2]$ **E** $\mathbb{R}^+$

4 If $f(x) = \frac{x}{x - 1}$, then $f\left(-\frac{1}{a}\right)$ can be simplified as

A $\frac{1}{-1 - a}$ **B** -1 **C** 0 **D** $\frac{a^2}{1 - a}$ **E** $\frac{1}{a + 1}$

5 If $f\colon [0, 2\pi] \to \mathbb{R}$ where $f(x) = \sin(2x)$ and $g\colon [0, 2\pi] \to \mathbb{R}$ where $g(x) = 2\sin x$, then the value of $(f + g)\left(\frac{3\pi}{2}\right)$ is

A 2 **B** 0 **C** -1 **D** 1 **E** -2

6 If $f(x) = 3x + 2$ and $g(x) = 2x^2$, then $f(g(3))$ equals

A 36 **B** 20 **C** 56 **D** 144 **E** 29

7 If $f(x) = 3x^2$, $0 \leq x \leq 6$ and $g(x) = \sqrt{2 - x}$, $x \leq 2$, then the domain of $f + g$ is

A $[0, 2]$ **B** $[0, 6]$ **C** $(-\infty, 2]$ **D** $\mathbb{R}^+ \cup \{0\}$ **E** $[2, 6]$

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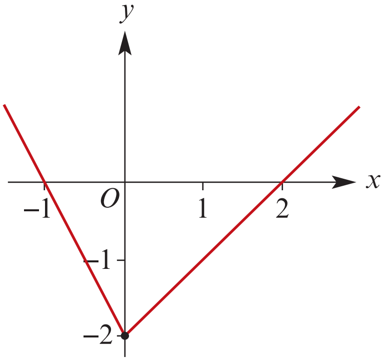
8 The graph shown has the equation



A $y = \begin{cases} x - 2, & x > 0 \\ -2x - 2, & x \le 0 \end{cases}$

B $y = \begin{cases} 2x - 2, & x \ge 0 \\ -2x - 2, & x < 0 \end{cases}$

C $y = \begin{cases} x - 2, & x > 0 \\ -2x - 1, & x \le 0 \end{cases}$

D $y = \begin{cases} x + 2, & x > 0 \\ -2x - 2, & x \le 0 \end{cases}$

E $y = \begin{cases} x - 2, & x > 0 \\ -x - 2, & x \le 0 \end{cases}$

9 If $g(x) = 2x^2 + 1$ and $f(x) = 3x + 2$, then the rule of the product function $(fg)(x)$ equals

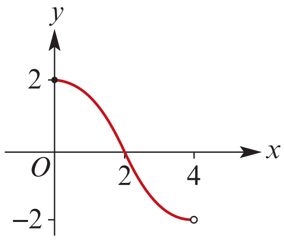
A $2x^2 + 3x + 3$ **B** $6x^3 + 4x^2 + 3x + 2$ **C** $6x^3 + 3$
D $6x^3 + 2x^2 + 3$ **E** $6x^3 + 2$

10 The implied domain for the function with rule $y = \sqrt{4 - x^2}$ is

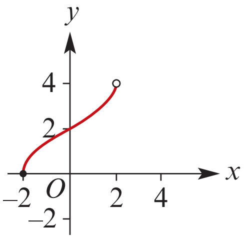
A $[2, \infty)$ **B** $\{x : -2 < x < 2\}$ **C** $[-2, 2]$ **D** $(-\infty, 2)$ **E** $\mathbb{R}^+$

11 The graph of the function with rule $y = f(x)$ is shown.

Which one of the following graphs is the graph of the inverse of f?



A



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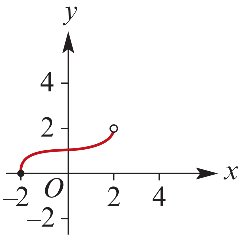

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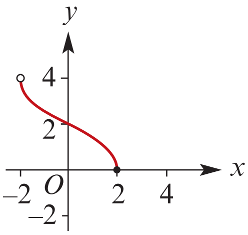

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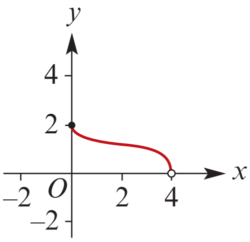

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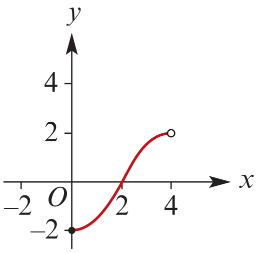

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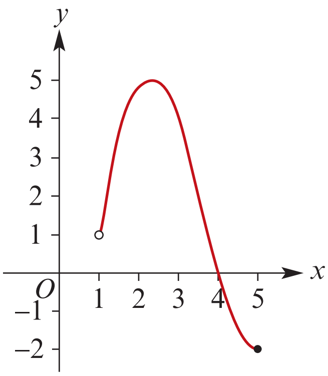
12 The domain of the function whose graph is shown is



A $[1, 5]$

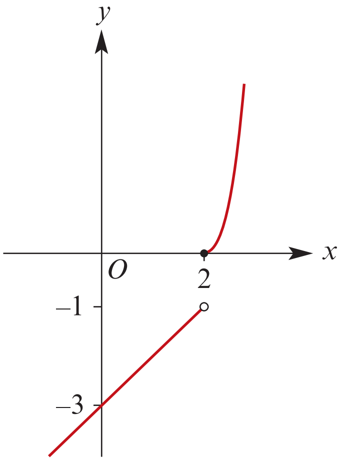
B $(1, 5]$

C $(-2, 5]$

D $(1, 5)$

E $(-2, 5)$

13 The graph shown has the rule



A $y = \begin{cases} (x-2)^2, & x \geq 2 \\ x - 3, & x < 2 \end{cases}$

B $y = \begin{cases} x-2, & x \geq 2 \\ x - 3, & x < 2 \end{cases}$

C $y = \begin{cases} (2-x)^2, & x \geq 2 \\ 2x - 3, & x < 2 \end{cases}$

D $y = \begin{cases} (x-2)^2, & x < 2 \\ 2x - 3, & x \geq 2 \end{cases}$

E $y = \begin{cases} (x-2)^2, & x \geq 2 \\ 2x - 3, & x < 2 \end{cases}$

14 The inverse, f^{-1}, of the function $f\colon [2, 3] \to \mathbb{R}$, $f(x) = 2x - 4$ is

A $f^{-1}\colon [0, 2] \to \mathbb{R}$, $f^{-1}(x) = \dfrac{x}{2} + 4$

B $f^{-1}\colon [3, 2] \to \mathbb{R}$, $f^{-1}(x) = \dfrac{x+4}{2}$

C $f^{-1}\colon [2, 3] \to \mathbb{R}$, $f^{-1}(x) = \dfrac{1}{2x-4}$

D $f^{-1}\colon [0, 2] \to \mathbb{R}$, $f^{-1}(x) = \dfrac{1}{2x-4}$

E $f^{-1}\colon [0, 2] \to \mathbb{R}$, $f^{-1}(x) = \dfrac{x+4}{2}$

15 Let f be the function defined by $f(x) = \dfrac{1}{x^2+2}$, $x \in \mathbb{R}$. A suitable restriction f^* of f such that $(f^*)^{-1}$ exists would be

A $f^*\colon [-1, 1] \to \mathbb{R}$, $f^*(x) = \dfrac{1}{x^2+2}$

B $f^*\colon \mathbb{R} \to \mathbb{R}$, $f^*(x) = \dfrac{1}{x^2+2}$

C $f^*\colon [-2, 2] \to \mathbb{R}$, $f^*(x) = \dfrac{1}{x^2+2}$

D $f^*\colon [0, \infty) \to \mathbb{R}$, $f^*(x) = \dfrac{1}{x^2+2}$

E $f^*\colon [-1, \infty) \to \mathbb{R}$, $f^*(x) = \dfrac{1}{x^2+2}$

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16 Let $h\colon [a, 2] \to \mathbb{R}$ where $h(x) = 2x - x^2$. If a is the smallest real value such that h has an inverse function, h^{-1}, then a equals

A -1 **B** 0 **C** 1 **D** -2 **E** $\frac{1}{2}$

17 If $f(x) = 3x - 2$, $x \in \mathbb{R}$, then $f^{-1}(x)$ equals

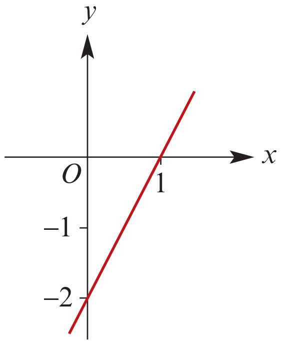
A $\frac{1}{3x-2}$ **B** $3x + 2$ **C** $\frac{1}{3}(x - 2)$ **D** $3x + 6$ **E** $\frac{1}{3}(x + 2)$

18 The solution of the equation $2x = \frac{3x}{2} - 4$ is

A 4 **B** -2 **C** -8 **D** 1 **E** 2

19 The graph shows

A $y + 2 = x$
B $y = 2x - 2$
C $y + 2x + 2 = 0$
D $y = -2x + 2$
E $y - 2 = x$



20 If $\frac{2(x-1)}{3} - \frac{x+4}{2} = \frac{5}{6}$, then x equals

A 5 **B** $\frac{7}{5}$ **C** $\frac{21}{5}$ **D** 21 **E** 3

21 The equation of the line that passes through the points $(-2, 3)$ and $(4, 0)$ is

A $2y = x + 4$ **B** $y = -\frac{1}{2}x - 2$ **C** $2y + x = 4$
D $y = \frac{1}{2}x - 2$ **E** $2y - x = 4$

22 The straight line with equation $y = \frac{4}{5}x - 4$ meets the x-axis at A and the y-axis at B. If O is the origin, the area of the triangle OAB is

A $3\frac{1}{5}$ square units **B** $9\frac{2}{5}$ square units **C** 10 square units
D 15 square units **E** 20 square units

23 If the equations $2x - 3y = 12$ and $3x - 2y = 13$ are simultaneously true, then $x + y$ equals

A -5 **B** -1 **C** 0 **D** 1 **E** 5

24 The graphs of the relations $7x - 6y = 20$ and $3x + 4y = 2$ are drawn on the same pair of axes. The x-coordinate of the point of intersection is

A -2 **B** -1 **C** 1 **D** 2 **E** 3

25 A possible equation for the graph shown is

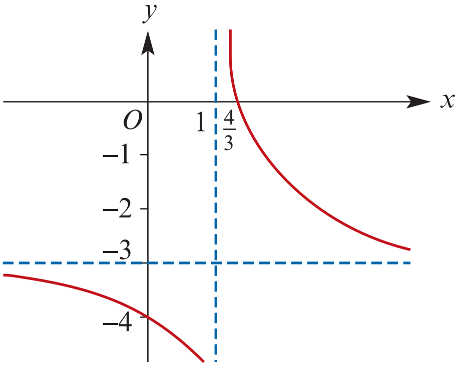
A $y - 3 = \dfrac{1}{x-1}$

B $y + 3 = \dfrac{1}{x+1}$

C $y - 3 = \dfrac{1}{x+1}$

D $y - 4 = \dfrac{1}{x+1}$

E $y = \dfrac{1}{x-1} - 3$



26 The function given by $f(x) = \dfrac{1}{x+3} - 2$ has the range

A $\mathbb{R} \setminus \{-2\}$ **B** $\mathbb{R}$ **C** $\mathbb{R} \setminus \{3\}$ **D** $\mathbb{R} \setminus \{2\}$ **E** $\mathbb{R} \setminus \{-3\}$

27 A parabola has its vertex at $(2, 3)$. A possible equation for this parabola is

A $y = (x + 2)^2 + 3$ **B** $y = (x - 2)^2 - 3$ **C** $y = (x + 2)^2 - 3$

D $y = (x - 2)^2 + 3$ **E** $y = 3 - (x + 2)^2$

28 Which one of the following is an even function of x?

A $f(x) = 3x + 1$ **B** $f(x) = x^3 - x$ **C** $f(x) = (1 - x)^2$

D $f(x) = -x^2$ **E** $f(x) = x^3 + x^2$

29 The graph of $y = 3\sqrt{x+2}$ can be obtained from the graph of $y = \sqrt{x}$ by

A a translation $(x, y) \to (x - 2, y)$ followed by a dilation of factor 3 from the x-axis

B a translation $(x, y) \to (x + 2, y)$ followed by a dilation of factor $\frac{1}{3}$ from the x-axis

C a translation $(x, y) \to (x + 3, y)$ followed by a dilation of factor 3 from the y-axis

D a translation $(x, y) \to (x - 2, y)$ followed by a dilation of factor 3 from the y-axis

E a translation $(x, y) \to (x + 2, y)$ followed by a dilation of factor 3 from the y-axis

30 A function with rule $f(x) = 3\sqrt{x-2} + 1$ has maximal domain

A $(-\infty, 2)$ **B** $[1, \infty)$ **C** $(2, \infty)$ **D** $[-2, \infty)$ **E** $[2, \infty)$

31 A possible equation for the graph shown is

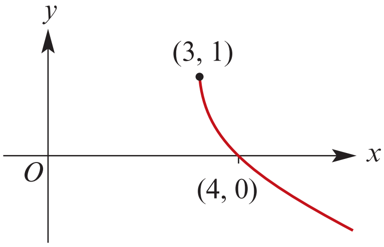
A $y = 2\sqrt{x-3} + 1$

B $y = -2\sqrt{x-3} + 1$

C $y = \sqrt{x-3} + 1$

D $y = -\sqrt{x-3} + 1$

E $y = -2\sqrt{x-3} + 2$



32 The range of the function $f\colon \mathbb{R} \setminus \{2\} \to \mathbb{R}$, $f(x) = \dfrac{3}{(x-2)^2} + 4$ is

A $(3, 4]$ **B** $(-\infty, 4)$ **C** $[3, 4)$ **D** $[4, \infty)$ **E** $(4, \infty)$

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33 If $3x^2 + kx + 1 = 0$ when $x = 1$, then k equals

A -4 **B** -1 **C** 1 **D** 4 **E** 0

34 The quadratic equation with solutions 5 and -7 is

A $x^2 + 2x - 35 = 0$ **B** $x^2 - 2x - 35 = 0$ **C** $x^2 + 12x - 35 = 0$

D $x^2 - 12x - 35 = 0$ **E** $-x^2 + 12x + 35 = 0$

35 If $x^3 - 5x^2 + x + k$ is divisible by $x + 1$, then k equals

A -7 **B** -5 **C** -2 **D** 5 **E** 7

36 Which one of the following could be the equation of the graph shown?

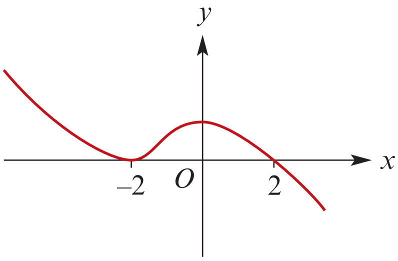
A $y = x(x - 2)(x + 2)$

B $y = -x(x + 2)(x - 2)$

C $y = -(x + 2)^2(x - 2)$

D $y = (x - 2)^2(x + 2)$

E $y = x(x - 2)^2$



37 The graph shown is

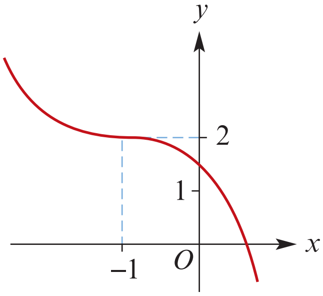
A $y + 2 = -2(x + 1)^3$

B $y - 2 = 2(x - 1)^3$

C $y = x^3 + 2$

D $y = -\frac{1}{2}(x + 1)^3 + 2$

E $y = 2(x - 1)^3 - 2$



38 $P(x) = x^3 + 2x^2 - 5x - 6$ has the factorisation

A $(x - 1)(x - 2)(x + 3)$ **B** $(x + 1)(x + 2)(x + 3)$ **C** $(x + 1)(x - 2)(x + 3)$

D $(x + 1)(x - 2)(x - 3)$ **E** $(x - 1)(x - 2)(x - 3)$

39 The graph shown is that of the function $f(x) = mx + 3$, where m is a constant. The inverse function is $f^{-1}: \mathbb{R} \to \mathbb{R}$, $f^{-1}(x) = ax + b$, where a and b are constants. Which one of the following statements is true?

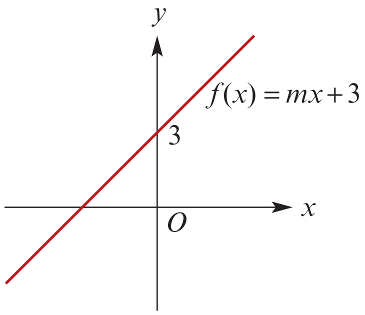
A $a = \dfrac{-3}{m},\ b = \dfrac{1}{m}$

B $a < 0$ and $b < 0$

C $a = -m,\ b = 3$

D $a > 0$ and $b > 0$

E $a = \dfrac{1}{m},\ b = \dfrac{-3}{m}$



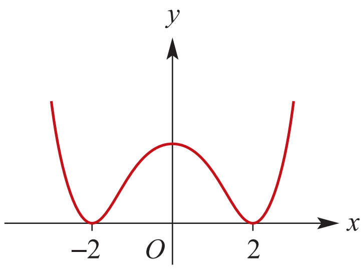
40 Let $P(x) = 2x^3 - 2x^2 + 3x + 1$. When $P(x)$ is divided by $x - 2$, the remainder is

A 31 **B** 15 **C** 1 **D** -2 **E** -29

41 If $x^3 + 2x^2 + ax - 4$ has remainder 1 when divided by $x + 1$, then a equals

A -8 **B** -4 **C** -2 **D** 0 **E** 2

42 Which of these equations is represented by the graph shown?



A $y = (x + 2)^2(x - 2)$

B $y = 16 - x^4$

C $y = (x^2 - 4)^2$

D $y = (x + 2)^2(2 - x)$

E $y = x^4 - 16$

43 The function $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = e^{-x} + 1$ has an inverse function f^{-1}. The domain of f^{-1} is

A $(0, \infty)$ **B** $\mathbb{R}$ **C** $[1, \infty)$ **D** $(1, \infty)$ **E** $[0, \infty)$

44 The function $f\colon \mathbb{R}^+ \to \mathbb{R}$, $f(x) = 2\log_e x + 1$ has an inverse function f^{-1}. The rule for f^{-1} is given by

A $f^{-1}(x) = 2e^{x-1}$ **B** $f^{-1}(x) = e^{\frac{1}{2}(x-1)}$ **C** $f^{-1}(x) = e^{\frac{x}{2}-1}$

D $f^{-1}(x) = 2e^{x+1}$ **E** $f^{-1}(x) = \dfrac{1}{2}e^{x-1}$

45 Let $f\colon \mathbb{R} \to \mathbb{R}$ where $f(x) = e^{-x}$ and let $g\colon (-1, \infty) \to \mathbb{R}$ where $g(x) = \log_e(x + 2)$. The function with the rule $y = f(g(x))$ has the range

A $(1, \infty)$ **B** $(0, 1)$ **C** $(0, 1]$ **D** $[1, \infty)$ **E** $[0, 1]$

46 The function $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = e^x - 1$ has an inverse whose rule is given by

A $f^{-1}(x) = \dfrac{1}{e^x - 1}$ **B** $f^{-1}(x) = -\log_e(x + 1)$ **C** $f^{-1}(x) = \log_e(x - 1)$

D $f^{-1}(x) = \log_e(1 - x)$ **E** $f^{-1}(x) = \log_e(x + 1)$

47 The function $f\colon [4, \infty) \to \mathbb{R}$, $f(x) = \log_e(x - 3)$ has an inverse. The domain of this inverse is

A $[0, \infty)$ **B** $(0, \infty)$ **C** $[4, \infty)$ **D** $(3, \infty)$ **E** $\mathbb{R}$

48 The function $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = e^{x-1}$ has an inverse whose rule is given by $f^{-1}(x) =$

A $e^{-(x-1)}$ **B** $-\log_e x$ **C** $1 + \log_e x$ **D** $\log_e(x + 1)$ **E** $\log_e(x - 1)$

49 The function $f\colon \mathbb{R}^+ \to \mathbb{R}$, $f(x) = \log_e\left(\dfrac{x}{2}\right)$ has an inverse function f^{-1}. The rule for f^{-1} is given by $f^{-1}(x) =$

A $e^{\frac{1}{2}x}$ **B** $\log_e\left(\dfrac{2}{x}\right)$ **C** $\dfrac{1}{2}e^{\frac{x}{2}}$ **D** $2e^x$ **E** $\dfrac{1}{\log_e\left(\dfrac{2}{x}\right)}$

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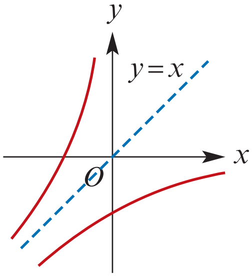
50 For which values of x is the function f with the rule $f(x) = -2 + \log_e(3x - 2)$ defined?

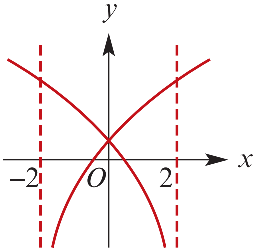
A $(-2, \infty)$ **B** $\left(\frac{2}{3}, \infty\right)$ **C** $[-2, \infty)$ **D** $\left[\frac{2}{3}, \infty\right)$ **E** $(2, \infty)$

51 The graphs of the function $f\colon (-2, \infty) \to \mathbb{R}$ where $f(x) = 2 + \log_e(x + 2)$ and its inverse f^{-1} are best shown by which one of the following?

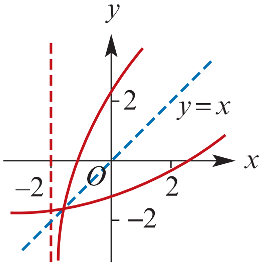
A



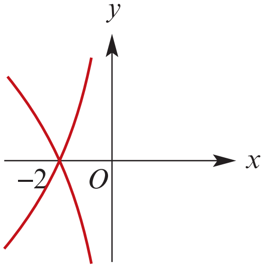
B



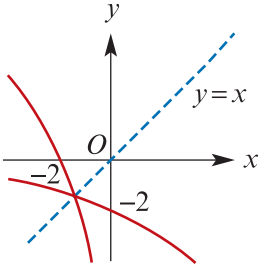
C



D



E



52 If $\log_2(8x) + \log_2(2x) = 6$, then $x =$

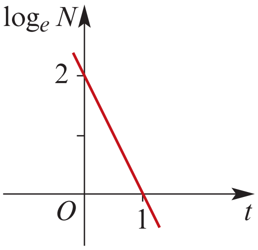
A 1.5 **B** ±1.5 **C** 2 **D** ±2 **E** 6.4

53 The equation $\log_{10} x = y(\log_{10} 3) + 1$ is equivalent to the equation

A $x = 10(3^y)$ **B** $x = 30^y$ **C** $x = 3^y + 10$

D $x = y^3 + 10$ **E** $x = 10y^3$

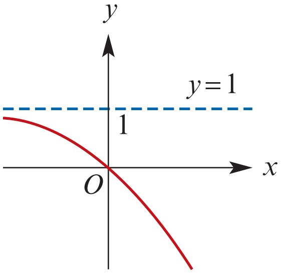
54 The graph indicates that the relationship between N and t is



A $N = 2 - e^{-2t}$ **B** $N = e^{2-2t}$ **C** $N = e^{2t} + 2$

D $N = \dfrac{e^{-2t}}{100}$ **E** $N = -2e^{2t}$

55 A possible equation for the graph is



A $y = 1 - e^x$

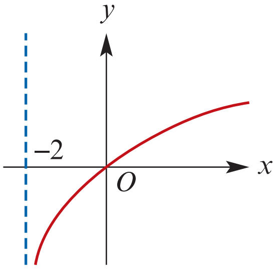
B $y = 1 - e^{-x}$

C $y = 1 + e^x$

D $y = 1 + e^{-x}$

E $y = e^{-x} - 1$

56 A possible equation for the graph is



A $y = \log_e(x - 2)$

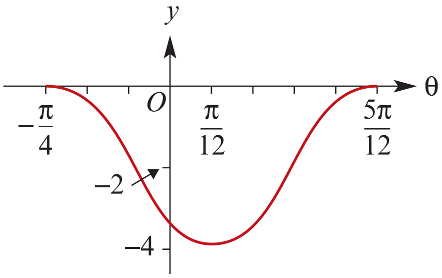
B $y = \log_e \frac{1}{2}(x + 2)$

C $y = \log_e 2(x + 1)$

D $y = 2\log_e(x + 1)$

E $y = \frac{1}{2}\log_e(x + 2)$

57 A possible equation for the graph shown is



A $y = 2\cos 3\left(\theta + \frac{\pi}{4}\right) - 4$

B $y = 2\cos 2\left(\theta + \frac{\pi}{4}\right) - 2$

C $y = 2\sin 3\left(\theta + \frac{\pi}{4}\right) - 2$

D $y = 2\cos 3\left(\theta + \frac{\pi}{4}\right) - 2$

E $y = 2\cos 3\left(\theta - \frac{\pi}{4}\right) - 2$

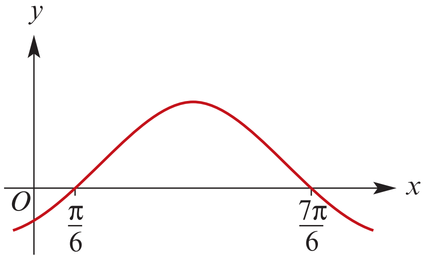
58 The function $f: \mathbb{R} \to \mathbb{R}$ where $f(x) = 2 - 3\cos 2\left(\theta + \frac{\pi}{2}\right)$ has range

A $[-3, 5]$ **B** $[2, 5]$ **C** $\mathbb{R}$ **D** $[-1, 5]$ **E** $[-3, 2]$

59 Two values between 0 and 2π for which $2\sin\theta + \sqrt{3} = 0$ are

A $\frac{\pi}{3}, \frac{2\pi}{3}$ **B** $60°, 240°$ **C** $\frac{2\pi}{3}, \frac{5\pi}{3}$ **D** $\frac{4\pi}{3}, \frac{5\pi}{3}$ **E** $\frac{7\pi}{6}, \frac{11\pi}{6}$

60 A possible equation for the graph shown is



A $y = \sin\left(x - \frac{\pi}{6}\right)$

B $y = \sin\left(x + \frac{\pi}{6}\right)$

C $y = -\sin\left(x - \frac{\pi}{6}\right)$

D $y = \cos\left(x - \frac{\pi}{6}\right)$

E $y = \cos\left(x + \frac{\pi}{6}\right)$

61 The function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 3\sin(2x)$ has

A amplitude 3 and period π

B amplitude 2 and period $\frac{\pi}{2}$

C amplitude 1 and period $\frac{\pi}{2}$

D amplitude $\frac{3}{2}$ and period 2π

E amplitude $1\frac{1}{2}$ and period 2π

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62 The function $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 3\sin(2x)$ has range

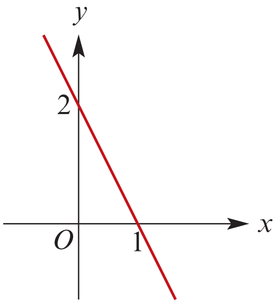
A $[0, 3]$ **B** $[-2, 2]$ **C** $[2, 3]$ **D** $[-3, 3]$ **E** $[-1, 5]$

63 Consider the polynomial $p(x) = (x - 2a)^2(x + a)(x^2 + a)$ where $a > 0$. The equation $p(x) = 0$ has exactly

A 1 distinct real solution **B** 2 distinct real solutions **C** 3 distinct real solutions
D 4 distinct real solutions **E** 5 distinct real solutions

64 The gradient of a straight line perpendicular to the line shown is

A 2 **B** -2 **C** $\frac{-1}{2}$ **D** $\frac{1}{2}$ **E** 3



65 The graph of a function f whose rule is $y = f(x)$ has exactly one asymptote, for which the equation is $y = 6$. The inverse function f^{-1} exists. The inverse function will have

A a horizontal asymptote with equation $y = 6$
B a vertical asymptote with equation $x = 6$
C a vertical asymptote with equation $x = -\frac{1}{6}$
D a horizontal asymptote with equation $y = -6$
E no asymptote

66 The function $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = a\sin(bx) + c$, where a, b and c are positive constants, has period

A a **B** b **C** c **D** $\frac{2\pi}{a}$ **E** $\frac{2\pi}{b}$

67 The functions $f\colon [18, 34] \to \mathbb{R}$, $f(x) = 2x - 4$ and $g\colon \mathbb{R}^+ \to \mathbb{R}$, $g(x) = \log_2 x$ are used to define the composite function $g \circ f$. The range of $g \circ f$ is

A $[2, \infty)$ **B** $\left[\frac{3}{2}, \infty\right)$ **C** $[5, 6]$ **D** $\mathbb{R}^+$ **E** $\mathbb{R}$

68 The rule for the inverse relation of the function with rule $y = x^2 - 4x + 5$ and domain $\mathbb{R}$ is

A $y = 2 \pm \sqrt{x + 1}$ **B** $y^2 = 2x + 5$ **C** $y = 2 \pm \sqrt{x - 1}$
D $y = \sqrt{4x - 5}$ **E** $y = 4x - 5$

69 The function $f\colon B \to \mathbb{R}$, $f(x) = x^2 - 4x + 3$ will have an inverse function for

A $B = \mathbb{R}$ **B** $B = (2, \infty)$ **C** $B = [-1, \infty)$ **D** $B = (-\infty, 4]$ **E** $B = \mathbb{R}^+$

70 Let $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 6x$ and let $g\colon [-5, -3) \to \mathbb{R}$, $g(x) = x + 6$. Then the domain of the inverse function of $h = f + g$ is

A $[-5, 3)$ **B** $[-5, 3) \cup \mathbb{R}$ **C** $(30, 56]$ **D** $[-5, 3) \cap \mathbb{R}$ **E** $[30, 56)$

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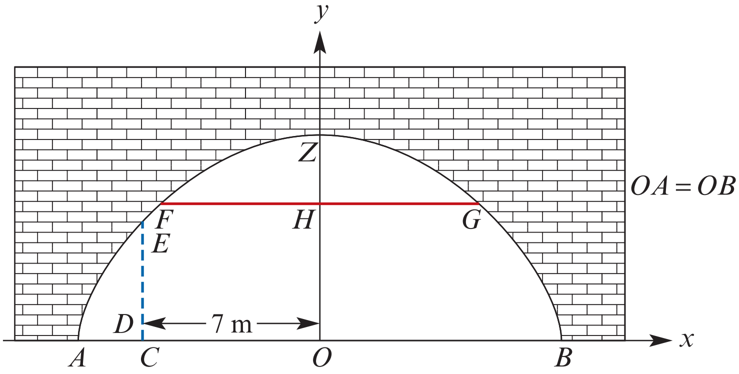
8C Extended-response questions

1 An arch is constructed as shown.



The height of the arch is 9 metres ($OZ = 9$ m). The width of the arch is 20 metres ($AB = 20$ m). The equation of the curve is of the form $y = ax^2 + b$, taking axes as shown.

- **a** Find the values of a and b.
- **b** A man of height 1.8 m stands at C ($OC = 7$ m). How far above his head is the point E on the arch? (That is, find the distance DE.)
- **c** A horizontal bar FG is placed across the arch as shown. The height, OH, of the bar above the ground is 6.3 m. Find the length of the bar.

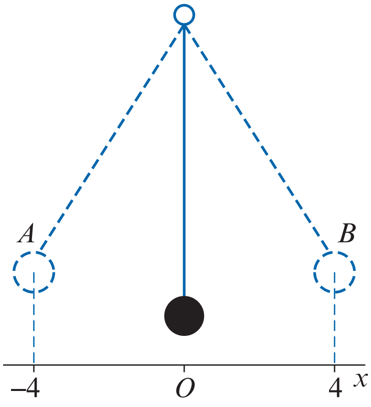
2
- **a** The expression $2x^3 + ax^2 - 72x - 18$ leaves a remainder of 17 when divided by $x + 5$. Determine the value of a.
- **b** Solve the equation $2x^3 = x^2 + 5x + 2$.
- **c**
  - **i** Given that the expression $x^2 - 5x + 7$ leaves the same remainder whether divided by $x - b$ or $x - c$, where $b \neq c$, show that $b + c = 5$.
  - **ii** Given further that $4bc = 21$ and $b > c$, find the values of b and c.

3 As a pendulum swings, its horizontal position, x cm, measured from the central position, varies from −4 cm (at A) to 4 cm (at B).

The position x is given by the rule

$$x = -4\sin(\pi t)$$

- **a** Sketch the graph of x against t for $t \in [0, 2]$.
- **b** Find the horizontal position of the pendulum for:
  - **i** $t = 0$ **ii** $t = \frac{1}{2}$ **iii** $t = 1$
- **c** Find the first time that the pendulum has horizontal position $x = 2$.
- **d** Find the period of the pendulum, i.e. the time it takes to go from A to B and back to A.



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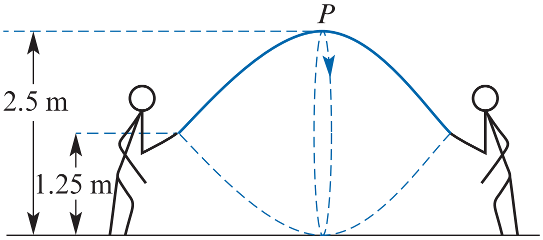
4 Two people are rotating a skipping rope. The rope is held 1.25 m above the ground. It reaches a height of 2.5 m above the ground, and just touches the ground.

The vertical position, y m, of the point P on the rope at time t seconds is given by the rule

$$y = -1.25\cos(2\pi t) + 1.25$$



a Find y when:

  i $t = 0$

  ii $t = \frac{1}{2}$

  iii $t = 1$

b How long does it take for one rotation of the rope?

c Sketch the graph of y against t.

d Find the first time that the point P on the rope reaches a height of 2 m above the ground.

5 The population of a country is found to be growing *continuously* at an annual rate of 2.96% after 1 January 1950. The population t years after 1 January 1950 is given by the formula

$$p(t) = (150 \times 10^6)e^{kt}$$

a Find the value of k.

b Find the population on 1 January 1950.

c Find the population on 1 January 2000.

d After how many years would the population be 300×10^6?

6 A large urn was filled with water. It was turned on, and the water was heated until its temperature reached 95°C. This occurred at exactly 2 p.m., at which time the urn was turned off and the water began to cool. The temperature of the room where the urn was located remained constant at 15°C.

Commencing at 2 p.m. and finishing at midnight, Jenny measured the temperature of the water every hour on the hour for the next 10 hours and recorded the results. At 4 p.m., Jenny recorded the temperature of the water as 55°C. She found that the temperature, T°C, of the water could be described by the equation

$$T = Ae^{-kt} + 15, \quad \text{for } 0 \le t \le 10$$

where t is the number of hours after 2 p.m.

a Find the values of A and k.

b Find the temperature of the water at midnight.

c At what time did Jenny first record a temperature less than 24°C?

d Sketch the graph of T against t.

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7 A football is kicked so that it leaves the player's foot with a velocity of V m/s. The total horizontal distance travelled by the football, x m, is given by

$$x = \frac{V^2 \sin(2\alpha)}{10}$$

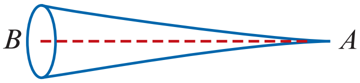
where α is the angle of projection.

a Find the horizontal distance travelled by the ball if $V = 25$ m/s and $\alpha = 45°$.

b For $V = 20$, sketch the graph of x against α for $0° \le \alpha \le 90°$.

c If the ball goes 30 m and the initial velocity is 20 m/s, find the angle of projection.

8 The diagram shows a conical glass fibre. The circular cross-sectional area at end B is 0.02 mm^2.



The cross-sectional area diminishes by a factor of $(0.92)^{\frac{1}{10}}$ per metre length of the fibre. The total length is 5 m.

a Write down a rule for the cross-sectional area of the fibre at a distance x m from B.

b What is the cross-sectional area of the fibre at a point one-third of its length from B?

c The fibre is constructed such that the strength increases in the direction B to A. At a distance of x m from B, the strength is given by the rule $S = (0.92)^{10-3x}$. If the load the fibre will take at each point before breaking is given by

load = strength × cross-sectional area

write down an expression, in terms of x, for the load the fibre will stand at a distance of x m from B.

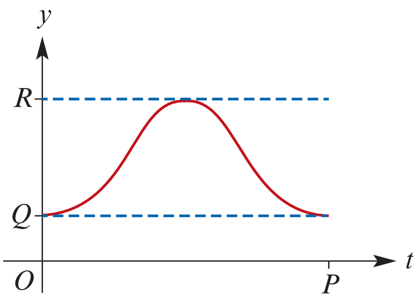
d A piece of glass fibre that will have to carry loads of up to $0.02 \times (0.92)^{2.5}$ units is needed. How much of the 5 m fibre could be used with confidence for this purpose?

9 **a** The graph is of one complete cycle of

$$y = h - k\cos\left(\frac{\pi t}{6}\right)$$

i How many units long is OP?

ii Express OQ and OR in terms of h and k.



b For a certain city in the northern hemisphere, the number of hours of daylight on the 21st day of each month is given by the table:

x	Dec	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
y	7.5	8.2	9.9	12.0	14.2	15.8	16.5	15.9	14.3	12.0	9.8	8.1	7.5

Using suitable scales, plot these points and draw a curve through them. Call December month 0, January month 1, etc., and treat all months as of equal length.

c Find the values of h and k so that your graph is approximately that of

$$y = h - k\cos\left(\frac{\pi t}{6}\right)$$

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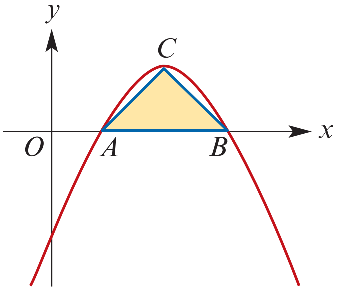
10 On an overnight interstate train, an electrical fault affected the illumination in two carriages, A and B. Before the fault occurred, the illumination in carriage A was I units and that in carriage B was $0.66I$ units. Every time the train stopped, the illumination in carriage A reduced by 17% and that in carriage B by 11%.

a Write down exponential expressions for the expected illumination in each carriage after the train had stopped for the nth time.

b At some time after the fault occurred, the illumination in both carriages was approximately the same. At how many stations did the train stop before this occurred?

11 **a** The curve $y = 1 - a(x - 3)^2$ in the figure intersects the x-axis at A and B. Point C is the vertex of the curve and a is a positive constant.



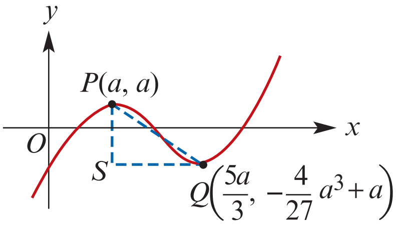
i Find the coordinates of A and B in terms of a.

ii Find the area of triangle ABC in terms of a.

b The graph shown has rule

$$y = (x - a)^2(x - 2a) + a$$

where $a > 0$.



i Use a calculator to sketch the graph for $a = 1, 2, 3$.

ii Find the values of a for which $\frac{-4}{27}a^3 + a = 0$.

iii Find the values of a for which $\frac{-4}{27}a^3 + a < 0$.

iv Find the value of a for which $\frac{-4}{27}a^3 + a = -1$.

v Find the value of a for which $\frac{-4}{27}a^3 + a = 1$.

vi Plot the graphs $y = (x - a)^2(x - 2a) + a$ for the values of a obtained in **iv** and **v**.

c Triangle PSQ is a right-angled triangle.

i Give the coordinates of S.

ii Find the length of PS and SQ in terms of a.

iii Give the area of triangle PSQ in terms of a.

iv Find the value of a for which the area of the triangle is 4.

v Find the value of a for which the area of the triangle is 1500.

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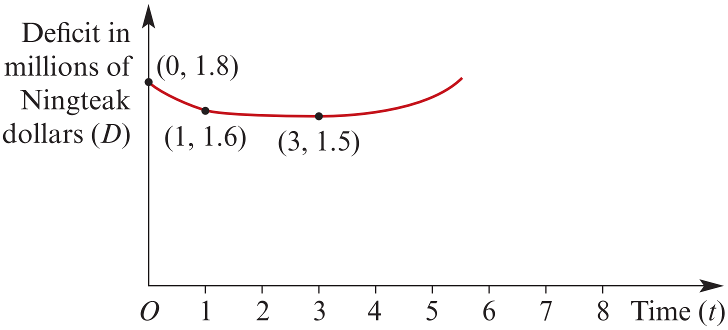
12 The deficit of a government department in Ningteak, a small monarchy east of Africa, is continually assessed over a period of 8 years. The following graph shows the deficit over these 8 years.



The graph is read as follows: The deficit at the beginning of the 8-year period is \$1.8 million. At the end of the third year the deficit is \$1.5 million, and this is the smallest deficit for the period $0 \leq t \leq 8$.

a Find the rule for D in terms of t, assuming that it is of the form $D = at^2 + bt + c$.

b Use this model to predict the deficit at the end of 8 years.

13 The rate of rainfall, R mm per hour, was recorded during a very rainy day in North Queensland. The recorded data are given in the table.

Assume a quadratic rule of the form

$$R = at^2 + bt + c$$

is applicable for $0 \leq t \leq 12$, where $t = 0$ is 4 a.m.

Use the quadratic model to predict the rate of rainfall at noon. At what time was the rate of rainfall greatest?

Time	Rainfall
4 a.m.	7.5 mm per hour
8 a.m.	9.0 mm per hour
10 a.m.	8.0 mm per hour

14 A population of insects is determined by a rule of the form

$$n = \frac{c}{1 + ae^{-bt}}, \quad t \geq 0$$

where n is the number of insects alive at time t days.

a Consider the population for $c = 5790$, $a = 4$ and $b = 0.03$.

- **i** Find the equation of the horizontal asymptote by considering values of n as t becomes large.
- **ii** Find n when $t = 0$.
- **iii** Sketch the graph of the function.
- **iv** Find the exact value of t for which $n = 4000$.

b **i** Use your calculator to find values of a, b and c such that the population growth yields the table on the right.

t	1	10	100
n	1500	2000	5000

- **ii** Sketch the graph for this population.

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Chapter 9

9 Differentiation

Objectives

- To understand the concept of **limit**.
- To understand the definition of **differentiation**.
- To understand and use the notation for the **derivative** of a polynomial function.
- To find the **gradient** of a tangent to the graph of a polynomial function by calculating its derivative.
- To understand and use the **chain rule**.
- To differentiate **rational powers**.
- To differentiate **exponential functions** and **natural logarithmic functions**.
- To differentiate **circular functions**.
- To understand and use the **product rule** and the **quotient rule**.
- To deduce the **graph of the derivative** from the graph of a function.

It is believed that calculus was discovered independently in the late seventeenth century by two great mathematicians: Isaac Newton and Gottfried Leibniz. Like most scientific breakthroughs, the discovery of calculus did not arise out of a vacuum. In fact, many mathematicians and philosophers going back to ancient times made discoveries relating to calculus.

In this chapter, we review some of the important ideas and results that have been introduced in earlier studies of calculus. We introduce the chain rule, the product rule and the quotient rule, along with the differentiation of exponential, logarithmic and circular functions.

9A The derivative

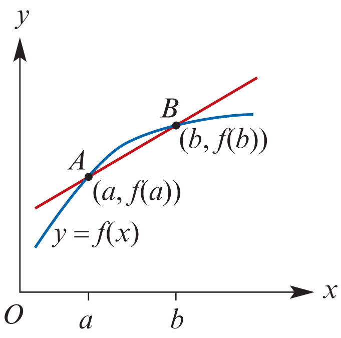
We begin this chapter by recalling the definition of average rate of change from Mathematical Methods Units 1 & 2.

Average rate of change

For any function $y = f(x)$, the **average rate of change** of y with respect to x over the interval $[a, b]$ is the gradient of the line through the two points $A(a, f(a))$ and $B(b, f(b))$.

That is:

$$\text{average rate of change} = \frac{f(b) - f(a)}{b - a}$$



Example 1

Find the average rate of change of the function with rule $f(x) = x^2 - 2x + 5$ as x changes from 1 to 5.

Solution

$$\text{Average rate of change} = \frac{\text{change in } y}{\text{change in } x}$$

$$f(1) = (1)^2 - 2(1) + 5 = 4$$

$$f(5) = (5)^2 - 2(5) + 5 = 20$$

$$\text{Average rate of change} = \frac{20 - 4}{5 - 1}$$

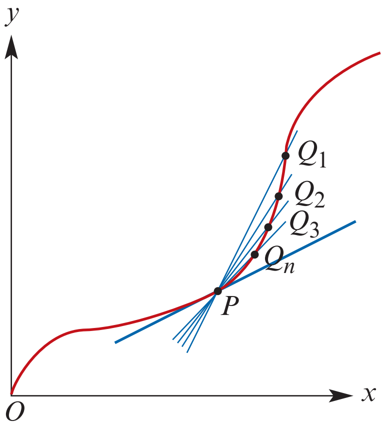
$$= 4$$

The tangent to a curve at a point

We first recall that a **chord** of a curve is a line segment joining points P and Q on the curve. A **secant** is a line through points P and Q on the curve.

The **instantaneous rate of change** at P can be defined by considering what happens when we look at a sequence of secants $PQ_1, PQ_2, PQ_3, \ldots, PQ_n, \ldots$, where the points Q_i get closer and closer to P.

Here we first focus our attention on the gradient of the tangent at P.



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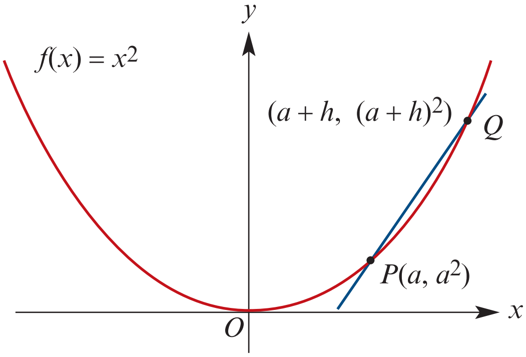
Consider the function $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$.

The gradient of the secant PQ shown on the graph is



$$\begin{aligned}\text{gradient of } PQ &= \frac{(a+h)^2 - a^2}{a+h-a} \\ &= \frac{a^2 + 2ah + h^2 - a^2}{h} \\ &= 2a + h\end{aligned}$$

The limit of $2a + h$ as h approaches 0 is $2a$, and so the gradient of the tangent at P is said to be $2a$.

Note: This also can be interpreted as the instantaneous rate of change of f at $(a, f(a))$.

The straight line that passes through the point P and has gradient $2a$ is called the **tangent** to the curve at P.

It can be seen that there is nothing special about a here. The same calculation works for any real number x. The gradient of the tangent to the graph of $y = x^2$ at any point x is $2x$.

We say that the **derivative of x^2 with respect to x is $2x$**, or more briefly, we can say that the **derivative of x^2 is $2x$**.

Limit notation

The notation for the limit of $2x + h$ as h approaches 0 is

$$\lim_{h \to 0} (2x + h)$$

The derivative of a function with rule $f(x)$ may be found by:

1. finding an expression for the gradient of the line through $P(x, f(x))$ and $Q(x + h, f(x + h))$
2. finding the limit of this expression as h approaches 0.

Example 2

Consider the function $f(x) = x^3$. By first finding the gradient of the secant through $P(2, 8)$ and $Q(2 + h, (2 + h)^3)$, find the gradient of the tangent to the curve at the point $(2, 8)$.

Solution

$$\begin{aligned}\text{Gradient of } PQ &= \frac{(2+h)^3 - 8}{2+h-2} \\ &= \frac{8 + 12h + 6h^2 + h^3 - 8}{h} \\ &= \frac{12h + 6h^2 + h^3}{h} \\ &= 12 + 6h + h^2\end{aligned}$$

The gradient of the tangent line at $(2, 8)$ is $\lim_{h \to 0} (12 + 6h + h^2) = 12$.

The following example provides practice in determining limits.

Example 3

Find:

a $\lim\limits_{h\to 0}(22x^2 + 20xh + h)$

b $\lim\limits_{h\to 0}\dfrac{3x^2h + 2h^2}{h}$

c $\lim\limits_{h\to 0} 12x$

d $\lim\limits_{h\to 0} 4$

Solution

a $\lim\limits_{h\to 0}(22x^2 + 20xh + h) = 22x^2$

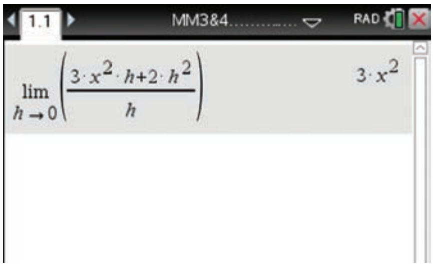
b
$$\begin{aligned}\lim_{h\to 0}\frac{3x^2h + 2h^2}{h} &= \lim_{h\to 0}(3x^2 + 2h)\\ &= 3x^2\end{aligned}$$

c $\lim\limits_{h\to 0} 12x = 12x$

d $\lim\limits_{h\to 0} 4 = 4$

Using the TI-Nspire

To calculate a limit, use menu > **Calculus** > **Limit** and complete as shown.



$\lim\limits_{h\to 0}\left(\dfrac{3\cdot x^2\cdot h+2\cdot h^2}{h}\right)$ $3\cdot x^2$

Note: The limit template can also be accessed from the 2D-template palette ⊞. When you insert the limit template, you will notice a superscript field (small box) on the template – generally this will be left empty.

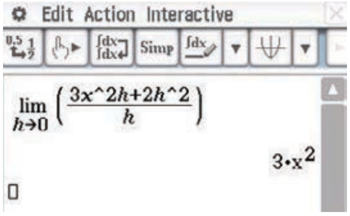
Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, enter and highlight the expression

$$\frac{3x^2h + 2h^2}{h}$$

Note: Use h from the Var keyboard.

- Select $\lim\limits_{\square\to\square}$ from the Math2 keyboard and tap EXE.
- Enter h and 0 in the spaces provided as shown.



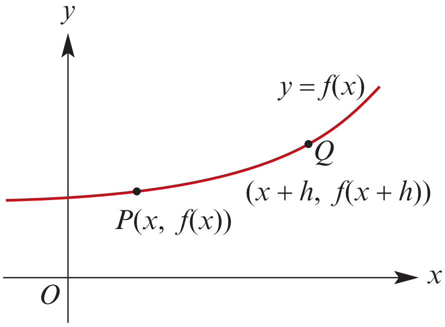
$\lim\limits_{h\to 0}\left(\dfrac{3x\hat{\ }2h+2h\hat{\ }2}{h}\right)$

$3\cdot x^2$

Definition of the derivative

In general, consider the graph $y = f(x)$ of a function $f : \mathbb{R} \to \mathbb{R}$.

$$\text{Gradient of secant } PQ = \frac{f(x+h) - f(x)}{x + h - x} = \frac{f(x+h) - f(x)}{h}$$



The gradient of the tangent to the graph of $y = f(x)$ at the point $P(x, f(x))$ is the limit of this expression as h approaches 0.

Derivative of a function

The **derivative** of the function f is denoted f' and is defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The **tangent line** to the graph of the function f at the point $(a, f(a))$ is defined to be the line through $(a, f(a))$ with gradient $f'(a)$.

Warning: This definition of the derivative assumes that the limit exists. For polynomial functions, such limits always exist. But it is not true that for every function you can find the derivative at every point of its domain. This is discussed further in Sections 9L and 9M.

Differentiation by first principles

Determining the derivative of a function by evaluating the limit is called **differentiation by first principles**.

Example 4

Find $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ for each of the following:

a $f(x) = 3x^2 + 2x + 2$ **b** $f(x) = 2 - x^3$

Solution

a
$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{3(x+h)^2 + 2(x+h) + 2 - (3x^2 + 2x + 2)}{h}\\ &= \frac{3x^2 + 6xh + 3h^2 + 2x + 2h + 2 - 3x^2 - 2x - 2}{h}\\ &= \frac{6xh + 3h^2 + 2h}{h}\\ &= 6x + 3h + 2\end{aligned}$$

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Therefore

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}(6x+3h+2) = 6x+2$$

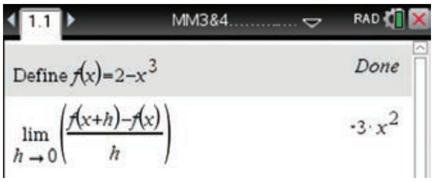
b
$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{2-(x+h)^3-(2-x^3)}{h}\\ &= \frac{2-(x^3+3x^2h+3xh^2+h^3)-2+x^3}{h}\\ &= \frac{-3x^2h-3xh^2-h^3}{h}\\ &= -3x^2-3xh-h^2\end{aligned}$$

Therefore

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}(-3x^2-3xh-h^2) = -3x^2$$

Using the TI-Nspire

- Define $f(x) = 2 - x^3$.
- Use menu > **Calculus** > **Limit** or the 2D-template palette, and complete as shown.



Define f(x)=2−x^3 Done

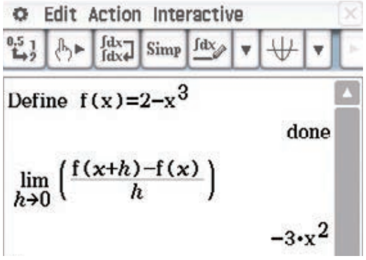
$\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$ $-3\cdot x^2$

Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, enter and highlight the expression $2 - x^3$. Select **Interactive** > **Define** and tap OK.
- Now enter and highlight the expression
$$\frac{f(x+h)-f(x)}{h}$$

Note: Select f from the abc keyboard and x, h from the Var keyboard.

- Select lim from the Math2 keyboard and tap EXE.
- Enter h and 0 in the spaces provided as shown.



Define f(x)=2−x^3

done

$\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$

$-3\cdot x^2$

Section summary

- The **derivative** of the function f is denoted f' and is defined by
$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
- The **tangent line** to the graph of the function f at the point $(a, f(a))$ is defined to be the line through $(a, f(a))$ with gradient $f'(a)$.

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Exercise 9A

Example 1 **1** Find the average rate of change of the function with rule $f(x) = -x^2 + 2x + 1$ as x changes from -1 to 4.

2 Find the average rate of change of the function with rule $f(x) = 6 - x^3$ as x changes from -1 to 1.

Example 2 **3** For the curve with equation $y = x^2 + 5x$:

a Find the gradient of the secant though points P and Q, where P is the point $(2, 14)$ and Q is the point $(2 + h, (2 + h)^2 + 5(2 + h))$.

b From the result of **a**, find the gradient of the tangent to the curve at the point $(2, 14)$.

Example 3 **4** Find:

a $\lim\limits_{h \to 0} \dfrac{4x^2h^2 + xh + h}{h}$

b $\lim\limits_{h \to 0} \dfrac{2x^3h - 2xh^2 + h}{h}$

c $\lim\limits_{h \to 0} (40 - 50h)$

d $\lim\limits_{h \to 0} 5h$

e $\lim\limits_{h \to 0} 5$

f $\lim\limits_{h \to 0} \dfrac{30h^2x^2 + 20h^2x + h}{h}$

g $\lim\limits_{h \to 0} \dfrac{3h^2x^3 + 2hx + h}{h}$

h $\lim\limits_{h \to 0} 3x$

i $\lim\limits_{h \to 0} \dfrac{3x^3h - 5x^2h^2 + xh}{h}$

j $\lim\limits_{h \to 0} (6x - 7h)$

5 For the curve with equation $y = x^3 - x$:

a Find the gradient of the chord PQ, where P is the point $(1, 0)$ and Q is the point $(1 + h, (1 + h)^3 - (1 + h))$.

b From the result of **a**, find the gradient of the tangent to the curve at the point $(1, 0)$.

6 If $f(x) = x^2 - 2$, simplify $\dfrac{f(x + h) - f(x)}{h}$. Hence find the derivative of $x^2 - 2$.

7 Let P and Q be points on the curve $y = x^2 + 2x + 5$ at which $x = 2$ and $x = 2 + h$ respectively. Express the gradient of the line PQ in terms of h, and hence find the gradient of the tangent to the curve $y = x^2 + 2x + 5$ at $x = 2$.

Example 4 **8** For each of the following, find $f'(x)$ by finding $\lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$:

a $f(x) = 5x^2$

b $f(x) = 3x + 2$

c $f(x) = 5$

d $f(x) = 3x^2 + 4x + 3$

e $f(x) = 5x^3 - 5$

f $f(x) = 5x^2 - 6x$

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9B Rules for differentiation

The derivative of x^n where n is a positive integer

Differentiating from first principles gives the following:

- For $f(x) = x$, $f'(x) = 1$.
- For $f(x) = x^2$, $f'(x) = 2x$.
- For $f(x) = x^3$, $f'(x) = 3x^2$.

This suggests the following general result:

For $f(x) = x^n$, $f'(x) = nx^{n-1}$, where $n = 1, 2, 3, \dots$

We can prove this result using the binomial theorem, which is discussed in Appendix A. The proof is not required to be known.

Proof Let $f(x) = x^n$, where $n \in \mathbb{N}$ with $n \geq 2$.

Then
$$\begin{aligned} f(x+h) - f(x) &= (x+h)^n - x^n \\ &= x^n + {}^nC_1 x^{n-1}h + {}^nC_2 x^{n-2}h^2 + \cdots + {}^nC_{n-1} x h^{n-1} + h^n - x^n \\ &= {}^nC_1 x^{n-1}h + {}^nC_2 x^{n-2}h^2 + \cdots + {}^nC_{n-1} x h^{n-1} + h^n \\ &= nx^{n-1}h + {}^nC_2 x^{n-2}h^2 + \cdots + {}^nC_{n-1} x h^{n-1} + h^n \end{aligned}$$

and so
$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{1}{h}\left(nx^{n-1}h + {}^nC_2 x^{n-2}h^2 + \cdots + {}^nC_{n-1} x h^{n-1} + h^n\right) \\ &= nx^{n-1} + {}^nC_2 x^{n-2}h + \cdots + {}^nC_{n-1} x h^{n-2} + h^{n-1} \end{aligned}$$

Thus
$$\begin{aligned} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0}\left(nx^{n-1} + {}^nC_2 x^{n-2}h + \cdots + {}^nC_{n-1} x h^{n-2} + h^{n-1}\right) \\ &= nx^{n-1} \end{aligned}$$

The derivative of a polynomial function

The following results are very useful when finding the derivative of a polynomial function.

- Constant function: If $f(x) = c$, then $f'(x) = 0$.
- Multiple: If $f(x) = k\,g(x)$, where k is a constant, then $f'(x) = k\,g'(x)$.
  That is, the derivative of a number multiple is the multiple of the derivative.
  For example: if $f(x) = 5x^2$, then $f'(x) = 5(2x) = 10x$.
- Sum: If $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$.
  That is, the derivative of the sum is the sum of the derivatives.
  For example: if $f(x) = x^2 + 2x$, then $f'(x) = 2x + 2$.
- Difference: If $f(x) = g(x) - h(x)$, then $f'(x) = g'(x) - h'(x)$.
  That is, the derivative of the difference is the difference of the derivatives.
  For example: if $f(x) = x^2 - 2x$, then $f'(x) = 2x - 2$.

You will meet rules for the derivatives of products and quotients later in this chapter.

The process of finding the derivative function is called **differentiation**.

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Example 5

Find the derivative of $x^5 - 2x^3 + 2$, i.e. differentiate $x^5 - 2x^3 + 2$ with respect to x.

Solution

Let $\quad f(x) = x^5 - 2x^3 + 2$

Then $\quad f'(x) = 5x^4 - 2(3x^2) + 0$

$\qquad\qquad = 5x^4 - 6x^2$

Explanation

We use the following results:

- the derivative of x^n is nx^{n-1}
- the derivative of a number is 0
- the multiple, sum and difference rules.

Example 6

Find the derivative of $f(x) = 3x^3 - 6x^2 + 1$ and thus find $f'(1)$.

Solution

Let $\quad f(x) = 3x^3 - 6x^2 + 1$

Then $\quad f'(x) = 3(3x^2) - 6(2x) + 0$

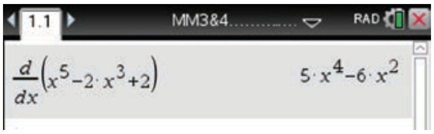
$\qquad\qquad = 9x^2 - 12x$

$\therefore \quad f'(1) = 9 - 12 = -3$

Using the TI-Nspire

For Example 5:

- Use menu > **Calculus** > **Derivative** and complete as shown.

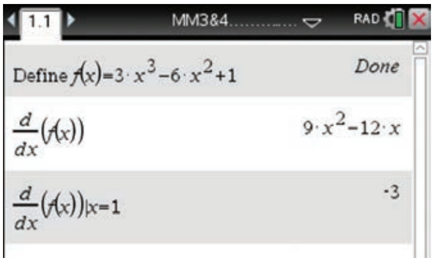


$\frac{d}{dx}(x^5-2\cdot x^3+2)$ $\quad 5\cdot x^4-6\cdot x^2$

Note: The derivative template can also be accessed from the 2D-template palette. Alternatively, using shift – will paste the derivative template to the screen.

For Example 6:

- Define $f(x) = 3x^3 - 6x^2 + 1$.
- Use menu > **Calculus** > **Derivative** to differentiate as shown.
- To evaluate the derivative at $x = 1$, use menu > **Calculus** > **Derivative at a Point**.



Define $f(x)=3\cdot x^3-6\cdot x^2+1$ $\quad$ Done

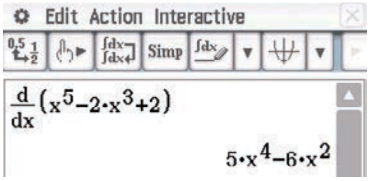
$\frac{d}{dx}(f(x))$ $\quad 9\cdot x^2-12\cdot x$

$\frac{d}{dx}(f(x))|x=1$ $\quad -3$

Using the Casio ClassPad

For Example 5:

- In Main $\sqrt{\alpha}$, enter and highlight the expression $x^5 - 2x^3 + 2$.
- Go to **Interactive** > **Calculation** > **diff** and tap OK.

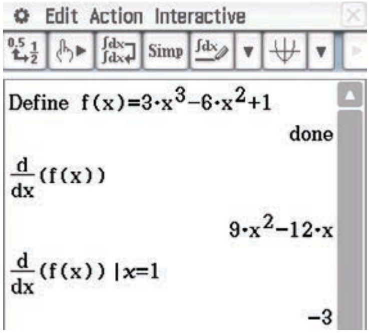


$\frac{d}{dx}(x^5-2\cdot x^3+2)$

$5\cdot x^4-6\cdot x^2$

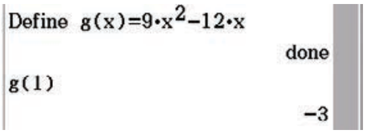
For Example 6:

- In $\overset{\text{Main}}{\sqrt{\alpha}}$, enter and highlight the expression $3x^3 - 6x^2 + 1$.
- Go to **Interactive** > **Calculation** > **diff** and tap OK; this will give the derivative only.
- To find the value of the derivative at $x = 1$, tap the stylus at the end of the entry line. Select | from the Math3 keyboard and type $x = 1$. Then tap EXE.



Define f(x)=3·x^3−6·x^2+1
done
$\frac{d}{dx}$(f(x))
9·x^2−12·x
$\frac{d}{dx}$(f(x))|x=1
−3

- Alternatively, define the derivative as $g(x)$ and find $g(1)$.



Define g(x)=9·x^2−12·x
done
g(1)
−3

Finding the gradient of a tangent line

We discussed the tangent line at a point on a graph in Section 9A. We recall the following:

> The **tangent line** to the graph of the function f at the point $(a, f(a))$ is defined to be the line through $(a, f(a))$ with gradient $f'(a)$.

Example 7

For the curve determined by the rule $f(x) = 3x^3 - 6x^2 + 1$, find the gradient of the tangent line to the curve at the point $(1, -2)$.

Solution

Now $f'(x) = 9x^2 - 12x$ and so $f'(1) = 9 - 12 = -3$.

The gradient of the tangent line at the point $(1, -2)$ is -3.

Alternative notations

It was mentioned in the introduction to this chapter that the German mathematician Gottfried Leibniz was one of the two people to whom the discovery of calculus is attributed. A form of the notation he introduced is still in use today.

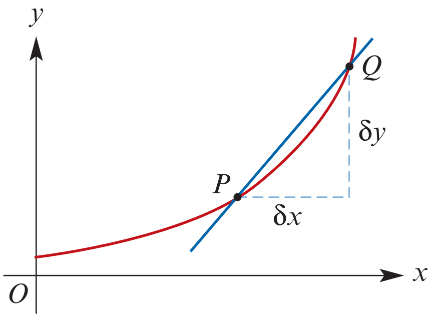
Leibniz notation

An alternative notation for the derivative is the following:

If $y = x^3$, then the derivative can be denoted by $\frac{dy}{dx}$, and so we write $\frac{dy}{dx} = 3x^2$.

In general, if y is a function of x, then the derivative of y with respect to x is denoted by $\frac{dy}{dx}$.

Similarly, if z is a function of t, then the derivative of z with respect to t is denoted $\frac{dz}{dt}$.



Warning: In Leibniz notation, the symbol d is not a factor and cannot be cancelled.

This notation came about because, in the eighteenth century, the standard diagram for finding the limiting gradient was labelled as shown:

- δx means a small difference in x
- δy means a small difference in y

where δ (delta) is the lowercase Greek letter d.

Example 8

a If $y = t^2$, find $\frac{dy}{dt}$. **b** If $x = t^3 + t$, find $\frac{dx}{dt}$. **c** If $z = \frac{1}{3}x^3 + x^2$, find $\frac{dz}{dx}$.

Solution

a $$y = t^2$$ $$\frac{dy}{dt} = 2t$$

b $$x = t^3 + t$$ $$\frac{dx}{dt} = 3t^2 + 1$$

c $$z = \frac{1}{3}x^3 + x^2$$ $$\frac{dz}{dx} = x^2 + 2x$$

Example 9

a For $y = (x+3)^2$, find $\frac{dy}{dx}$.

b For $z = (2t-1)^2(t+2)$, find $\frac{dz}{dt}$.

c For $y = \frac{x^2+3x}{x}$, find $\frac{dy}{dx}$.

d Differentiate $y = 2x^3 - 1$ with respect to x.

Solution

a First write $y = (x+3)^2$ in expanded form:

$$y = x^2 + 6x + 9$$
$$\therefore \quad \frac{dy}{dx} = 2x + 6$$

b Expanding:

$$\begin{aligned} z &= (4t^2 - 4t + 1)(t+2) \\ &= 4t^3 - 4t^2 + t + 8t^2 - 8t + 2 \\ &= 4t^3 + 4t^2 - 7t + 2 \end{aligned}$$
$$\therefore \quad \frac{dz}{dt} = 12t^2 + 8t - 7$$

c First simplify:

$$y = x + 3 \quad (\text{for } x \neq 0)$$
$$\therefore \quad \frac{dy}{dx} = 1 \quad (\text{for } x \neq 0)$$

d $$y = 2x^3 - 1$$
$$\therefore \quad \frac{dy}{dx} = 6x^2$$

Operator notation

'Find the derivative of $2x^2 - 4x$ with respect to x' can also be written as 'find $\frac{d}{dx}(2x^2 - 4x)$'.

In general: $\frac{d}{dx}(f(x)) = f'(x)$.

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Example 10

Find:

a $\frac{d}{dx}(5x - 4x^3)$ **b** $\frac{d}{dz}(5z^2 - 4z)$ **c** $\frac{d}{dz}(6z^3 - 4z^2)$

Solution

a $\frac{d}{dx}(5x - 4x^3)$
$= 5 - 12x^2$

b $\frac{d}{dz}(5z^2 - 4z)$
$= 10z - 4$

c $\frac{d}{dz}(6z^3 - 4z^2)$
$= 18z^2 - 8z$

Example 11

For each of the following curves, find the coordinates of the points on the curve at which the gradient of the tangent line at that point has the given value:

a $y = x^3$, gradient = 8 **b** $y = x^2 - 4x + 2$, gradient = 0

c $y = 4 - x^3$, gradient = −6

Solution

a $y = x^3$ implies $\frac{dy}{dx} = 3x^2$

$$\therefore \quad 3x^2 = 8$$

$$\therefore \quad x = \pm\sqrt{\frac{8}{3}} = \frac{\pm 2\sqrt{6}}{3}$$

The points are $\left(\frac{2\sqrt{6}}{3}, \frac{16\sqrt{6}}{9}\right)$ and $\left(\frac{-2\sqrt{6}}{3}, \frac{-16\sqrt{6}}{9}\right)$.

b $y = x^2 - 4x + 2$ implies $\frac{dy}{dx} = 2x - 4$

$$\therefore \quad 2x - 4 = 0$$

$$\therefore \quad x = 2$$

The only point is (2, −2).

c $y = 4 - x^3$ implies $\frac{dy}{dx} = -3x^2$

$$\therefore \quad -3x^2 = -6$$

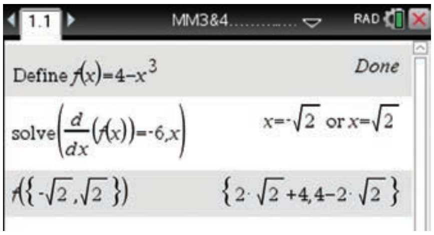
$$\therefore \quad x^2 = 2$$

$$\therefore \quad x = \pm\sqrt{2}$$

The points are $\left(2^{\frac{1}{2}}, 4 - 2^{\frac{3}{2}}\right)$ and $\left(-2^{\frac{1}{2}}, 4 + 2^{\frac{3}{2}}\right)$.

Using the TI-Nspire

- Define $f(x) = 4 - x^3$.
- Solve the equation $\frac{d}{dx}(f(x)) = -6$.
- Substitute in $f(x)$ to find the y-coordinates.



Define $f(x)=4-x^3$ *Done*

$\text{solve}\left(\frac{d}{dx}(f(x))=-6,x\right)$ $x=-\sqrt{2}$ or $x=\sqrt{2}$

$f(\{-\sqrt{2},\sqrt{2}\})$ $\{2\cdot\sqrt{2}+4,4-2\cdot\sqrt{2}\}$

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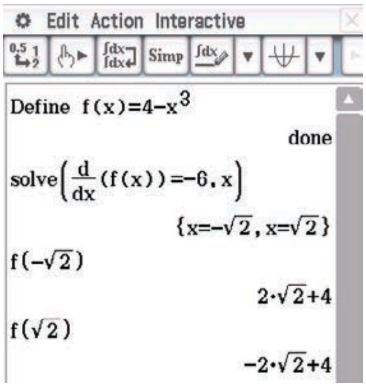
Using the Casio ClassPad

- In $\overset{\text{Main}}{\sqrt{\alpha}}$, enter and highlight the expression $4 - x^3$.
- Go to **Interactive** > **Define** and tap OK.
- In the next entry line, type and highlight $f(x)$.
- Go to **Interactive** > **Calculation** > **diff** and tap OK.
- Type $= -6$ after $\dfrac{d}{dx}(f(x))$. Highlight the equation and use **Interactive** > **Equation/Inequality** > **solve**.
- Enter $f(-\sqrt{2})$ and $f(\sqrt{2})$ to find the required y-values.



Define $f(x)=4-x^3$

done

solve$\left(\frac{d}{dx}(f(x))=-6, x\right)$

$\{x=-\sqrt{2}, x=\sqrt{2}\}$

$f(-\sqrt{2})$

$2\cdot\sqrt{2}+4$

$f(\sqrt{2})$

$-2\cdot\sqrt{2}+4$

An angle associated with the gradient of a curve at a point

The gradient of a curve at a point is the gradient of the tangent at that point. A straight line, the tangent, is associated with each point on the curve.

If α is the angle a straight line makes with the positive direction of the x-axis, then the gradient, m, of the straight line is equal to $\tan\alpha$. That is, $m = \tan\alpha$.

For example, if $\alpha = 135°$, then $\tan\alpha = -1$ and so the gradient is -1.

Example 12

Find the coordinates of the points on the curve with equation $y = x^2 - 7x + 8$ at which the tangent line:

a makes an angle of 45° with the positive direction of the x-axis

b is parallel to the line $y = -2x + 6$.

Solution

a $\dfrac{dy}{dx} = 2x - 7$

$$2x - 7 = 1 \qquad (\text{as } \tan 45° = 1)$$
$$2x = 8$$
$$\therefore \quad x = 4$$
$$y = 4^2 - 7 \times 4 + 8 = -4$$

The coordinates are $(4, -4)$.

b The line $y = -2x + 6$ has gradient -2.

$$2x - 7 = -2$$
$$2x = 5$$
$$\therefore \quad x = \frac{5}{2}$$

The coordinates are $\left(\dfrac{5}{2}, -\dfrac{13}{4}\right)$.

Increasing and decreasing functions

We have discussed strictly increasing and strictly decreasing functions in previous chapters:

- A function f is **strictly increasing** on an interval if $x_2 > x_1$ implies $f(x_2) > f(x_1)$.
- A function f is **strictly decreasing** on an interval if $x_2 > x_1$ implies $f(x_2) < f(x_1)$.

We have the following very important results.

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If $f'(x) > 0$, for all x in the interval, then the function is strictly increasing.
(Think of the tangents at each point – they each have positive gradient.)

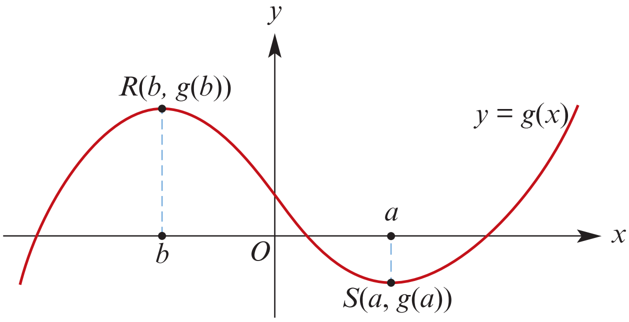
If $f'(x) < 0$, for all x in the interval, then the function is strictly decreasing.
(Think of the tangents at each point – they each have negative gradient.)

Warning: The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ is strictly increasing, but $f'(0) = 0$. This means that *strictly increasing does not imply* $f'(x) > 0$.

▶ Sign of the derivative

Gradients of tangents can, of course, be negative or zero. They are not always positive.

At a point $(a, g(a))$ on the graph of $y = g(x)$, the gradient of the tangent is $g'(a)$.



Some features of the graph shown are:

- For $x < b$, the gradient of any tangent is positive, i.e. $g'(x) > 0$.
- For $x = b$, the gradient of the tangent is zero, i.e. $g'(b) = 0$.
- For $b < x < a$, the gradient of any tangent is negative, i.e. $g'(x) < 0$.
- For $x = a$, the gradient of the tangent is zero, i.e. $g'(a) = 0$.
- For $x > a$, the gradient of any tangent is positive, i.e. $g'(x) > 0$.

Note: This function g is strictly decreasing on the open interval (b, a), but it is also strictly decreasing on the closed interval $[b, a]$. Similarly, the function g is strictly increasing on the intervals $[a, \infty)$ and $(-\infty, b]$.

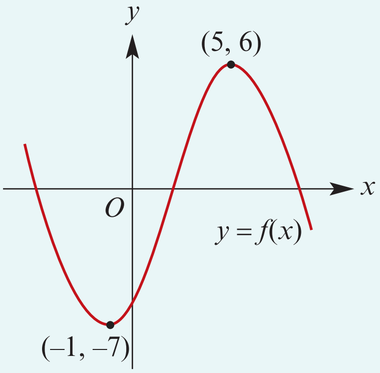
Example 13

For the graph of $f : \mathbb{R} \to \mathbb{R}$, find:

a $\{ x : f'(x) > 0 \}$

b $\{ x : f'(x) < 0 \}$

c $\{ x : f'(x) = 0 \}$



Solution

a $\{ x : f'(x) > 0 \} = \{ x : -1 < x < 5 \} = (-1, 5)$

b $\{ x : f'(x) < 0 \} = \{ x : x < -1 \} \cup \{ x : x > 5 \} = (-\infty, -1) \cup (5, \infty)$

c $\{ x : f'(x) = 0 \} = \{-1, 5\}$

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Section summary

- For $f(x) = x^n$, $f'(x) = nx^{n-1}$, where $n = 1, 2, 3, \ldots$
- Constant function: If $f(x) = c$, then $f'(x) = 0$.
- Multiple: If $f(x) = k\,g(x)$, where k is a constant, then $f'(x) = k\,g'(x)$.
  That is, the derivative of a number multiple is the multiple of the derivative.
- Sum: If $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$.
  That is, the derivative of the sum is the sum of the derivatives.
- Difference: If $f(x) = g(x) - h(x)$, then $f'(x) = g'(x) - h'(x)$.
  That is, the derivative of the difference is the difference of the derivatives.
- Angle of inclination of tangent
  - A straight line, the tangent, is associated with each point on a smooth curve.
  - If α is the angle that a straight line makes with the positive direction of the x-axis, then the gradient of the line is given by $m = \tan\alpha$.
- Increasing and decreasing functions
  - A function f is **strictly increasing** on an interval if $x_2 > x_1$ implies $f(x_2) > f(x_1)$.
  - A function f is **strictly decreasing** on an interval if $x_2 > x_1$ implies $f(x_2) < f(x_1)$.
  - If $f'(x) > 0$ for all x in the interval, then the function is strictly increasing.
  - If $f'(x) < 0$ for all x in the interval, then the function is strictly decreasing.

Exercise 9B

Skillsheet
Example 5

1 For each of the following, find the derivative with respect to x:

a x^5 **b** $4x^7$ **c** $6x$

d $5x^2 - 4x + 3$ **e** $4x^3 + 6x^2 + 2x - 4$ **f** $5x^4 + 3x^3$

g $-2x^2 + 4x + 6$ **h** $6x^3 - 2x^2 + 4x - 6$

Example 6

2 For each of the following, find the derivative of $f(x)$ and thus find $f'(1)$:

a $f(x) = 2x^3 - 5x^2 + 1$ **b** $f(x) = -2x^3 - x^2 - 1$

c $f(x) = x^4 - 2x^3 + 1$ **d** $f(x) = x^5 - 3x^3 + 2$

Example 7

3 **a** For the curve determined by the rule $f(x) = 2x^3 - 5x^2 + 2$, find the gradient of the tangent line to the curve at the point $(1, -1)$.

b For the curve determined by the rule $f(x) = -2x^3 - 3x^2 + 2$, find the gradient of the tangent line to the curve at the point $(2, -26)$.

Example 8

4 **a** If $y = t^3$, find $\dfrac{dy}{dt}$.

b If $x = t^3 - t^2$, find $\dfrac{dx}{dt}$.

c If $z = \dfrac{1}{4}x^4 + 3x^3$, find $\dfrac{dz}{dx}$.

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Example 9 **5** For each of the following, find $\frac{dy}{dx}$:

a $y = -2x$

b $y = 7$

c $y = 5x^3 - 3x^2 + 2x + 1$

d $y = \frac{2}{5}(x^3 - 4x + 6)$

e $y = (2x + 1)(x - 3)$

f $y = 3x(2x - 4)$

g $y = \frac{10x^7 + 2x^2}{x^2}, \ x \neq 0$

h $y = \frac{9x^4 + 3x^2}{x}, \ x \neq 0$

Example 10 **6** Find:

a $\frac{d}{dx}(2x^2 - 5x^3)$

b $\frac{d}{dz}(-2z^2 - 6z)$

c $\frac{d}{dz}(6z^3 - 4z^2 + 3)$

d $\frac{d}{dx}(-2x - 5x^3)$

e $\frac{d}{dz}(-2z^2 - 6z + 7)$

f $\frac{d}{dz}(-z^3 - 4z^2 + 3)$

Example 11 **7** Find the coordinates of the points on the curves given by the following equations at which the gradient has the given value:

a $y = 2x^2 - 4x + 1$, gradient = -6

b $y = 4x^3$, gradient = 48

c $y = x(5 - x)$, gradient = 1

d $y = x^3 - 3x^2$, gradient = 0

Example 12 **8** Find the coordinates of the points on the curve with equation $y = 2x^2 - 3x + 8$ at which the tangent line:

a makes an angle of 45° with the positive direction of the x-axis

b is parallel to the line $y = 2x + 8$.

9 Find the value of x such that the tangent line to the curve $f(x) = x^2 - x$ at $(x, f(x))$:

a makes an angle of 45° with the positive direction of the x-axis

b makes an angle of 135° with the positive direction of the x-axis

c makes an angle of 60° with the positive direction of the x-axis

d makes an angle of 30° with the positive direction of the x-axis

e makes an angle of 120° with the positive direction of the x-axis.

10 For each of the following, find the angle that the tangent line to the curve $y = f(x)$ makes with the positive direction of the x-axis at the given point:

a $y = x^2 + 3x, \ (1, 4)$

b $y = -x^2 + 2x, \ (1, 1)$

c $y = x^3 + x, \ (0, 0)$

d $y = -x^3 - x, \ (0, 0)$

e $y = x^4 - x^2, \ (1, 0)$

f $y = x^4 - x^2, \ (-1, 0)$

11 **a** Differentiate $y = (2x - 1)^2$ with respect to x.

b For $y = \frac{x^3 + 2x^2}{x}$, $x \neq 0$, find $\frac{dy}{dx}$.

c Given that $y = 2x^3 - 6x^2 + 18x$, find $\frac{dy}{dx}$. Hence show that $\frac{dy}{dx} > 0$ for all x.

d Given that $y = \frac{x^3}{3} - x^2 + x$, find $\frac{dy}{dx}$. Hence show that $\frac{dy}{dx} \geq 0$ for all x.

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12 At the points on the following curves corresponding to the given values of x, find the y-coordinate and the gradient:

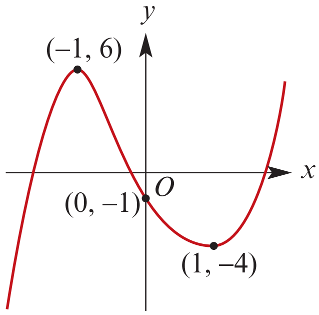
a $y = x^2 + 2x + 1,\ x = 3$
b $y = x^2 - x - 1,\ x = 0$
c $y = 2x^2 - 4x,\ x = -1$
d $y = (2x + 1)(3x - 1)(x + 2),\ x = 4$
e $y = (2x + 5)(3 - 5x)(x + 1),\ x = 1$
f $y = (2x - 5)^2,\ x = 2\frac{1}{2}$

13 For the function $f(x) = 3(x - 1)^2$, find the value(s) of x for which:

a $f(x) = 0$
b $f'(x) = 0$
c $f'(x) > 0$
d $f'(x) < 0$
e $f'(x) = 10$
f $f(x) = 27$

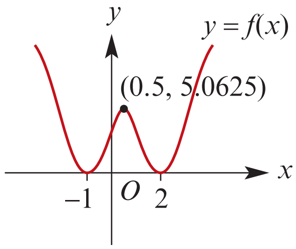
Example 13 **14** For the graph of $y = h(x)$ illustrated, find:

a $\{x : h'(x) > 0\}$
b $\{x : h'(x) < 0\}$
c $\{x : h'(x) = 0\}$



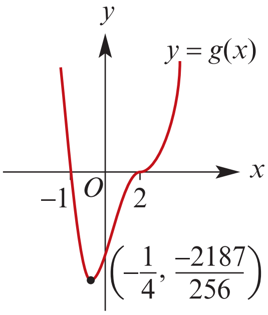
15 For the graph of $y = f(x)$ shown, find:

a $\{x : f'(x) > 0\}$
b $\{x : f'(x) < 0\}$
c $\{x : f'(x) = 0\}$



16 For the graph of $y = g(x)$ shown, find:

a $\{x : g'(x) > 0\}$
b $\{x : g'(x) < 0\}$
c $\{x : g'(x) = 0\}$



17 Find the coordinates of the points on the parabola $y = x^2 - 4x - 8$ at which:

a the gradient is zero
b the tangent is parallel to $y = 2x + 6$
c the tangent is parallel to $3x + 2y = 8$.

18 **a** Show that $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^3$ is a strictly increasing function for $\mathbb{R}$ by showing that $f'(x) > 0$, for all non-zero x, and showing that, if $b > 0$, then $f(b) > f(0)$ and, if $0 > b$, then $f(0) > f(b)$.

b Show that $f : \mathbb{R} \to \mathbb{R},\ f(x) = -x^3$ is a strictly decreasing function for $\mathbb{R}$.

19 **a** Show that $f\colon [0, \infty) \to \mathbb{R}$, $f(x) = x^2$ is a strictly increasing function.
b Show that $f\colon (-\infty, 0] \to \mathbb{R}$, $f(x) = x^2$ is a strictly decreasing function.

20 For the function $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - x - 12$, show that the largest interval for which f is strictly increasing is $[\frac{1}{2}, \infty)$.

21 For each of the following, find the largest interval for which the function is strictly decreasing:

a $y = x^2 + 2x$ **b** $y = -x^2 + 4x$ **c** $y = 2x^2 + 3$ **d** $y = -2x^2 + 6x$

9C Differentiating x^n where n is a negative integer

In the previous sections we have seen how to differentiate polynomial functions. In this section we add to the family of functions that we can differentiate. In particular, we will consider functions which involve linear combinations of powers of x, where the indices may be negative integers.

e.g. $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R},\ f(x) = x^{-1}$
$f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R},\ f(x) = 2x + x^{-1}$
$f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R},\ f(x) = x + 3 + x^{-2}$

Note: We have reintroduced function notation to emphasise the need to consider domains.

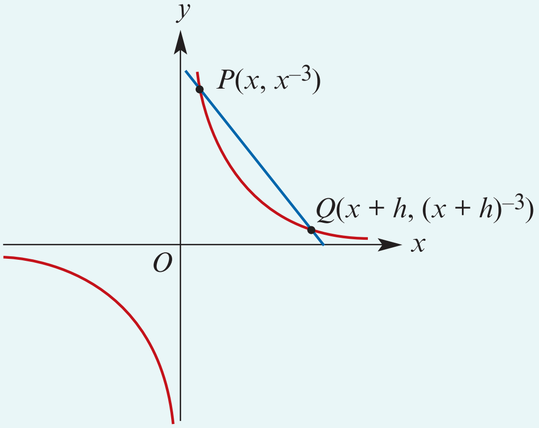
Example 14

Let $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = x^{-3}$. Find $f'(x)$ by first principles.

Solution

The gradient of secant PQ is given by

$$\begin{aligned}
&\frac{(x+h)^{-3} - x^{-3}}{h}\\
&= \frac{x^3 - (x+h)^3}{(x+h)^3 x^3} \times \frac{1}{h}\\
&= \frac{x^3 - (x^3 + 3x^2h + 3xh^2 + h^3)}{(x+h)^3 x^3} \times \frac{1}{h}\\
&= \frac{-3x^2h - 3xh^2 - h^3}{(x+h)^3 x^3} \times \frac{1}{h}\\
&= \frac{-3x^2 - 3xh - h^2}{(x+h)^3 x^3}
\end{aligned}$$



So the gradient of the curve at P is given by

$$\lim_{h \to 0} \frac{-3x^2 - 3xh - h^2}{(x+h)^3 x^3} = \frac{-3x^2}{x^6} = -3x^{-4}$$

Hence $f'(x) = -3x^{-4}$.

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We are now in a position to state the generalisation of the result we found in Section 9B. This result can be proved by again using the binomial theorem.

For $f(x) = x^n$, $f'(x) = nx^{n-1}$, where n is a non-zero integer.

For $f(x) = c$, $f'(x) = 0$, where c is a constant.

When n is positive, we take the domain of f to be $\mathbb{R}$, and when n is negative, we take the domain of f to be $\mathbb{R} \setminus \{0\}$.

Example 15

Find the derivative of $x^4 - 2x^{-3} + x^{-1} + 2$, $x \neq 0$.

Solution

$$\begin{aligned} \text{If} \quad f(x) &= x^4 - 2x^{-3} + x^{-1} + 2 && (\text{for } x \neq 0) \\ \text{then} \quad f'(x) &= 4x^3 - 2(-3x^{-4}) + (-x^{-2}) + 0 \\ &= 4x^3 + 6x^{-4} - x^{-2} && (\text{for } x \neq 0) \end{aligned}$$

Example 16

Find the derivative f' of $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = 3x^2 - 6x^{-2} + 1$.

Solution

$$f' : \mathbb{R} \setminus \{0\} \to \mathbb{R}, \quad \begin{aligned} f'(x) &= 3(2x) - 6(-2x^{-3}) + 0 \\ &= 6x + 12x^{-3} \end{aligned}$$

Example 17

Find the gradient of the tangent to the curve determined by the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = x^2 + \dfrac{1}{x}$ at the point $(1, 2)$.

Solution

$$f' : \mathbb{R} \setminus \{0\} \to \mathbb{R}, \quad \begin{aligned} f'(x) &= 2x + (-x^{-2}) \\ &= 2x - x^{-2} \end{aligned}$$

Therefore $f'(1) = 2 - 1 = 1$. The gradient of the curve is 1 at the point $(1, 2)$.

Example 18

Show that the derivative of the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = x^{-3}$ is always negative.

Solution

$$f' : \mathbb{R} \setminus \{0\} \to \mathbb{R}, \quad f'(x) = -3x^{-4} = \frac{-3}{x^4}$$

Since x^4 is positive for all $x \neq 0$, we have $f'(x) < 0$ for all $x \neq 0$.

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Section summary

For $f(x) = x^n$, $f'(x) = nx^{n-1}$, where n is a non-zero integer.

For $f(x) = c$, $f'(x) = 0$, where c is a constant.

Exercise 9C

1 **a** Sketch the graph of $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}, f(x) = \dfrac{2}{x^2}$.

b Let P be the point $(1, 2)$ and Q the point $(1 + h, f(1 + h))$. Find the gradient of the secant PQ.

c Hence find the gradient of the tangent to the curve $f(x) = \dfrac{2}{x^2}$ at $(1, 2)$.

Example 14

2 **a** Let $f\colon \mathbb{R} \setminus \{3\} \to \mathbb{R}, f(x) = \dfrac{1}{x-3}$. Find $f'(x)$ by first principles.

b Let $f\colon \mathbb{R} \setminus \{-2\} \to \mathbb{R}, f(x) = \dfrac{1}{x+2}$. Find $f'(x)$ by first principles.

3 Let $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}, f(x) = x^{-4}$. Find $f'(x)$ by first principles.
Hint: Remember that $(x + h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$.

Example 15, 16

4 Differentiate each of the following with respect to x:

a $3x^{-2} + 5x^{-1} + 6$ **b** $\dfrac{5}{x^3} + 6x^2$ **c** $\dfrac{-5}{x^3} + \dfrac{4}{x^2} + 1$

d $6x^{-3} + 3x^{-2}$ **e** $\dfrac{4x^2 + 2x}{x^2}$

5 Find the derivative of each of the following:

a $\dfrac{2z^2 - 4z}{z^2}$, $z \neq 0$ **b** $\dfrac{6 + z}{z^3}$, $z \neq 0$ **c** $16 - z^{-3}$, $z \neq 0$

d $\dfrac{4z + z^3 - z^4}{z^2}$, $z \neq 0$ **e** $\dfrac{6z^2 - 2z}{z^4}$, $z \neq 0$ **f** $\dfrac{6}{x} - 3x^2$, $x \neq 0$

Example 17

6 Find the gradient of the tangent to each of the following curves at the stated point:

a $y = x^{-2} + x^3$, $x \neq 0$, at $(2, 8\frac{1}{4})$ **b** $y = x^{-2} - \dfrac{1}{x}$, $x \neq 0$, at $(4, \frac{1}{2})$

c $y = x^{-2} - \dfrac{1}{x}$, $x \neq 0$, at $(1, 0)$ **d** $y = x(x^{-1} + x^2 - x^{-3})$, $x \neq 0$, at $(1, 1)$

Example 18

7 Show that the derivative of the function $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}, f(x) = -2x^{-5}$ is always positive.

8 Find the x-coordinates of the points on the curve $y = \dfrac{x^2 - 1}{x}$ at which the gradient of the curve is 5.

9 Given that the curve $y = ax^2 + \dfrac{b}{x}$ has a gradient of -5 at the point $(2, -2)$, find the values of a and b.

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10 Find the gradient of the curve $y = \frac{2x - 4}{x^2}$ at the point where the curve crosses the x-axis.

11 The gradient of the curve $y = \frac{a}{x} + bx^2$ at the point $(3, 6)$ is 7. Find the values of a and b.

12 For the curve with equation $y = \frac{5}{3}x + kx^2 - \frac{8}{9}x^3$, calculate the possible values of k such that the tangents at the points with x-coordinates 1 and $-\frac{1}{2}$ are perpendicular.

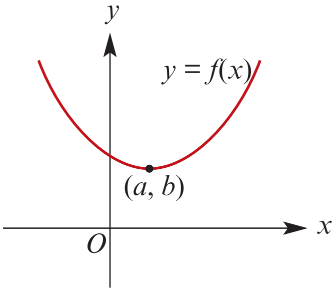
9D The graph of the derivative function

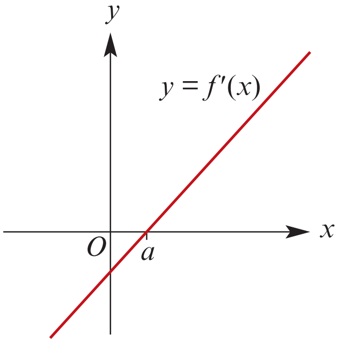
First consider the quadratic function with rule $y = f(x)$ shown in the graph on the left. The vertex is at the point with coordinates (a, b).

- For $x < a$, $f'(x) < 0$.
- For $x = a$, $f'(x) = 0$.
- For $x > a$, $f'(x) > 0$.

The graph of the derivative function with rule $y = f'(x)$ is therefore as shown on the right.

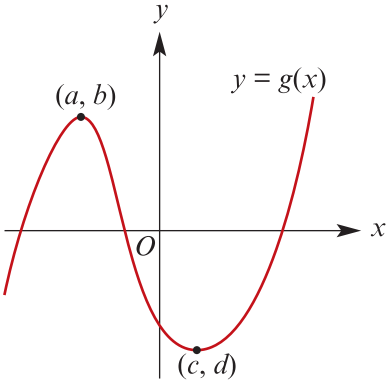
The derivative f' is known to be linear as f is quadratic.



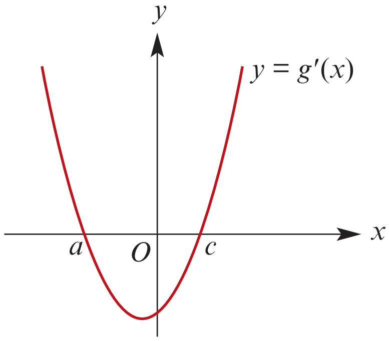


Now consider the cubic function with rule $y = g(x)$ shown in the graph.

- For $x < a$, $g'(x) > 0$.
- For $x = a$, $g'(x) = 0$.
- For $a < x < c$, $g'(x) < 0$.
- For $x = c$, $g'(x) = 0$.
- For $x > c$, $g'(x) > 0$.

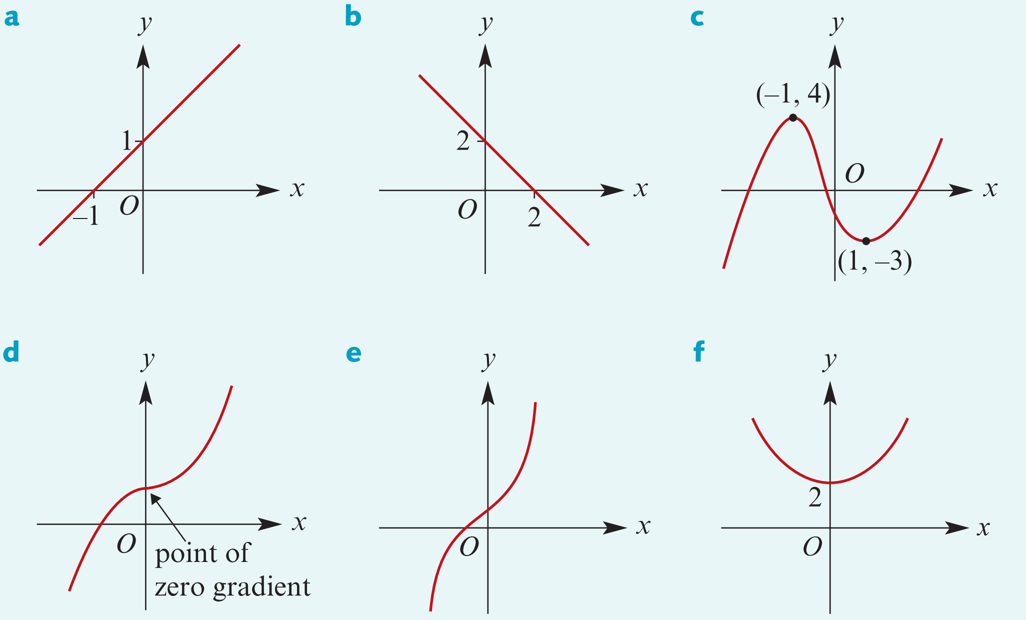


The graph of the derivative function with rule $y = g'(x)$ is therefore as shown to the right. The derivative g' is known to be quadratic as g is cubic.



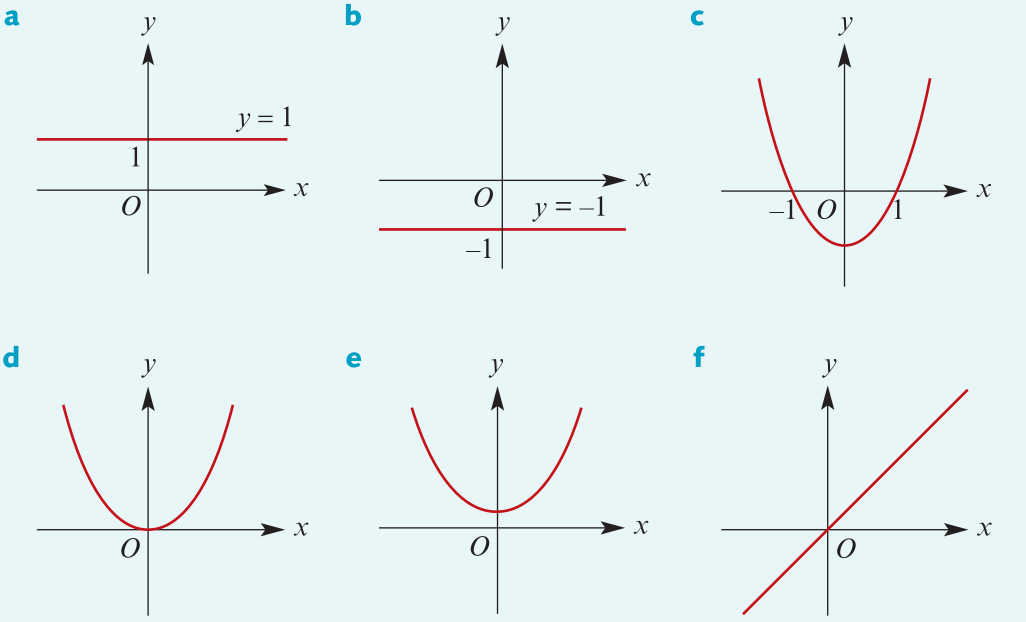
Example 19

Sketch the graph of the derivative function for each of the functions of the graphs shown:


a
y
1
–1
O
x
b
y
2
O
2
x
c
y
(–1, 4)
O
x
(1, –3)
d
y
x
O
point of
zero gradient
e
y
x
O
f
y
2
O
x

Solution

Note: Not all features of the graphs are known.


a
y
y = 1
1
O
x
b
y
x
O
y = –1
–1
c
y
x
–1
O
1
d
y
x
O
e
y
x
O
f
y
x
O

For some functions f, there are values of x for which the derivative $f'(x)$ is not defined. We will consider differentiability informally here and more formally in Section 9M.

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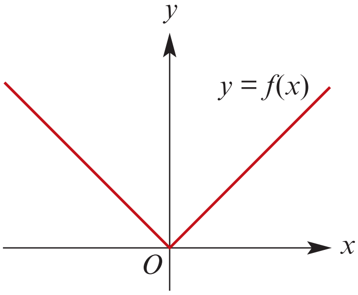
Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$



Now consider the gradient of the secant through the points $(0, 0)$ and $(h, f(h))$ on the graph of $y = f(x)$:

$$\text{gradient} = \frac{f(0+h) - f(0)}{h} = \begin{cases} \dfrac{h}{h} & \text{for } h > 0 \\ \dfrac{-h}{h} & \text{for } h < 0 \end{cases}$$

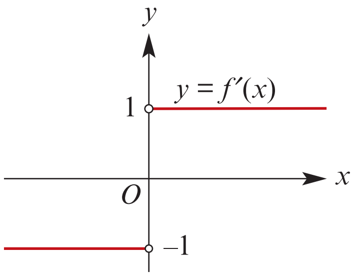
$$= \begin{cases} 1 & \text{for } h > 0 \\ -1 & \text{for } h < 0 \end{cases}$$

The gradient does not approach a unique value as $h \to 0$, and so we say $\lim\limits_{h\to 0} \dfrac{f(0+h) - f(0)}{h}$ does not exist. The function f is not differentiable at $x = 0$.

The gradient of the curve $y = f(x)$ is -1 to the left of 0, and 1 to the right of 0. Therefore the derivative function $f' : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is given by

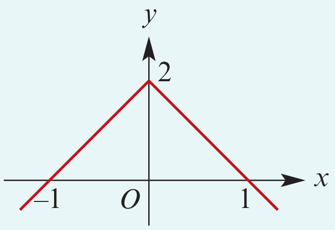
$$f'(x) = \begin{cases} 1 & \text{for } x > 0 \\ -1 & \text{for } x < 0 \end{cases}$$

The graph of f' is shown on the right.



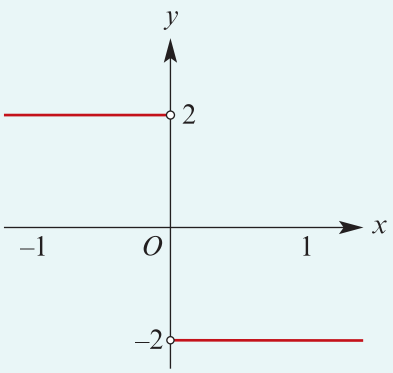
Example 20

Draw a sketch graph of f' where the graph of f is as illustrated. Indicate where f' is not defined.



Solution

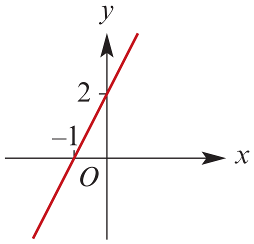
The derivative does not exist at $x = 0$,
i.e. the function is not differentiable at $x = 0$.



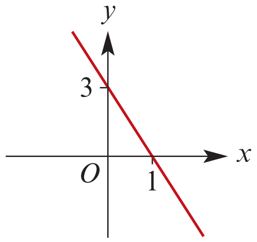
Exercise 9D

Example 19 **1** Sketch the graph of the derivative function for each of the following functions:

a



b



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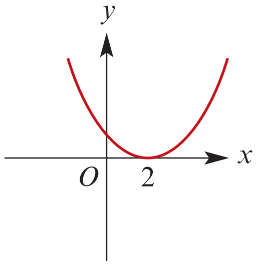

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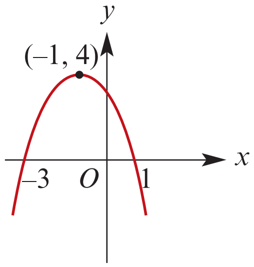
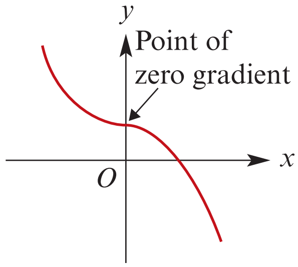
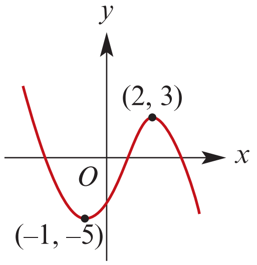


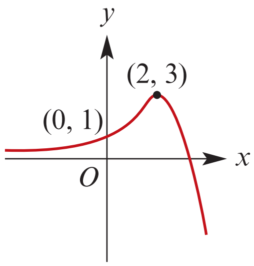
e


y
Point of zero gradient
O
x

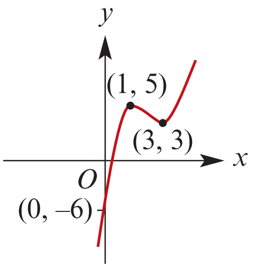
f



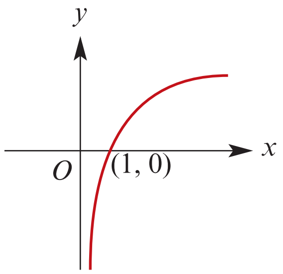
g



h

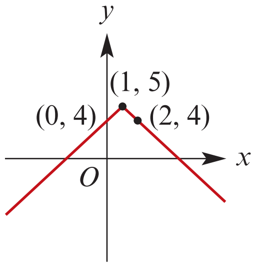


i

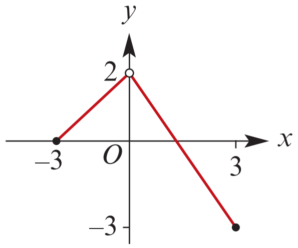


Example 20 **2** Sketch the graph of the derivative function for each of the following functions:

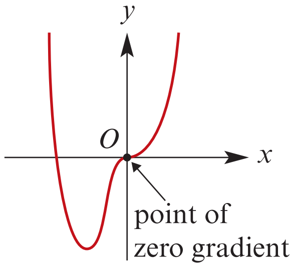
a



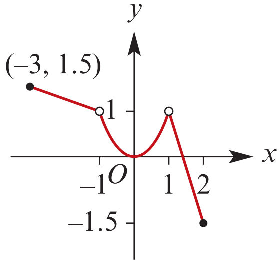
b



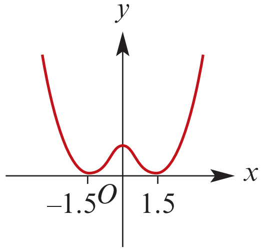
c


y
O
x
point of zero gradient

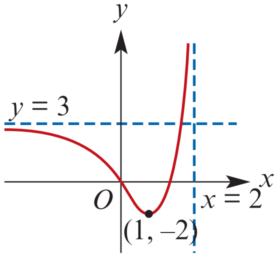
d


y
(–3, 1.5)
1
–1
O
1
2
–1.5
x

e



f



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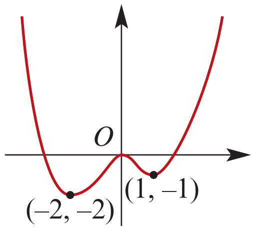
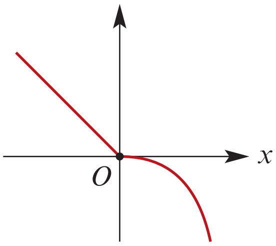
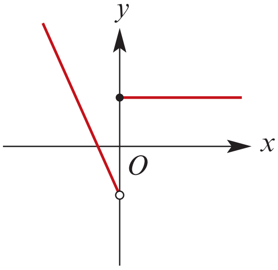
g



h

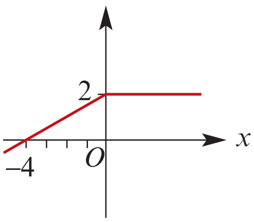


i

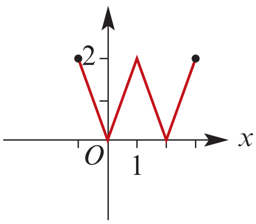


3 Match the graphs of the functions **a**–**f** with the graphs of their derivatives **A**–**F**:

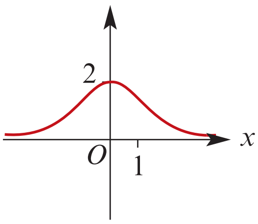
a



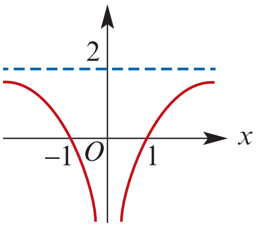
b



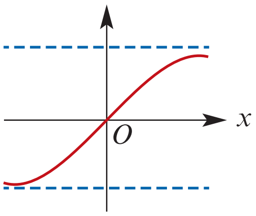
c



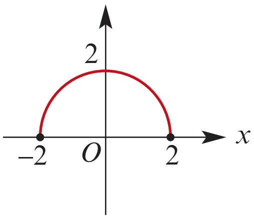
d



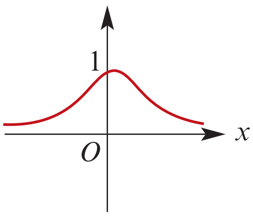
e



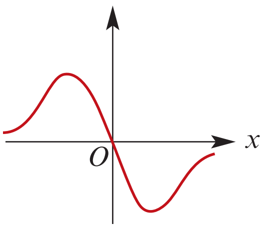
f



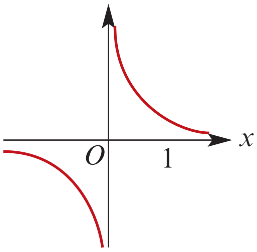
A



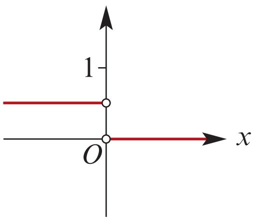
B



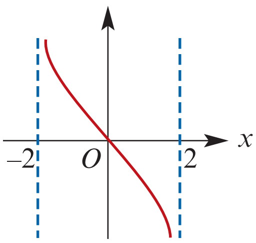
C



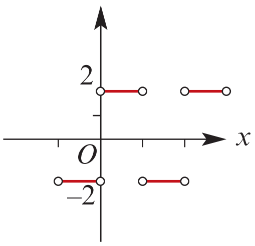
D



E



F



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4 **a** Use a calculator to plot the graph of $y = f(x)$ where $f(x) = (x^2 - 2x)^2$.

b Using the same screen, plot the graph of $y = f'(x)$. (Do not attempt to determine the rule for $f'(x)$ first.)

c Use a calculator to determine $f'(x)$ for:

i $x = 0$ **ii** $x = 2$ **iii** $x = 1$ **iv** $x = 4$

d For $0 \le x \le 1$, find the value of x for which:

i $f(x)$ is a maximum **ii** $f'(x)$ is a maximum.

5 For $f(x) = \frac{x^3}{3} - x^2 + x + 1$, plot the graphs of $y = f(x)$ and $y = f'(x)$ on the same screen. Comment.

6 For $g(x) = x^3 + 2x + 1$, plot the graphs of $y = g(x)$ and $y = g'(x)$ on the same screen. Comment.

7 **a** For $h(x) = x^4 + 2x + 1$, plot the graphs of $y = h(x)$ and $y = h'(x)$ on the same screen.

b Find the value(s) of x such that:

i $h(x) = 3$ **ii** $h'(x) = 3$

9E The chain rule

An expression such as $q(x) = (x^3 + 1)^2$ may be differentiated by expanding and then differentiating each term separately. This method is a great deal more tiresome for an expression such as $q(x) = (x^3 + 1)^{30}$.

We can express $q(x) = (x^3 + 1)^2$ as the composition of two simpler functions defined by

$$u = g(x) = x^3 + 1 \quad \text{and} \quad y = f(u) = u^2$$

which are ‘chained’ together:

$$x \xrightarrow{g} u \xrightarrow{f} y$$

That is, $q(x) = (x^3 + 1)^2 = f(g(x))$, and so q is expressed as the composition $f \circ g$.

The chain rule gives a method of differentiating such functions.

The chain rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $q(x) = f(g(x))$ is differentiable at x and

$$q'(x) = f'(g(x))\, g'(x)$$

Or using Leibniz notation, where $u = g(x)$ and $y = f(u)$,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

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Proof To find the derivative of $q = f \circ g$ where $x = a$, consider the secant through the points $(a, f \circ g(a))$ and $(a + h, f \circ g(a + h))$. The gradient of this secant is

$$\frac{f \circ g(a + h) - f \circ g(a)}{h}$$

We carry out the trick of multiplying the numerator and the denominator by $g(a + h) - g(a)$. This gives

$$\frac{f(g(a + h)) - f(g(a))}{h} \times \frac{g(a + h) - g(a)}{g(a + h) - g(a)}$$

provided $g(a + h) - g(a) \neq 0$.

Now write $b = g(a)$ and $b + k = g(a + h)$ so that $k = g(a + h) - g(a)$. The expression for the gradient becomes

$$\frac{f(b + k) - f(b)}{k} \times \frac{g(a + h) - g(a)}{h}$$

The function g is continuous, since its derivative exists, and therefore

$$\lim_{h \to 0} k = \lim_{h \to 0} [g(a + h) - g(a)] = 0$$

Thus, as h approaches 0, so does k. Hence $q'(a) = f'(g(a))\, g'(a)$.

Note that this proof does not hold for a function g such that $g(a + h) - g(a) = 0$ for arbitrarily chosen small h. However, a fully rigorous proof is beyond the scope of this course.

Example 21

Differentiate $y = (4x^3 - 5x)^{-2}$.

Solution

The differentiation is undertaken using both notations:

Let $u = 4x^3 - 5x$

Then $y = u^{-2}$

We have

$$\frac{dy}{du} = -2u^{-3}$$

$$\frac{du}{dx} = 12x^2 - 5$$

Therefore

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= -2u^{-3} \cdot (12x^2 - 5)$$

$$= \frac{-2(12x^2 - 5)}{(4x^3 - 5x)^3}$$

Let $h(x) = 4x^3 - 5x$

and $g(x) = x^{-2}$

Then $f(x) = g(h(x))$

We have

$$h'(x) = 12x^2 - 5$$

$$g'(x) = -2x^{-3}$$

Therefore

$$\begin{aligned} f'(x) &= g'(h(x))\, h'(x) \\ &= -2(h(x))^{-3}\, h'(x) \\ &= -2(4x^3 - 5x)^{-3} \times (12x^2 - 5) \\ &= \frac{-2(12x^2 - 5)}{(4x^3 - 5x)^3} \end{aligned}$$

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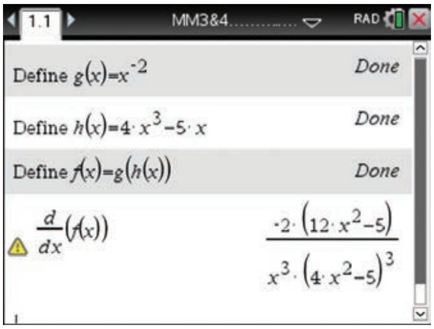
Using the TI-Nspire

- Define $g(x)$ and $h(x)$.
- Then define $f(x) = g(h(x))$.
- Use (menu) > **Calculus** > **Derivative** and complete as shown.



Define $g(x)=x^{-2}$ Done

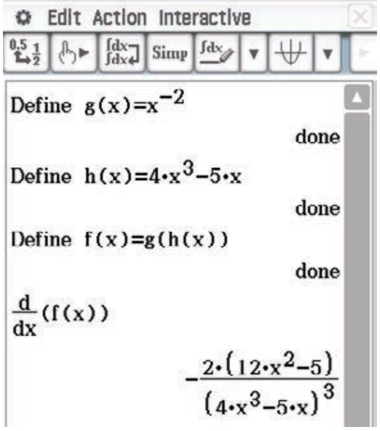
Define $h(x)=4\cdot x^3-5\cdot x$ Done

Define $f(x)=g(h(x))$ Done

$\frac{d}{dx}(f(x))$ $\quad \frac{-2\cdot(12\cdot x^2-5)}{x^3\cdot(4\cdot x^2-5)^3}$

Using the Casio ClassPad

- Define $g(x)$ and $h(x)$.
- Then define $f(x) = g(h(x))$.
- Find the derivative of $f(x)$.



Define $g(x)=x^{-2}$

done

Define $h(x)=4\cdot x^3-5\cdot x$

done

Define $f(x)=g(h(x))$

done

$\frac{d}{dx}(f(x))$

$-\frac{2\cdot(12\cdot x^2-5)}{(4\cdot x^3-5\cdot x)^3}$

Example 22

Find the gradient of the tangent to the curve with equation $y = \dfrac{16}{3x^2+1}$ at the point $(1, 4)$.

Solution

Let $\quad u = 3x^2 + 1 \quad$ then $\quad y = 16u^{-1}$

So $\quad \dfrac{du}{dx} = 6x \quad$ and $\quad \dfrac{dy}{du} = -16u^{-2}$

$$\begin{aligned}\therefore \quad \frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx}\\ &= -16u^{-2}\cdot 6x\\ &= \frac{-96x}{(3x^2+1)^2}\end{aligned}$$

$\therefore$ At $x = 1$, the gradient is $\dfrac{-96}{16} = -6$.

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Section summary

The chain rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $q(x) = f(g(x))$ is differentiable at x and

$$q'(x) = f'(g(x))\, g'(x)$$

Or using Leibniz notation, where $u = g(x)$ and $y = f(u)$,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Exercise 9E

Example 21

1 Differentiate each of the following with respect to x:

- **a** $(x^2 + 1)^4$
- **b** $(2x^2 - 3)^5$
- **c** $(6x + 1)^4$
- **d** $(ax + b)^n$
- **e** $(ax^2 + b)^n$
- **f** $(1 - x^2)^{-3}$
- **g** $\left(x^2 - \dfrac{1}{x^2}\right)^{-3}$
- **h** $(1 - x)^{-1}$

2 Differentiate each of the following with respect to x:

- **a** $(x^2 + 2x + 1)^3$
- **b** $(x^3 + 2x^2 + x)^4$
- **c** $\left(6x^3 + \dfrac{2}{x}\right)^4$
- **d** $(x^2 + 2x + 1)^{-2}$

Example 22

3 Find the gradient of the tangent to the curve with equation $y = \dfrac{16}{3x^3 + x}$ at the point $(1, 4)$.

4 Find the gradient of the tangent to the curve with equation $y = \dfrac{1}{x^2 + 1}$ at the points $(1, \frac{1}{2})$ and $(-1, \frac{1}{2})$.

5 Given that $f'(x) = \sqrt{3x + 4}$ and $g(x) = x^2 - 1$, find $F'(x)$ where $F(x) = f(g(x))$.

6 Differentiate each of the following with respect to x, giving the answer in terms of $f(x)$ and $f'(x)$:

- **a** $[f(x)]^n$, where n is a positive integer
- **b** $\dfrac{1}{f(x)}$, where $f(x) \neq 0$

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9F Differentiating rational powers

Before using the chain rule to differentiate rational powers, we will show how to differentiate $x^{\frac{1}{2}}$ and $x^{\frac{1}{3}}$ by first principles.

Example 23

Differentiate each of the following by first principles:

a $f(x) = x^{\frac{1}{2}},\ x > 0$

b $g(x) = x^{\frac{1}{3}},\ x \neq 0$

Solution

a

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{\sqrt{x+h}-\sqrt{x}}{h}\\ &= \frac{\sqrt{x+h}-\sqrt{x}}{h} \times \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\\ &= \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}\\ &= \frac{1}{\sqrt{x+h}+\sqrt{x}}\end{aligned}$$

$$\therefore\quad f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

b We use the identity

$$a^3 - b^3 = (a-b)(a^2+ab+b^2)$$

By observing that $\left(a^{\frac{1}{3}}\right)^3 = a$ and $\left(b^{\frac{1}{3}}\right)^3 = b$, we obtain

$$a - b = \left(a^{\frac{1}{3}} - b^{\frac{1}{3}}\right)\left(a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}\right)$$

and therefore

$$a^{\frac{1}{3}} - b^{\frac{1}{3}} = \frac{a-b}{a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}}$$

We now have

$$\begin{aligned}\frac{g(x+h)-g(x)}{h} &= \frac{(x+h)^{\frac{1}{3}} - x^{\frac{1}{3}}}{h}\\ &= \frac{x+h-x}{h\left((x+h)^{\frac{2}{3}} + (x+h)^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{2}{3}}\right)}\\ &= \frac{1}{(x+h)^{\frac{2}{3}} + (x+h)^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{2}{3}}}\end{aligned}$$

Hence

$$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{1}{(x+h)^{\frac{2}{3}} + (x+h)^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{2}{3}}} = \frac{1}{3x^{\frac{2}{3}}}$$

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Note: We can prove that $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1})$ for $n \geq 2$. We could use this result to find the derivative of $x^{\frac{1}{n}}$ by first principles, but instead we will use the chain rule.

▶ Using the chain rule

If y is a one-to-one function of x, then using the chain rule in the form $\dfrac{dy}{du} = \dfrac{dy}{dx} \cdot \dfrac{dx}{du}$ with $y = u$, we have

$$1 = \frac{dy}{dx} \cdot \frac{dx}{dy}$$

Thus $$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \qquad \text{for } \frac{dx}{dy} \neq 0$$

Now let $y = x^{\frac{1}{n}}$, where $n \in \mathbb{Z} \setminus \{0\}$ and $x > 0$.

We have $y^n = x$ and so $\dfrac{dx}{dy} = ny^{n-1}$. Therefore

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{ny^{n-1}} = \frac{1}{n\left(x^{\frac{1}{n}}\right)^{n-1}} = \frac{1}{n}x^{\frac{1}{n}-1}$$

For $y = x^{\frac{1}{n}}$, $\dfrac{dy}{dx} = \dfrac{1}{n}x^{\frac{1}{n}-1}$, where $n \in \mathbb{Z} \setminus \{0\}$ and $x > 0$.

This result may now be extended to rational powers.

Let $y = x^{\frac{p}{q}}$, where $p, q \in \mathbb{Z} \setminus \{0\}$.

Write $y = \left(x^{\frac{1}{q}}\right)^p$. Let $u = x^{\frac{1}{q}}$. Then $y = u^p$. The chain rule yields

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= pu^{p-1} \cdot \frac{1}{q}x^{\frac{1}{q}-1} \\ &= p\left(x^{\frac{1}{q}}\right)^{p-1} \cdot \frac{1}{q}x^{\frac{1}{q}-1} \\ &= \frac{p}{q}x^{\frac{p}{q}-\frac{1}{q}}x^{\frac{1}{q}-1} \\ &= \frac{p}{q}x^{\frac{p}{q}-1} \end{aligned}$$

Thus the result for integer powers has been extended to rational powers. In fact, the analogous result holds for any non-zero real power:

For $f(x) = x^a$, $f'(x) = ax^{a-1}$, where $a \in \mathbb{R} \setminus \{0\}$ and $x > 0$.

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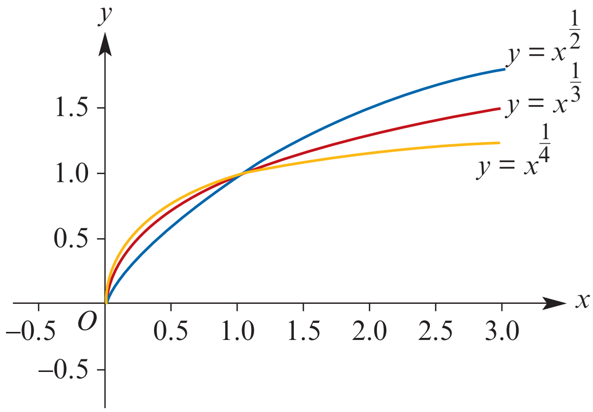
This result is stated for $x > 0$, as $(-3)^{\frac{1}{2}}$ is not defined, although $(-2)^{\frac{1}{3}}$ is defined.

The graphs of $y = x^{\frac{1}{2}}$, $y = x^{\frac{1}{3}}$ and $y = x^{\frac{1}{4}}$ are shown.

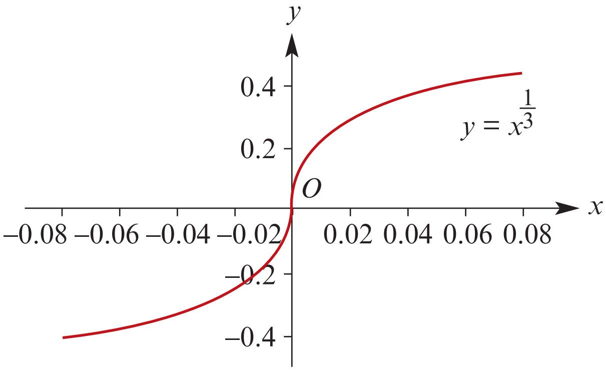
The domain of each has been taken to be $\mathbb{R}^+$.



The figure to the right is the graph of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^{\frac{1}{3}}$.

Note that the values shown here are $-0.08 \leq x \leq 0.08$.

From this it can be seen that the tangent to $y = x^{\frac{1}{3}}$ at the origin is on the y-axis.



Example 24

Find the derivative of each of the following with respect to x:

a $2x^{-\frac{1}{5}} + 3x^{\frac{2}{7}}$

b $\sqrt[3]{x^2 + 2x}$

Solution

a

$$\frac{d}{dx}\left(2x^{-\frac{1}{5}} + 3x^{\frac{2}{7}}\right)$$

$$= 2\left(\frac{-1}{5}x^{-\frac{6}{5}}\right) + 3\left(\frac{2}{7}x^{-\frac{5}{7}}\right)$$

$$= -\frac{2}{5}x^{-\frac{6}{5}} + \frac{6}{7}x^{-\frac{5}{7}}$$

b

$$\frac{d}{dx}\left(\sqrt[3]{x^2 + 2x}\right)$$

$$= \frac{d}{dx}\left((x^2 + 2x)^{\frac{1}{3}}\right)$$

$$= \frac{1}{3}(x^2 + 2x)^{-\frac{2}{3}}(2x + 2) \qquad \text{(chain rule)}$$

$$= \frac{2x + 2}{3\sqrt[3]{(x^2 + 2x)^2}}$$

Section summary

For any non-zero rational number $r = \frac{p}{q}$, if $f(x) = x^r$, then $f'(x) = rx^{r-1}$.

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Exercise 9F

Example 23 **1** Differentiate $2x^{\frac{1}{2}}$ by first principles.

Example 24a **2** Find the derivative of each of the following with respect to x:

a $x^{\frac{1}{5}}$ **b** $x^{\frac{5}{2}}$ **c** $x^{\frac{5}{2}} - x^{\frac{3}{2}}, \ x > 0$

d $3x^{\frac{1}{2}} - 4x^{\frac{5}{3}}$ **e** $x^{-\frac{6}{7}}$ **f** $x^{-\frac{1}{4}} + 4x^{\frac{1}{2}}$

3 Find the gradient of the tangent to the curve for each of the following at the stated value for x:

a $f(x) = x^{\frac{1}{3}}$ where $x = 27$ **b** $f(x) = x^{\frac{1}{3}}$ where $x = -8$

c $f(x) = x^{\frac{2}{3}}$ where $x = 27$ **d** $f(x) = x^{\frac{5}{4}}$ where $x = 16$

Example 24b **4** Find the derivative of each of the following with respect to x:

a $\sqrt{2x+1}$ **b** $\sqrt{4-3x}$ **c** $\sqrt{x^2+2}$

d $\sqrt[3]{4-3x}$ **e** $\dfrac{x^2+2}{\sqrt{x}}$ **f** $3\sqrt{x}\,(x^2+2x)$

5 **a** Show that $\dfrac{d}{dx}\left(\sqrt{x^2 \pm a^2}\right) = \dfrac{x}{\sqrt{x^2 \pm a^2}}$.

b Show that $\dfrac{d}{dx}\left(\sqrt{a^2 - x^2}\right) = \dfrac{-x}{\sqrt{a^2 - x^2}}$.

6 If $y = (x + \sqrt{x^2+1})^2$, show that $\dfrac{dy}{dx} = \dfrac{2y}{\sqrt{x^2+1}}$.

7 Find the derivative with respect to x of each of the following:

a $\sqrt{x^2+2}$ **b** $\sqrt[3]{x^2-5x}$ **c** $\sqrt[5]{x^2+2x}$

9G Differentiation of e^x

In this section we investigate the derivative of functions of the form $f(x) = a^x$. We will see that Euler's number e has the special property that $f'(x) = f(x)$ where $f(x) = e^x$.

First consider $f\colon \mathbb{R} \to \mathbb{R}, \ f(x) = 2^x$.

To find the derivative of f we recall that:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{2^{x+h} - 2^x}{h} \\ &= 2^x \lim_{h\to 0} \frac{2^h - 1}{h} \\ &= 2^x f'(0) \end{aligned}$$

We can investigate this limit numerically to find that $f'(0) \approx 0.6931$ and therefore

$$f'(x) \approx 0.6931 \times 2^x$$

Now consider $g\colon \mathbb{R} \to \mathbb{R}$, $g(x) = 3^x$. Then, as for f, it may be shown that $g'(x) = 3^x g'(0)$. We find $g'(0) \approx 1.0986$ and hence

$$g'(x) \approx 1.0986 \times 3^x$$

The question now arises:

> Can we find a number b between 2 and 3 such that, if $f(x) = b^x$, then $f'(0) = 1$ and therefore $f'(x) = b^x$?

Using a calculator or a spreadsheet, we can investigate the limit as $h \to 0$ of $\dfrac{b^h - 1}{h}$, for various values of b between 2 and 3.

This investigation is carried out in the spreadsheet shown on the right.

Start by taking values for b between 2.71 and 2.72 (first table) and finding $f'(0)$ for each of these values. From these results it may be seen that the required value of b lies between 2.718 and 2.719.

The investigation is continued with values of b between 2.718 and 2.719 (second table). From this the required value of b is seen to lie between 2.7182 and 2.7183.

b	$f'(0)$
2.710	0.996949
2.711	0.997318
2.712	0.997686
2.713	0.998055
2.714	0.998424
2.715	0.998792
2.716	0.999160
2.717	0.999528
2.718	0.999896
2.719	1.000264
2.720	1.000632

b	$f'(0)$
2.7180	0.999896
2.7181	0.999933
2.7182	0.999970
2.7183	1.000007
2.7184	1.000043
2.7185	1.000080
2.7186	1.000117
2.7187	1.000154
2.7188	1.000191
2.7189	1.000227
2.7190	1.000264

The required value of b is in fact Euler's number e, which was introduced in Chapter 5.

Our results can be recorded:

> For $f(x) = e^x$, $f'(x) = e^x$.

Next consider $y = e^{kx}$ where $k \in \mathbb{R}$. The chain rule can be used to find the derivative:

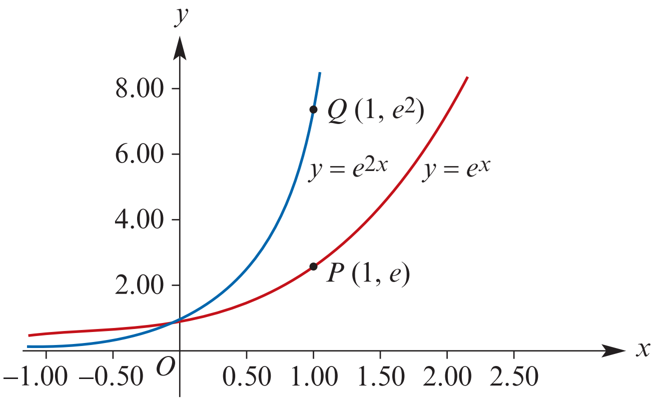
Let $u = kx$. Then $y = e^u$. The chain rule yields

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx}\\ &= e^u \cdot k\\ &= ke^{kx}\end{aligned}$$

> For $f(x) = e^{kx}$, $f'(x) = ke^{kx}$, where $k \in \mathbb{R}$.

The graph illustrates the case where $k = 2$:

- the gradient of $y = e^x$ at the point $P(1, e)$ is e
- the gradient of $y = e^{2x}$ at the point $Q(1, e^2)$ is $2e^2$.



Example 25

Find the derivative of each of the following with respect to x:

a e^{3x} **b** e^{-2x} **c** e^{2x+1} **d** $\dfrac{1}{e^{2x}} + e^{3x}$

Solution

a Let $y = e^{3x}$. Then $\dfrac{dy}{dx} = 3e^{3x}$.

b Let $y = e^{-2x}$. Then $\dfrac{dy}{dx} = -2e^{-2x}$.

c Let $y = e^{2x+1}$. Then

$$\begin{aligned} y &= e^{2x} \cdot e && \text{(index laws)} \\ &= e \cdot e^{2x} \\ \therefore \quad \frac{dy}{dx} &= 2e \cdot e^{2x} \\ &= 2e^{2x+1} \end{aligned}$$

d Let $y = \dfrac{1}{e^{2x}} + e^{3x}$. Then

$$\begin{aligned} y &= e^{-2x} + e^{3x} \\ \therefore \quad \frac{dy}{dx} &= -2e^{-2x} + 3e^{3x} \end{aligned}$$

Example 26

Find the derivative of each of the following with respect to x:

a e^{x^2} **b** e^{x^2+4x}

Solution

a Let $y = e^{x^2}$ and $u = x^2$.
Then $y = e^u$ and the chain rule yields

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= e^u \cdot 2x \\ &= 2xe^{x^2} \end{aligned}$$

b Let $y = e^{x^2+4x}$ and $u = x^2 + 4x$.
Then $y = e^u$ and the chain rule yields

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= e^u(2x + 4) \\ &= (2x + 4)\, e^{x^2+4x} \end{aligned}$$

In general, for $h(x) = e^{f(x)}$, the chain rule gives $h'(x) = f'(x)\, e^{f(x)}$.

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Example 27

Find the gradient of the tangent to the curve $y = e^{2x} + 4$ at the point:

a $(0, 5)$

b $(1, e^2 + 4)$

Solution

We have $\dfrac{dy}{dx} = 2e^{2x}$.

a When $x = 0$, $\dfrac{dy}{dx} = 2$.

The gradient at $(0, 5)$ is 2.

b When $x = 1$, $\dfrac{dy}{dx} = 2e^2$.

The gradient at $(1, e^2 + 4)$ is $2e^2$.

Example 28

For each of the following, first find the derivative with respect to x. Then evaluate the derivative at $x = 2$, given that $f(2) = 0$, $f'(2) = 4$ and $f'(e^2) = 5$.

a $e^{f(x)}$

b $f(e^x)$

Solution

a Let $y = e^{f(x)}$ and $u = f(x)$. Then $y = e^u$.

By the chain rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx}\\ &= e^u f'(x)\\ &= e^{f(x)} f'(x)\end{aligned}$$

When $x = 2$, $\dfrac{dy}{dx} = e^0 \times 4 = 4$.

b Let $y = f(e^x)$ and $u = e^x$. Then $y = f(u)$.

By the chain rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx}\\ &= f'(u) \cdot e^x\\ &= f'(e^x) \cdot e^x\end{aligned}$$

When $x = 2$, $\dfrac{dy}{dx} = f'(e^2) \cdot e^2 = 5e^2$.

Section summary

For $f(x) = e^{kx}$, $f'(x) = ke^{kx}$, where $k \in \mathbb{R}$.

Exercise 9G

Example 25

1 Find the derivative of each of the following with respect to x:

a e^{5x}

b $7e^{-3x}$

c $3e^{-4x} + e^x - x^2$

d $\dfrac{e^{2x} - e^x + 1}{e^x}$

e $\dfrac{4e^{2x} - 2e^x + 1}{2e^{2x}}$

f $e^{2x} + e^4 + e^{-2x}$

Example 26

2 Find the derivative of each of the following with respect to x:

a e^{-2x^3}

b $e^{x^2} + 3x + 1$

c $e^{x^2-4x} + 3x + 1$

d $e^{x^2-2x+3} - x$

e $e^{\frac{1}{x}}, \quad x \neq 0$

f $e^{x^{\frac{1}{2}}}$

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Example 27 **3** Find the gradient of the tangent to the curve $y = e^{\frac{x}{2}} + 4x$ at the point:

a $(0, 1)$ **b** $(1, e^{\frac{1}{2}} + 4)$

4 Find the gradient of the tangent to the curve $y = e^{x^2+3x} + 2x$ at the point:

a $(0, 1)$ **b** $(1, e^4 + 2)$

Example 28 **5** Find the derivative with respect to x of:

a $e^{2f(x)}$ **b** $f(e^{2x})$

6 Find the derivative with respect to x of:

a $(e^{2x} - 1)^4$ **b** $e^{\sqrt{x}}$ **c** $\sqrt{e^x - 1}$

d $e^{x^{\frac{2}{3}}}$ **e** $e^{(x-1)(x-2)}$ **f** e^{e^x}

9H Differentiation of the natural logarithm function

For the function with rule $f(x) = e^x$, we have seen that $f'(x) = e^x$.

This will be used to find the derivative of $g: \mathbb{R}^+ \to \mathbb{R}$, $g(x) = \log_e(kx)$ where $k > 0$.

Let $y = \log_e(kx)$ and solve for x:

$$e^y = kx$$

$$\therefore \quad x = \frac{1}{k}e^y$$

From our observation above:

$$\frac{dx}{dy} = \frac{1}{k}e^y$$

Since $e^y = kx$, this gives

$$\frac{dx}{dy} = \frac{kx}{k} = x$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{x}$$

Let $f: \mathbb{R}^+ \to \mathbb{R}$, $f(x) = \log_e(kx)$ where $k > 0$.

Then $f': \mathbb{R}^+ \to \mathbb{R}$, $f'(x) = \dfrac{1}{x}$.

Example 29

Find the derivative of each of the following with respect to x:

a $\log_e(5x)$, $x > 0$ **b** $\log_e(5x + 3)$, $x > \dfrac{-3}{5}$

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Solution

a Let $y = \log_e(5x)$ for $x > 0$.

Then $\dfrac{dy}{dx} = \dfrac{1}{x}$.

Alternatively, let $u = 5x$. Then $y = \log_e u$ and the chain rule gives

$$\begin{aligned}\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} &= \frac{1}{u} \times 5\\ &= \frac{5}{u}\\ &= \frac{1}{x}\end{aligned}$$

b Let $y = \log_e(5x + 3)$ for $x > \dfrac{-3}{5}$.

Let $u = 5x + 3$. Then $y = \log_e u$ and the chain rule gives

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\ &= \frac{1}{u} \times 5\\ &= \frac{5}{u}\\ &= \frac{5}{5x+3}\end{aligned}$$

In general, if $y = \log_e(ax + b)$ for $x > \dfrac{-b}{a}$, then $\dfrac{dy}{dx} = \dfrac{a}{ax + b}$.

Note: Let $y = \log_e(-x)$, $x < 0$. Using the chain rule with $u = -x$ gives $\dfrac{dy}{dx} = \dfrac{1}{-x} \times (-1) = \dfrac{1}{x}$.

Example 30

Differentiate each of the following with respect to x:

a $\log_e(x^2 + 2)$

b $(\log_e x)^2$, $x > 0$

Solution

a We use the chain rule.

Let $y = \log_e(x^2 + 2)$ and $u = x^2 + 2$.

Then $y = \log_e u$.

$$\begin{aligned}\therefore \quad \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx}\\ &= \frac{1}{u} \cdot 2x\\ &= \frac{2x}{x^2 + 2}\end{aligned}$$

b We use the chain rule.

Let $y = (\log_e x)^2$ and $u = \log_e x$.

Then $y = u^2$.

$$\begin{aligned}\therefore \quad \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx}\\ &= 2u \cdot \frac{1}{x}\\ &= \frac{2\log_e x}{x}\end{aligned}$$

Section summary

- If $y = \log_e(ax + b)$ for $x > \dfrac{-b}{a}$, then $\dfrac{dy}{dx} = \dfrac{a}{ax + b}$.
- If $h(x) = \log_e(f(x))$, then the chain rule gives $h'(x) = \dfrac{f'(x)}{f(x)}$.

Exercise 9H

Example 29

1 Find the derivative of each of the following with respect to x:

a $y = 2\log_e x$

b $y = 2\log_e(2x)$

c $y = x^2 + 3\log_e(2x)$

d $y = 3\log_e x + \frac{1}{x}$

e $y = 3\log_e(4x) + x$

f $y = \log_e(x + 1)$

g $y = \log_e(2x + 4)$

h $y = \log_e(3x - 1)$

i $y = \log_e(6x - 1)$

Example 30

2 Find the derivative of each of the following with respect to x:

a $y = \log_e(x^3)$

b $y = (\log_e x)^3$

c $y = \log_e(x^2 + x - 1)$

d $y = \log_e(x^3 + x^2)$

e $y = \log_e((2x + 3)^2)$

f $y = \log_e((3 - 2x)^2)$

3 For each of the following, find $f'(x)$:

a $f(x) = \log_e(x^2 + 1)$

b $f(x) = \log_e(e^x)$

4 Find the y-coordinate and the gradient of the tangent to the curve at the point corresponding to the given value of x:

a $y = \log_e x$, $x > 0$, at $x = e$

b $y = \log_e(x^2 + 1)$ at $x = e$

c $y = \log_e(-x)$, $x < 0$, at $x = -e$

d $y = x + \log_e x$ at $x = 1$

e $y = \log_e(x^2 - 2x + 2)$ at $x = 1$

f $y = \log_e(2x - 1)$ at $x = \frac{3}{2}$

5 Find $f'(1)$ if $f(x) = \log_e \sqrt{x^2 + 1}$.

6 Differentiate $\log_e(1 + x + x^2)$.

7 If $f(x) = \log_e(x^2 + 1)$, find $f'(3)$.

8 Given that $f(0) = 2$ and $f'(0) = 4$, find $\frac{d}{dx}\big(\log_e(f(x))\big)$ when $x = 0$.

9I Derivatives of circular functions

In this section we find the derivatives of sin, cos and tan.

▶ The derivative of $\sin(k\theta)$

We first consider the sine function. The following proof uses a trigonometric identity from Specialist Mathematics Units 1 & 2 and is beyond the scope of this course, but it is important to know that the result can be proved.

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If $f: \mathbb{R} \to \mathbb{R}$, $f(\theta) = \sin\theta$, then $f'(\theta) = \cos\theta$.

Proof We use the identity

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

Consider points $P(\theta, \sin\theta)$ and $Q(\theta+h, \sin(\theta+h))$ on the graph of $f(\theta) = \sin\theta$. The gradient of the secant PQ is

$$\begin{aligned}\frac{\sin(\theta+h)-\sin\theta}{h} &= \frac{\sin\theta\cos h+\cos\theta\sin h-\sin\theta}{h}\\ &= \frac{\sin\theta\cdot(\cos h-1)}{h}+\frac{\cos\theta\sin h}{h}\end{aligned}$$

We now consider what happens as $h \to 0$. We use two limit results (the second limit is proved below and the first limit then follows using a trigonometric identity):

$$\lim_{h\to 0}\frac{\cos h-1}{h} = 0 \quad\text{and}\quad \lim_{h\to 0}\frac{\sin h}{h} = 1$$

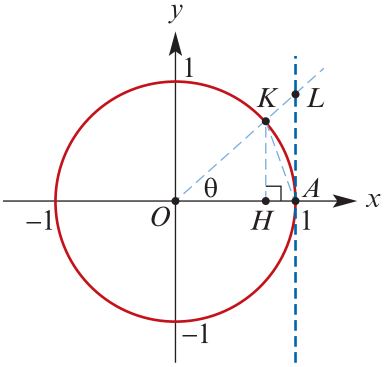
Therefore

$$\begin{aligned}f'(\theta) &= \lim_{h\to 0}\left(\frac{\sin\theta\cdot(\cos h-1)}{h}+\frac{\cos\theta\sin h}{h}\right)\\ &= \sin\theta\times 0+\cos\theta\times 1\\ &= \cos\theta\end{aligned}$$

We now prove the following result.

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

Proof Let K be a point on the unit circle as shown, and let $\angle KOH = \theta$. The coordinates of K are $(\cos\theta, \sin\theta)$. Point H is on the x-axis such that $\angle KHO$ is a right angle.



Draw a tangent to the circle at $A(1, 0)$. The line OK intersects this tangent at $L(1, \tan\theta)$.

The area of sector OAK is $\frac{1}{2}\theta$.

Thus $\quad$ area $\triangle OAK \le \frac{1}{2}\theta \le$ area $\triangle OAL$

i.e. $\quad \frac{1}{2}OA\cdot HK \le \frac{1}{2}\theta \le \frac{1}{2}OA\cdot AL$

This implies that $\sin\theta \le \theta \le \tan\theta$.

For $0 < \theta < \frac{\pi}{2}$, we have $\sin\theta > 0$, and so we can divide both inequalities by $\sin\theta$ to obtain

$$1 \le \frac{\theta}{\sin\theta} \le \frac{1}{\cos\theta}$$

As θ approaches 0, the value of $\cos\theta$ approaches 1, and so $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$.

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We now turn our attention to the function $f(\theta) = \sin(k\theta)$. The graph of $y = \sin(k\theta)$ is obtained from the graph of $y = \sin\theta$ by a dilation of factor $\frac{1}{k}$ from the y-axis (and so this immediately suggests that the gradient will change by a factor of k).

We use the chain rule to determine $f'(\theta)$.

Let $y = \sin(k\theta)$ and let $u = k\theta$. Then $y = \sin u$ and therefore

$$\frac{dy}{d\theta} = \frac{dy}{du} \cdot \frac{du}{d\theta} = \cos u \cdot k = k\cos(k\theta)$$

For $f : \mathbb{R} \to \mathbb{R},\ f(\theta) = \sin(k\theta)$
$f' : \mathbb{R} \to \mathbb{R},\ f'(\theta) = k\cos(k\theta)$

The derivative of $\cos(k\theta)$

We next find the derivative of $\cos(k\theta)$. We first note the following:

$$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right) \quad \text{and} \quad \sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$$

These results will be used in the following way.

Let $y = \cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$.

Now let $u = \frac{\pi}{2} - \theta$. Then $y = \sin u$. The chain rule gives

$$\begin{aligned}\frac{dy}{d\theta} = \frac{dy}{du} \cdot \frac{du}{d\theta} &= \cos u \cdot (-1)\\ &= -\cos\left(\frac{\pi}{2} - \theta\right)\\ &= -\sin\theta\end{aligned}$$

We have the following results:

- For $f : \mathbb{R} \to \mathbb{R},\ f(\theta) = \cos\theta$
  $f' : \mathbb{R} \to \mathbb{R},\ f'(\theta) = -\sin\theta$
- For $f : \mathbb{R} \to \mathbb{R},\ f(\theta) = \cos(k\theta)$
  $f' : \mathbb{R} \to \mathbb{R},\ f'(\theta) = -k\sin(k\theta)$

The derivative of $\tan(k\theta)$

For convenience, we introduce a new function, called **secant**, given by

$$\sec\theta = \frac{1}{\cos\theta}$$

We can write $\sin^n\theta = (\sin\theta)^n$ and $\cos^n\theta = (\cos\theta)^n$.

Here we find the derivative of $\tan\theta$ by first principles. In Section 9K we show another method.

If $f(\theta) = \tan(k\theta)$, then $f'(\theta) = k\sec^2(k\theta)$.

Proof Consider points $P(\theta, \tan\theta)$ and $Q(\theta + h, \tan(\theta + h))$ on the graph of $f(\theta) = \tan\theta$. The gradient of the secant PQ is

$$\begin{aligned}\frac{\tan(\theta+h)-\tan\theta}{h} &= \left(\frac{\sin(\theta+h)}{\cos(\theta+h)} - \frac{\sin\theta}{\cos\theta}\right)\times\frac{1}{h}\\ &= \left(\frac{\sin(\theta+h)\cos(\theta)-\cos(\theta+h)\sin(\theta)}{\cos(\theta+h)\cos(\theta)}\right)\times\frac{1}{h}\\ &= \frac{\sin h}{h\cos(\theta+h)\cos(\theta)}\end{aligned}$$

We now consider what happens as $h \to 0$:

$$\lim_{h\to 0}\cos(\theta+h) = \cos\theta \quad \text{and} \quad \lim_{h\to 0}\frac{\sin h}{h} = 1$$

Therefore

$$f'(\theta) = \lim_{h\to 0}\left(\frac{\sin h}{h\cos(\theta+h)\cos(\theta)}\right) = \frac{1}{\cos^2\theta} = \sec^2\theta$$

We can use the chain rule to show that, if $f(\theta) = \tan(k\theta)$, then $f'(\theta) = k\sec^2(k\theta)$.

Example 31

Find the derivative with respect to θ of each of the following:

a $\sin(2\theta)$ **b** $\tan(3\theta)$ **c** $\sin^2(2\theta)$

d $\sin^2(2\theta + 1)$ **e** $\cos^3(4\theta + 1)$ **f** $\tan(3\theta^2 + 1)$

Solution

a Let $y = \sin(2\theta)$. Then $\frac{dy}{d\theta} = 2\cos(2\theta)$.

b Let $y = \tan(3\theta)$. Then $\frac{dy}{d\theta} = 3\sec^2(3\theta)$.

c Let $y = \sin^2(2\theta)$ and $u = \sin(2\theta)$. Then $y = u^2$. Using the chain rule:

$$\begin{aligned}\frac{dy}{d\theta} &= \frac{dy}{du}\cdot\frac{du}{d\theta}\\ &= 2u\cdot 2\cos(2\theta)\\ &= 4u\cos(2\theta)\\ &= 4\sin(2\theta)\cos(2\theta)\end{aligned}$$

d Let $y = \sin^2(2\theta + 1)$ and $u = \sin(2\theta + 1)$. Then $y = u^2$. Using the chain rule:

$$\begin{aligned}\frac{dy}{d\theta} &= \frac{dy}{du}\cdot\frac{du}{d\theta}\\ &= 2u\cdot 2\cos(2\theta+1)\\ &= 4\sin(2\theta+1)\cos(2\theta+1)\end{aligned}$$

e Let $y = \cos^3(4\theta + 1)$ and $u = \cos(4\theta + 1)$. Then $y = u^3$. Using the chain rule:

$$\begin{aligned}\frac{dy}{d\theta} &= \frac{dy}{du}\cdot\frac{du}{d\theta}\\ &= 3u^2\cdot(-4)\sin(4\theta+1)\\ &= -12\cos^2(4\theta+1)\sin(4\theta+1)\end{aligned}$$

f Let $y = \tan(3\theta^2 + 1)$ and $u = 3\theta^2 + 1$. Then $y = \tan u$. Using the chain rule:

$$\begin{aligned}\frac{dy}{d\theta} &= \frac{dy}{du}\cdot\frac{du}{d\theta}\\ &= \sec^2 u\cdot 6\theta\\ &= 6\theta\sec^2(3\theta^2+1)\end{aligned}$$

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Example 32

Find the y-coordinate and the gradient of the tangent at the points on the following curves corresponding to the given values of θ:

a $y = \cos\theta$ at $\theta = \dfrac{\pi}{4}$ and $\theta = \dfrac{\pi}{2}$ **b** $y = \tan\theta$ at $\theta = 0$ and $\theta = \dfrac{\pi}{4}$

Solution

a Let $y = \cos\theta$. Then $\dfrac{dy}{d\theta} = -\sin\theta$.

When $\theta = \dfrac{\pi}{4}$, we have $y = \cos\left(\dfrac{\pi}{4}\right) = \dfrac{1}{\sqrt{2}}$ and $\dfrac{dy}{d\theta} = -\sin\left(\dfrac{\pi}{4}\right) = \dfrac{-1}{\sqrt{2}}$.

So the gradient at $\left(\dfrac{\pi}{4}, \dfrac{1}{\sqrt{2}}\right)$ is $\dfrac{-1}{\sqrt{2}}$.

When $\theta = \dfrac{\pi}{2}$, we have $y = 0$ and $\dfrac{dy}{d\theta} = -1$. The gradient at $\left(\dfrac{\pi}{2}, 0\right)$ is -1.

b Let $y = \tan\theta$. Then $\dfrac{dy}{d\theta} = \sec^2\theta$.

When $\theta = 0$, we have $y = 0$ and $\dfrac{dy}{d\theta} = 1$. The gradient at $(0, 0)$ is 1.

When $\theta = \dfrac{\pi}{4}$, we have $y = 1$ and $\dfrac{dy}{d\theta} = 2$. The gradient at $\left(\dfrac{\pi}{4}, 1\right)$ is 2.

Section summary

- If $f\colon \mathbb{R} \to \mathbb{R}$, $f(\theta) = \sin(k\theta)$, then $f'\colon \mathbb{R} \to \mathbb{R}$, $f'(\theta) = k\cos(k\theta)$.
- If $f\colon \mathbb{R} \to \mathbb{R}$, $f(\theta) = \cos(k\theta)$, then $f'\colon \mathbb{R} \to \mathbb{R}$, $f'(\theta) = -k\sin(k\theta)$.
- If $f(\theta) = \tan(k\theta)$, then $f'(\theta) = k\sec^2(k\theta)$.

Exercise 9I

Example 31

1 Find the derivative with respect to x of each of the following:

a $\sin(5x)$ **b** $\cos(5x)$ **c** $\tan(5x)$ **d** $\sin^2 x$ **e** $\tan(3x + 1)$

f $\cos(x^2 + 1)$ **g** $\sin^2\left(x - \dfrac{\pi}{4}\right)$ **h** $\cos^2\left(x - \dfrac{\pi}{3}\right)$ **i** $\sin^3\left(2x + \dfrac{\pi}{6}\right)$ **j** $\cos^3\left(2x - \dfrac{\pi}{4}\right)$

Example 32

2 Find the y-coordinate and the gradient of the tangent at the points on the following curves corresponding to the given values of x:

a $y = \sin(2x)$ at $x = \dfrac{\pi}{8}$ **b** $y = \sin(3x)$ at $x = \dfrac{\pi}{6}$ **c** $y = 1 + \sin(3x)$ at $x = \dfrac{\pi}{6}$

d $y = \cos^2(2x)$ at $x = \dfrac{\pi}{4}$ **e** $y = \sin^2(2x)$ at $x = \dfrac{\pi}{4}$ **f** $y = \tan(2x)$ at $x = \dfrac{\pi}{8}$

3 For each of the following, find $f'(x)$:

a $f(x) = 5\cos x - 2\sin(3x)$ **b** $f(x) = \cos x + \sin x$

c $f(x) = \sin x + \tan x$ **d** $f(x) = \tan^2 x$

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4 Find the derivative of each of the following. (Change degrees to radians first.)

a $2\cos x^\circ$ **b** $3\sin x^\circ$ **c** $\tan(3x)^\circ$

5 **a** If $y = -\log_e(\cos x)$, find $\dfrac{dy}{dx}$. **b** If $y = -\log_e(\tan x)$, find $\dfrac{dy}{dx}$.

6 **a** If $y = e^{2\sin x}$, find $\dfrac{dy}{dx}$. **b** If $y = e^{\cos(2x)}$, find $\dfrac{dy}{dx}$.

9J The product rule

In the next two sections, we introduce two more rules for differentiation. The first of these is the **product rule**.

Let $F(x) = f(x) \cdot g(x)$. If $f'(x)$ and $g'(x)$ exist, then

$$F'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

For example, consider $F(x) = (x^2 + 3x)(4x + 5)$. Then F is the product of two functions f and g, where $f(x) = x^2 + 3x$ and $g(x) = 4x + 5$. The product rule gives:

$$\begin{aligned} F'(x) &= f(x) \cdot g'(x) + g(x) \cdot f'(x) \\ &= (x^2 + 3x) \cdot 4 + (4x + 5) \cdot (2x + 3) \\ &= 4x^2 + 12x + 8x^2 + 22x + 15 \\ &= 12x^2 + 34x + 15 \end{aligned}$$

This could also have been found by multiplying $x^2 + 3x$ by $4x + 5$ and then differentiating.

The product rule (function notation)

Let $F(x) = f(x) \cdot g(x)$. If $f'(x)$ and $g'(x)$ exist, then

$$F'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

Proof By the definition of the derivative of F, we have

$$\begin{aligned} F'(x) &= \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} \\ &= \lim_{h \to 0} \frac{f(x+h)\,g(x+h) - f(x)\,g(x)}{h} \end{aligned}$$

Adding and subtracting $f(x+h)\,g(x)$:

$$\begin{aligned} F'(x) &= \lim_{h \to 0} \frac{f(x+h)\,g(x+h) - f(x)\,g(x) + [f(x+h)\,g(x) - f(x+h)\,g(x)]}{h} \\ &= \lim_{h \to 0} \left[f(x+h) \cdot \left(\frac{g(x+h) - g(x)}{h} \right) + g(x) \cdot \left(\frac{f(x+h) - f(x)}{h} \right) \right] \end{aligned}$$

Since f and g are differentiable, we obtain

$$\begin{aligned} F'(x) &= \lim_{h \to 0} f(x+h) \cdot \lim_{h \to 0} \left(\frac{g(x+h) - g(x)}{h} \right) + \lim_{h \to 0} g(x) \cdot \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right) \\ &= f(x) \cdot g'(x) + g(x) \cdot f'(x) \end{aligned}$$

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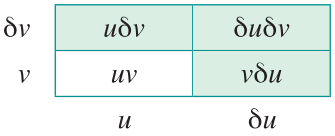
We can state the product rule in Leibniz notation and give a geometric interpretation.

The product rule (Leibniz notation)

If $y = uv$, where u and v are functions of x, then

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

In the following figure, the white region represents $y = uv$ and the shaded region δy, as explained below.



$$\begin{aligned}\delta y &= (u + \delta u)(v + \delta v) - uv \\ &= uv + v\delta u + u\delta v + \delta u\delta v - uv \\ &= v\delta u + u\delta v + \delta u\delta v\end{aligned}$$

$$\therefore \quad \frac{\delta y}{\delta x} = v\frac{\delta u}{\delta x} + u\frac{\delta v}{\delta x} + \frac{\delta u}{\delta x}\frac{\delta v}{\delta x}\delta x$$

In the limit, as $\delta x \to 0$, we have

$$\frac{\delta u}{\delta x} = \frac{du}{dx}, \qquad \frac{\delta v}{\delta x} = \frac{dv}{dx} \qquad \text{and} \qquad \frac{\delta y}{\delta x} = \frac{dy}{dx}$$

Therefore

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

Example 33

Differentiate each of the following with respect to x:

a $(2x^2 + 1)(5x^3 + 16)$

b $x^3(3x - 5)^4$

Solution

a Let $y = (2x^2 + 1)(5x^3 + 16)$. Let $u = 2x^2 + 1$ and $v = 5x^3 + 16$.

Then $\dfrac{du}{dx} = 4x$ and $\dfrac{dv}{dx} = 15x^2$.

The product rule gives:

$$\begin{aligned}\frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx} \\ &= (2x^2 + 1) \cdot 15x^2 + (5x^3 + 16) \cdot 4x \\ &= 30x^4 + 15x^2 + 20x^4 + 64x \\ &= 50x^4 + 15x^2 + 64x\end{aligned}$$

b Let $y = x^3(3x-5)^4$. Let $u = x^3$ and $v = (3x-5)^4$.

Then $\frac{du}{dx} = 3x^2$ and $\frac{dv}{dx} = 12(3x-5)^3$, using the chain rule.

The product rule gives:

$$\begin{aligned}\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} &= 12x^3(3x-5)^3 + (3x-5)^4 \cdot 3x^2\\ &= (3x-5)^3[12x^3 + 3x^2(3x-5)]\\ &= (3x-5)^3[12x^3 + 9x^3 - 15x^2]\\ &= (3x-5)^3(21x^3 - 15x^2)\\ &= 3x^2(7x-5)(3x-5)^3\end{aligned}$$

Example 34

For $F : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $F(x) = x^{-3}(10x^2-5)^3$, find $F'(x)$.

Solution

Let $f(x) = x^{-3}$ and $g(x) = (10x^2-5)^3$.

Then $f'(x) = -3x^{-4}$ and $g'(x) = 60x(10x^2-5)^2$ using the chain rule.

By the product rule:

$$\begin{aligned}F'(x) &= x^{-3} \cdot 60x(10x^2-5)^2 + (10x^2-5)^3 \cdot (-3x^{-4})\\ &= (10x^2-5)^2\,[60x^{-2} + (10x^2-5) \cdot (-3x^{-4})]\\ &= (10x^2-5)^2\left(\frac{60x^2 - 30x^2 + 15}{x^4}\right)\\ &= \frac{(10x^2-5)^2(30x^2+15)}{x^4}\end{aligned}$$

Example 35

Differentiate each of the following with respect to x:

a $e^x(2x^2+1)$

b $e^x\sqrt{x-1}$

Solution

a Use the product rule.

Let $y = e^x(2x^2+1)$. Then

$$\begin{aligned}\frac{dy}{dx} &= e^x(2x^2+1) + 4xe^x\\ &= e^x(2x^2+4x+1)\end{aligned}$$

b Use the product rule and the chain rule.

Let $y = e^x\sqrt{x-1}$. Then

$$\begin{aligned}\frac{dy}{dx} &= e^x\sqrt{x-1} + \frac{1}{2}e^x(x-1)^{-\frac{1}{2}}\\ &= e^x\sqrt{x-1} + \frac{e^x}{2\sqrt{x-1}}\\ &= \frac{2e^x(x-1) + e^x}{2\sqrt{x-1}}\\ &= \frac{2xe^x - e^x}{2\sqrt{x-1}}\end{aligned}$$

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Example 36

Find the derivative of each of the following with respect to x:

a $2x^2 \sin(2x)$ **b** $e^{2x} \sin(2x+1)$ **c** $\cos(4x)\sin(2x)$

Solution

a Let $y = 2x^2 \sin(2x)$.

Applying the product rule:

$$\frac{dy}{dx} = 4x\sin(2x) + 4x^2\cos(2x)$$

b Let $y = e^{2x}\sin(2x+1)$.

Applying the product rule:

$$\frac{dy}{dx} = 2e^{2x}\sin(2x+1) + 2e^{2x}\cos(2x+1)$$
$$= 2e^{2x}[\sin(2x+1) + \cos(2x+1)]$$

c Let $y = \cos(4x)\sin(2x)$. Then the product rule gives

$$\frac{dy}{dx} = -4\sin(4x)\sin(2x) + 2\cos(2x)\cos(4x)$$

Section summary

The product rule

Let $F(x) = f(x) \cdot g(x)$. If $f'(x)$ and $g'(x)$ exist, then

$$F'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

Exercise 9J

Example 33, 34

1 Find the derivative of each of the following with respect to x, using the product rule:

a $(2x^2+6)(2x^3+1)$ **b** $3x^{\frac{1}{2}}(2x+1)$ **c** $3x(2x-1)^3$

d $4x^2(2x^2+1)^2$ **e** $(3x+1)^{\frac{3}{2}}(2x+4)$ **f** $(x^2+1)\sqrt{2x-4}$

g $x^3(3x^2+2x+1)^{-1}$ **h** $x^4\sqrt{2x^2-1}$ **i** $x^2\sqrt[3]{x^2+2x}$

j $x^{-2}(5x^2-4)^3$ **k** $x^{-3}(x^3-4)^2$ **l** $x^3\sqrt[5]{x^3-x}$

Example 35

2 Find $f'(x)$ for each of the following:

a $f(x) = e^x(x^2+1)$ **b** $f(x) = e^{2x}(x^3+3x+1)$

c $f(x) = e^{4x+1}(x+1)^2$ **d** $f(x) = e^{-4x}\sqrt{x+1},\ x \geq -1$

3 For each of the following, find $f'(x)$:

a $f(x) = x\log_e x,\ x > 0$ **b** $f(x) = 2x^2\log_e x,\ x > 0$

c $f(x) = e^x\log_e x,\ x > 0$ **d** $f(x) = x\log_e(-x),\ x < 0$

4 Differentiate each of the following with respect to x:

a x^4e^{-2x} **b** e^{2x+3} **c** $(e^{2x}+x)^{\frac{3}{2}}$ **d** $\frac{1}{x}e^x$

e $e^{\frac{1}{2}x^2}$ **f** $(x^2+2x+2)e^{-x}$

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5 Find each of the following:

a $\dfrac{d}{dx}\left(e^x f(x)\right)$ **b** $\dfrac{d}{dx}\left(\dfrac{e^x}{f(x)}\right)$ **c** $\dfrac{d}{dx}\left(e^{f(x)}\right)$ **d** $\dfrac{d}{dx}\left(e^x (f(x))^2\right)$

Example 36

6 Differentiate each of the following with respect to x:

a $x^3 \cos x$ **b** $(1 + x^2)\cos x$ **c** $e^{-x}\sin x$ **d** $6x\cos x$ **e** $\sin(3x)\cos(4x)$

f $\tan(2x)\sin(2x)$ **g** $12x\sin x$ **h** $x^2 e^{\sin x}$ **i** $x^2\cos^2 x$ **j** $e^x \tan x$

7 For each of the following, find $f'(\pi)$:

a $f(x) = e^x \sin x$ **b** $f(x) = \cos^2(2x)$

8 Given that $f(1) = 2$ and $f'(1) = 4$, find the derivative of $f(x)\log_e(x)$ when $x = 1$.

9K The quotient rule

Let $F(x) = \dfrac{f(x)}{g(x)}$, where $g(x) \neq 0$. If $f'(x)$ and $g'(x)$ exist, then

$$F'(x) = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{[g(x)]^2}$$

For example, if

$$F(x) = \frac{x^3 + 2x}{x^5 + 2}$$

then F can be considered as a quotient of two functions f and g, where $f(x) = x^3 + 2x$ and $g(x) = x^5 + 2$. The quotient rule gives

$$\begin{aligned} F'(x) &= \frac{(x^5+2)(3x^2+2) - (x^3+2x)5x^4}{(x^5+2)^2} \\ &= \frac{3x^7 + 6x^2 + 2x^5 + 4 - 5x^7 - 10x^5}{(x^5+2)^2} \\ &= \frac{-2x^7 - 8x^5 + 6x^2 + 4}{(x^5+2)^2} \end{aligned}$$

The quotient rule (function notation)

Let $F(x) = \dfrac{f(x)}{g(x)}$, where $g(x) \neq 0$. If $f'(x)$ and $g'(x)$ exist, then

$$F'(x) = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{[g(x)]^2}$$

Proof The quotient rule can be proved from first principles, but instead we will use the product rule and the chain rule.

We can write $F(x) = f(x)\cdot h(x)$, where $h(x) = [g(x)]^{-1}$. Using the chain rule, we have

$$h'(x) = -[g(x)]^{-2}\cdot g'(x)$$

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Therefore, using the product rule, we obtain

$$\begin{aligned} F'(x) &= f(x) \cdot h'(x) + h(x) \cdot f'(x) \\ &= -f(x) \cdot [g(x)]^{-2} \cdot g'(x) + [g(x)]^{-1} \cdot f'(x) \\ &= [g(x)]^{-2}\big(-f(x) \cdot g'(x) + g(x) \cdot f'(x)\big) \\ &= \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2} \end{aligned}$$

The quotient rule (Leibniz notation)

If $y = \dfrac{u}{v}$, where u and v are functions of x and $v \neq 0$, then

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

Example 37

Find the derivative of $\dfrac{x-2}{x^2+4x+1}$ with respect to x.

Solution

Let $y = \dfrac{x-2}{x^2+4x+1}$. The quotient rule gives

$$\begin{aligned} \frac{dy}{dx} &= \frac{x^2+4x+1-(x-2)(2x+4)}{(x^2+4x+1)^2} \\ &= \frac{x^2+4x+1-(2x^2-8)}{(x^2+4x+1)^2} \\ &= \frac{-x^2+4x+9}{(x^2+4x+1)^2} \end{aligned}$$

Example 38

Differentiate each of the following with respect to x:

a $\dfrac{e^x}{e^{2x}+1}$

b $\dfrac{\sin x}{x+1}$, $x \neq -1$

Solution

a Let $y = \dfrac{e^x}{e^{2x}+1}$.

Applying the quotient rule:

$$\begin{aligned} \frac{dy}{dx} &= \frac{(e^{2x}+1)e^x - e^x \cdot 2e^{2x}}{(e^{2x}+1)^2} \\ &= \frac{e^{3x}+e^x-2e^{3x}}{(e^{2x}+1)^2} \\ &= \frac{e^x-e^{3x}}{(e^{2x}+1)^2} \end{aligned}$$

b Let $y = \dfrac{\sin x}{x+1}$ for $x \neq -1$.

Applying the quotient rule:

$$\frac{dy}{dx} = \frac{(x+1)\cos x - \sin x}{(x+1)^2}$$

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Using the quotient rule to find the derivative of tan θ

Let $y = \tan\theta$. We write $y = \dfrac{\sin\theta}{\cos\theta}$ and apply the quotient rule to find the derivative:

$$\begin{aligned}\frac{dy}{d\theta} &= \frac{\cos\theta\cos\theta - \sin\theta\cdot(-\sin\theta)}{(\cos\theta)^2}\\ &= \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}\\ &= \frac{1}{\cos^2\theta} \qquad \text{(by the Pythagorean identity)}\\ &= \sec^2\theta\end{aligned}$$

Section summary

The quotient rule

Let $F(x) = \dfrac{f(x)}{g(x)}$, where $g(x) \neq 0$. If $f'(x)$ and $g'(x)$ exist, then

$$F'(x) = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{[g(x)]^2}$$

Exercise 9K

Example 37

1 Find the derivative of each of the following with respect to x:

a $\dfrac{x}{x+4}$ **b** $\dfrac{x^2-1}{x^2+1}$ **c** $\dfrac{x^{\frac{1}{2}}}{1+x}$ **d** $\dfrac{(x+2)^3}{x^2+1}$

e $\dfrac{x-1}{x^2+2}$ **f** $\dfrac{x^2+1}{x^2-1}$ **g** $\dfrac{3x^2+2x+1}{x^2+x+1}$ **h** $\dfrac{2x+1}{2x^3+2x}$

2 Find the y-coordinate and the gradient at the point on the curve corresponding to the given value of x:

a $y = (2x+1)^4x^2$ at $x = 1$ **b** $y = x^2\sqrt{x+1}$ at $x = 0$

c $y = x^2(2x+1)^{\frac{1}{2}}$ at $x = 0$ **d** $y = \dfrac{x}{x^2+1}$ at $x = 1$

e $y = \dfrac{2x+1}{x^2+1}$ at $x = 1$

3 For each of the following, find $f'(x)$:

a $f(x) = (x+1)\sqrt{x^2+1}$ **b** $f(x) = (x^2+1)\sqrt{x^3+1},\ x > -1$

c $f(x) = \dfrac{2x+1}{x+3},\ x \neq -3$

Example 38

4 For each of the following, find $f'(x)$:

a $f(x) = \dfrac{e^x}{e^{3x}+3}$ **b** $f(x) = \dfrac{\cos x}{x+1},\ x \neq -1$ **c** $f(x) = \dfrac{\log_e x}{x+1},\ x > 0$

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5 For each of the following, find $f'(x)$:

a $f(x) = \dfrac{\log_e x}{x}, \quad x > 0$ **b** $f(x) = \dfrac{\log_e x}{x^2 + 1}, \quad x > 0$

6 Find $f'(x)$ for each of the following:

a $f(x) = \dfrac{e^{3x}}{e^{3x} + 3}$ **b** $f(x) = \dfrac{e^x + 1}{e^x - 1}$

c $f(x) = \dfrac{e^{2x} + 2}{e^{2x} - 2}$

7 For each of the following, find $f'(\pi)$:

a $f(x) = \dfrac{2x}{\cos x}$ **b** $f(x) = \dfrac{3x^2 + 1}{\cos x}$

c $f(x) = \dfrac{e^x}{\cos x}$ **d** $f(x) = \dfrac{\sin x}{x}$

9L Limits and continuity

▶ Limits

It is not the intention of this course to provide a formal introduction to limits. We require only an intuitive understanding of limits and some fairly obvious rules for how to handle them.

> The notation $\lim_{x \to a} f(x) = p$ says that the limit of $f(x)$, as x approaches a, is p. We can also say: 'As x approaches a, $f(x)$ approaches p.'
>
> This means that we can make the value of $f(x)$ as close as we like to p, provided we choose x-values close enough to a.

We have met a similar idea earlier in the course. For example, we have seen that $\lim_{x \to \infty} f(x) = 4$ for the function with rule $f(x) = \dfrac{1}{x} + 4$. The graph of $y = f(x)$ can get as close as we like to the line $y = 4$, just by taking larger and larger values of x.

As we will see, for many functions (in particular, for polynomial functions), the limit at a particular point is simply the value of the function at that point.

Example 39

Find $\lim_{x \to 2} 3x^2$.

Solution

$\lim_{x \to 2} 3x^2 = 3(2)^2 = 12$

Explanation

As x gets closer and closer to 2, the value of $3x^2$ gets closer and closer to 12.

If the function is not defined at the value for which the limit is to be found, a different procedure is used.

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Example 40

For $f(x) = \dfrac{2x^2 - 5x + 2}{x - 2}$, $x \neq 2$, find $\lim_{x \to 2} f(x)$.

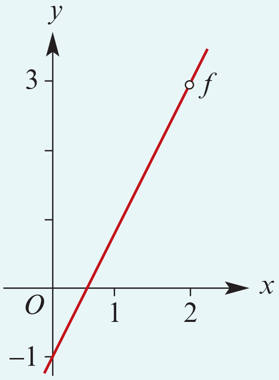
Solution

Observe that

$$\begin{aligned} f(x) &= \frac{2x^2 - 5x + 2}{x - 2} \\ &= \frac{(2x - 1)(x - 2)}{x - 2} \\ &= 2x - 1 \qquad \text{(for } x \neq 2\text{)} \end{aligned}$$

Hence $\lim_{x \to 2} f(x) = 3$.

The graph of $f : \mathbb{R} \setminus \{2\} \to \mathbb{R}$, $f(x) = 2x - 1$ is shown.



We can investigate Example 40 further by looking at the values of the function as we take x-values closer and closer to 2.

Observe that $f(x)$ is defined for $x \in \mathbb{R} \setminus \{2\}$. Examine the behaviour of $f(x)$ for values of x close to 2.

$x < 2$	$x > 2$
$f(1.7) = 2.4$	$f(2.3) = 3.6$
$f(1.8) = 2.6$	$f(2.2) = 3.4$
$f(1.9) = 2.8$	$f(2.1) = 3.2$
$f(1.99) = 2.98$	$f(2.01) = 3.02$
$f(1.999) = 2.998$	$f(2.001) = 3.002$

From the table, it is apparent that, as x takes values closer and closer to 2 (regardless of whether x approaches 2 from the left or from the right), the values of $f(x)$ become closer and closer to 3. That is, $\lim_{x \to 2} f(x) = 3$.

Note that the limit exists, but the function is not defined at $x = 2$.

Algebra of limits

The following important results are useful for the evaluation of limits.

Assume that both $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist.

- **Sum:** $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
  That is, the limit of the sum is the sum of the limits.
- **Multiple:** $\lim_{x \to a} kf(x) = k \lim_{x \to a} f(x)$, where k is a given real number.
- **Product:** $\lim_{x \to a} (f(x)\, g(x)) = \lim_{x \to a} f(x) \, \lim_{x \to a} g(x)$
  That is, the limit of the product is the product of the limits.
- **Quotient:** $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided $\lim_{x \to a} g(x) \neq 0$.
  That is, the limit of the quotient is the quotient of the limits.

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Example 41

Find:

a $\lim\limits_{h\to 0}(3h+4)$

b $\lim\limits_{x\to 2}4x(x+2)$

c $\lim\limits_{x\to 3}\dfrac{5x+2}{x-2}$

Solution

a $$\begin{aligned}\lim_{h\to 0}(3h+4) &= \lim_{h\to 0}(3h)+\lim_{h\to 0}(4)\\ &= 0+4\\ &= 4\end{aligned}$$

b $$\begin{aligned}\lim_{x\to 2}4x(x+2) &= \lim_{x\to 2}(4x)\ \lim_{x\to 2}(x+2)\\ &= 8\times 4\\ &= 32\end{aligned}$$

c $$\begin{aligned}\lim_{x\to 3}\frac{5x+2}{x-2} &= \lim_{x\to 3}(5x+2)\div\lim_{x\to 3}(x-2)\\ &= 17\div 1\\ &= 17\end{aligned}$$

Example 42

Find:

a $\lim\limits_{x\to 3}\dfrac{x^2-3x}{x-3}$

b $\lim\limits_{x\to 2}\dfrac{x^2-x-2}{x-2}$

c $\lim\limits_{x\to 3}\dfrac{x^2-7x+10}{x^2-25}$

Solution

a $$\lim_{x\to 3}\frac{x^2-3x}{x-3}=\lim_{x\to 3}\frac{x(x-3)}{x-3}=\lim_{x\to 3}x=3$$

b $$\lim_{x\to 2}\frac{x^2-x-2}{x-2}=\lim_{x\to 2}\frac{(x-2)(x+1)}{x-2}=\lim_{x\to 2}(x+1)=3$$

c $$\lim_{x\to 3}\frac{x^2-7x+10}{x^2-25}=\lim_{x\to 3}\frac{(x-2)(x-5)}{(x+5)(x-5)}=\frac{\lim\limits_{x\to 3}(x-2)}{\lim\limits_{x\to 3}(x+5)}=\frac{1}{8}$$

▶ Left and right limits

An idea which is useful in the following discussion is the existence of limits from the left and from the right. This is particularly useful when talking about piecewise-defined functions.

If the value of $f(x)$ approaches the number p as x approaches a from the right-hand side, then it is written as $\lim\limits_{x\to a^+}f(x)=p$.

If the value of $f(x)$ approaches the number p as x approaches a from the left-hand side, then it is written as $\lim\limits_{x\to a^-}f(x)=p$.

The limit as x approaches a exists only if both the limit from the left and the limit from the right exist and are equal. Then $\lim\limits_{x\to a}f(x)=p$.

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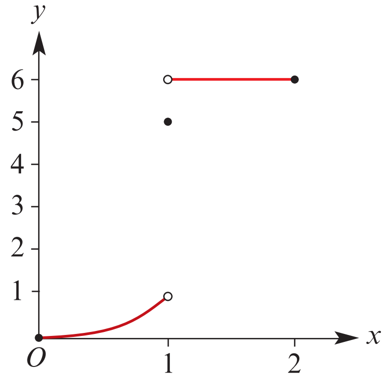
Piecewise-defined function

The following is an example of a piecewise-defined function where the limit does not exist for a particular value.



Let $f(x) = \begin{cases} x^3 & \text{if } 0 \le x < 1 \\ 5 & \text{if } x = 1 \\ 6 & \text{if } 1 < x \le 2 \end{cases}$

It is clear from the graph of f that $\lim_{x\to 1} f(x)$ does not exist. However, if x is allowed to approach 1 from the left, then $f(x)$ approaches 1. On the other hand, if x is allowed to approach 1 from the right, then $f(x)$ approaches 6. Also note that $f(1) = 5$.

Rectangular hyperbola

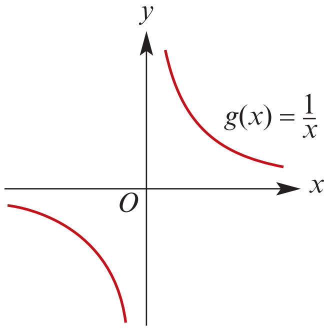
As mentioned at the start of this section, the notation of limits is used to describe the asymptotic behaviour of graphs.

First consider $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = \dfrac{1}{x^2}$. Observe that, as x approaches 0 both from the left and from the right, $f(x)$ increases without bound. The limit notation for this is $\lim_{x\to 0} f(x) = \infty$.

Now consider $g\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $g(x) = \dfrac{1}{x}$. The behaviour of $g(x)$ as x approaches 0 from the left is different from the behaviour as x approaches 0 from the right.

With limit notation this is written as:

$$\lim_{x\to 0^-} g(x) = -\infty \qquad \text{and} \qquad \lim_{x\to 0^+} g(x) = \infty$$



Now examine this function as the magnitude of x becomes very large. It can be seen that, as x increases without bound through positive values, the corresponding values of $g(x)$ approach zero. Likewise, as x decreases without bound through negative values, the corresponding values of $g(x)$ also approach zero.

Symbolically this is written as:

$$\lim_{x\to\infty} g(x) = 0^+ \qquad \text{and} \qquad \lim_{x\to -\infty} g(x) = 0^-$$

Many functions approach a limiting value or limit as x approaches ±∞.

▶ Continuity at a point

We only require an intuitive understanding of continuity.

> A function with rule $f(x)$ is said to be continuous at $x = a$ if the graph of $y = f(x)$ can be drawn through the point with coordinates $(a, f(a))$ without a break. Otherwise, there is said to be a discontinuity at $x = a$.

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We can give a more formal definition of continuity using limits. A function f is continuous at the point $x = a$ provided $f(a)$, $\lim\limits_{x \to a^+} f(x)$ and $\lim\limits_{x \to a^-} f(x)$ all exist and are equal.

We can state this equivalently as follows:

A function f is **continuous** at the point $x = a$ if the following conditions are met:

- $f(x)$ is defined at $x = a$
- $\lim\limits_{x \to a} f(x) = f(a)$

The function is **discontinuous** at a point if it is not continuous at that point.

A function is said to be **continuous everywhere** if it is continuous for all real numbers. All the polynomial functions are continuous everywhere. In contrast, the function

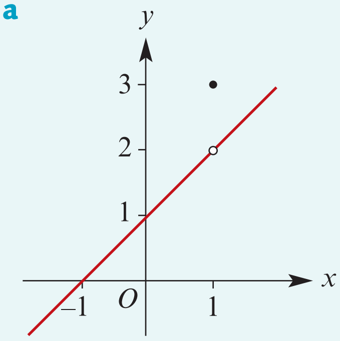
$$f(x) = \begin{cases} x^3 & \text{if } x < 1 \\ 5 & \text{if } x = 1 \\ 6 & \text{if } x > 1 \end{cases}$$

is defined for all real numbers but is not continuous at $x = 1$.

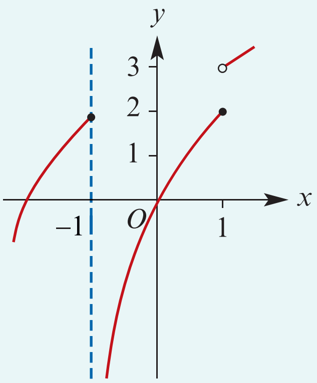
Example 43

State the values for x for which the functions shown below have a discontinuity:

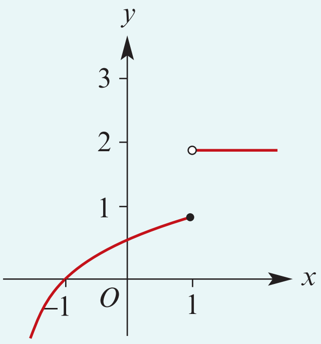
a



b



c



Solution

a Discontinuity at $x = 1$, as $f(1) = 3$ but $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^-} f(x) = 2$.

b Discontinuity at $x = -1$, as $f(-1) = 2$ and $\lim\limits_{x \to -1^-} f(x) = 2$ but $\lim\limits_{x \to -1^+} f(x) = -\infty$, and a discontinuity at $x = 1$, as $f(1) = 2$ and $\lim\limits_{x \to 1^-} f(x) = 2$ but $\lim\limits_{x \to 1^+} f(x) = 3$.

c Discontinuity at $x = 1$, as $f(1) = 1$ and $\lim\limits_{x \to 1^-} f(x) = 1$ but $\lim\limits_{x \to 1^+} f(x) = 2$.

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Example 44

For each function, state the values of x for which there is a discontinuity, and use the definition of continuity in terms of $f(a)$, $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ to explain why:

a $f(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ -2x+1 & \text{if } x < 0 \end{cases}$

b $f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -2x+1 & \text{if } x < 0 \end{cases}$

c $f(x) = \begin{cases} x & \text{if } x \leq -1 \\ x^2 & \text{if } -1 < x < 0 \\ -2x+1 & \text{if } x \geq 0 \end{cases}$

d $f(x) = \begin{cases} x^2+1 & \text{if } x \geq 0 \\ -2x+1 & \text{if } x < 0 \end{cases}$

e $f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{cases}$

Solution

a $f(0) = 0$ but $\lim_{x\to 0^-} f(x) = 1$, therefore there is a discontinuity at $x = 0$.

b $f(0) = 0$ but $\lim_{x\to 0^-} f(x) = 1$, therefore there is a discontinuity at $x = 0$.

c $f(-1) = -1$ but $\lim_{x\to -1^+} f(x) = 1$, therefore there is a discontinuity at $x = -1$.

$f(0) = 1$ but $\lim_{x\to 0^-} f(x) = 0$, therefore there is a discontinuity at $x = 0$.

d No discontinuity. **e** No discontinuity.

Section summary

- A function f is **continuous** at the point $x = a$ if the following conditions are met:
  - $f(x)$ is defined at $x = a$
  - $\lim_{x\to a} f(x) = f(a)$
- A function is **discontinuous** at a point if it is not continuous at that point.
- A function is said to be **continuous everywhere** if it is continuous for all real numbers. All the polynomial functions are continuous everywhere.
- Algebra of limits Assume that both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist.
  - $\lim_{x\to a} (f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
    That is, the limit of the sum is the sum of the limits.
  - $\lim_{x\to a} kf(x) = k \lim_{x\to a} f(x)$, where k is a given real number.
  - $\lim_{x\to a} (f(x)\, g(x)) = \lim_{x\to a} f(x) \, \lim_{x\to a} g(x)$
    That is, the limit of the product is the product of the limits.
  - $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided $\lim_{x\to a} g(x) \neq 0$.
    That is, the limit of the quotient is the quotient of the limits.

Exercise 9L

Example 41, 42

1 Find the following limits:

a $\lim_{x \to 2} 17$ **b** $\lim_{x \to 6} (x - 3)$ **c** $\lim_{x \to \frac{1}{2}} (2x - 5)$

d $\lim_{t \to -3} \dfrac{t + 2}{t - 5}$ **e** $\lim_{t \to 2} \dfrac{t^2 + 2t + 1}{t + 1}$ **f** $\lim_{x \to 0} \dfrac{(x + 2)^2 - 4}{x}$

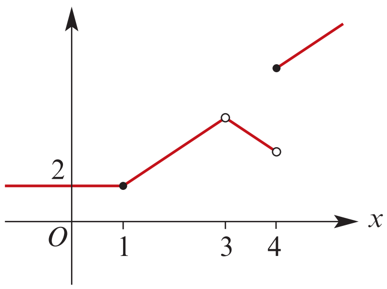
g $\lim_{t \to 1} \dfrac{t^2 - 1}{t - 1}$ **h** $\lim_{x \to 9} \sqrt{x + 3}$ **i** $\lim_{x \to 0} \dfrac{x^2 - 2x}{x}$

j $\lim_{x \to 2} \dfrac{x^3 - 8}{x - 2}$ **k** $\lim_{x \to 2} \dfrac{3x^2 - x - 10}{x^2 + 5x - 14}$ **l** $\lim_{x \to 1} \dfrac{x^2 - 3x + 2}{x^2 - 6x + 5}$

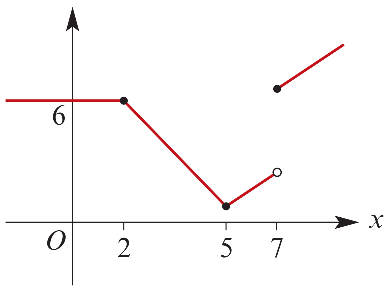
Example 43

2 For each of the following graphs, give the values of x at which a discontinuity occurs. Give reasons.

a



b



Example 44

3 For each of the following functions, state the values of x for which there is a discontinuity and use the definition of continuity in terms of $f(a)$, $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ to explain why each stated value of x corresponds to a discontinuity:

a $f(x) = \begin{cases} 3x & \text{if } x \ge 0 \\ -2x + 2 & \text{if } x < 0 \end{cases}$

b $f(x) = \begin{cases} x^2 + 2 & \text{if } x \ge 1 \\ -2x + 1 & \text{if } x < 1 \end{cases}$

c $f(x) = \begin{cases} -x & \text{if } x \le -1 \\ x^2 & \text{if } -1 < x < 0 \\ -3x + 1 & \text{if } x \ge 0 \end{cases}$

4 The rule of a particular function is given below. For what values of x is the graph of this function continuous?

$$y = \begin{cases} 2, & x < 1 \\ (x - 4)^2 - 9, & 1 \le x < 7 \\ x - 7, & x \ge 7 \end{cases}$$

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9M When is a function differentiable?

A function f is said to be **differentiable** at $x = a$ if $\lim\limits_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$ exists.

Many of the functions considered in this chapter are differentiable for their implicit domains. However, this is not true for all functions. We noted in Section 9D that $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$$

is not differentiable at $x = 0$. The gradient is -1 to the left of 0, and 1 to the right of 0.

It was shown in the previous section that some piecewise-defined functions are continuous everywhere. Similarly, some piecewise-defined functions are differentiable everywhere. The smoothness of the 'joins' determines whether this is the case.

Example 45

For the function with following rule, find $f'(x)$ and sketch the graph of $y = f'(x)$:

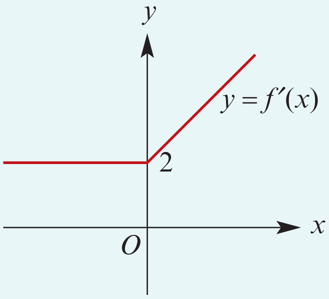
$$f(x) = \begin{cases} x^2 + 2x + 1 & \text{if } x \ge 0 \\ 2x + 1 & \text{if } x < 0 \end{cases}$$

Solution

$$f'(x) = \begin{cases} 2x + 2 & \text{if } x \ge 0 \\ 2 & \text{if } x < 0 \end{cases}$$

In particular, $f'(0)$ is defined and is equal to 2.

The two sections of the graph of $y = f(x)$ join smoothly at the point $(0, 1)$.



Example 46

For the function with the following rule, state the set of values for which the derivative is defined, find $f'(x)$ for this set of values and sketch the graph of $y = f'(x)$:

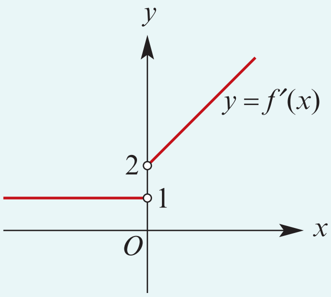
$$f(x) = \begin{cases} x^2 + 2x + 1 & \text{if } x \ge 0 \\ x + 1 & \text{if } x < 0 \end{cases}$$

Solution

$$f'(x) = \begin{cases} 2x + 2 & \text{if } x > 0 \\ 1 & \text{if } x < 0 \end{cases}$$

$f'(0)$ is not defined as the limits from the left and right are not equal.

The function f is differentiable for $\mathbb{R} \setminus \{0\}$.



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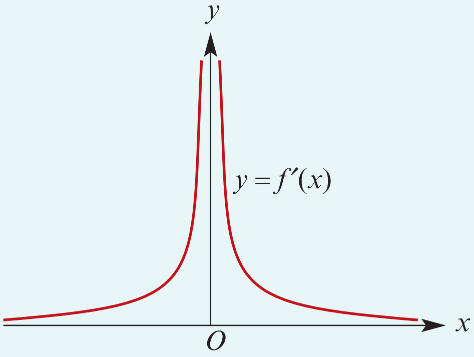
If a function is differentiable at $x = a$, then it is also continuous at $x = a$. But the converse is not true. The function f from Example 46 is continuous at $x = 0$, as $\lim_{x \to 0} f(x) = f(0)$, but f is not differentiable at $x = 0$.

Example 47

For the function with rule $f(x) = x^{\frac{1}{3}}$, state when the derivative is defined and sketch the graph of the derivative function.

Solution

By the rule for differentiating powers, $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$. The derivative is not defined at $x = 0$.



Explanation

We can also see that $f'(0)$ is not defined from first principles:

$$\frac{f(0+h) - f(0)}{h} = \frac{(0+h)^{\frac{1}{3}} - 0^{\frac{1}{3}}}{h}$$

$$= \frac{h^{\frac{1}{3}}}{h} = h^{-\frac{2}{3}}$$

But $h^{-\frac{2}{3}} \to \infty$ as $h \to 0$. Thus $\lim_{h \to 0} h^{-\frac{2}{3}}$ does not exist and so $f'(0)$ is not defined.

The function $f(x) = x^{\frac{1}{3}}$ is continuous everywhere, but not differentiable at $x = 0$.

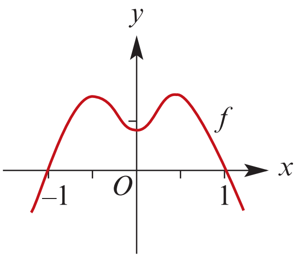
Section summary

- A function f is said to be **differentiable** at $x = a$ if $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists.
- If a function is differentiable at a point, then it is also continuous at that point.

Exercise 9M

1 In each of the following figures, the graph of a function f is given. Sketch the graph of f'. Obviously your sketch of f' cannot be exact; but $f'(x)$ should be zero at values of x for which the gradient of f is zero, and $f'(x)$ should be negative where the original graph slopes downwards, and so on.

a



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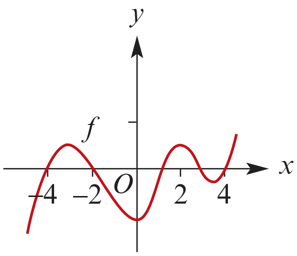

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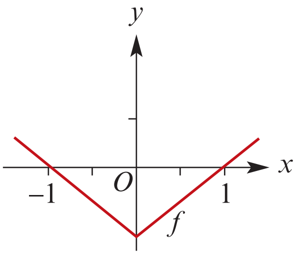
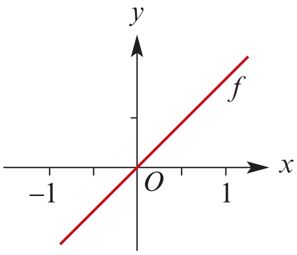
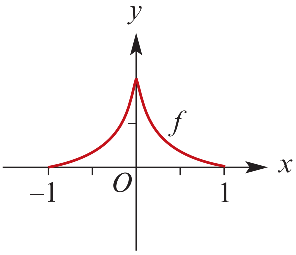


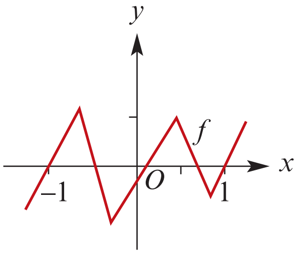
d



e



f



Example 45

2 For the function with following rule, find $f'(x)$ and sketch the graph of $y = f'(x)$:

$$f(x) = \begin{cases} -x^2 + 3x + 1 & \text{if } x \geq 0 \\ 3x + 1 & \text{if } x < 0 \end{cases}$$

Example 46

3 For the function with the following rule, state the set of values for which the derivative is defined, find $f'(x)$ for this set of values and sketch the graph of $y = f'(x)$:

$$f(x) = \begin{cases} x^2 + 2x + 1 & \text{if } x \geq 1 \\ -2x + 3 & \text{if } x < 1 \end{cases}$$

4 For the function with the following rule, state the set of values for which the derivative is defined, find $f'(x)$ for this set of values and sketch the graph of $y = f'(x)$:

$$f(x) = \begin{cases} -x^2 - 2x + 1 & \text{if } x \geq -1 \\ -2x + 3 & \text{if } x < -1 \end{cases}$$

Example 47

5 For each of the following, give the set of values for which the derivative is defined, give the derivative and sketch the graph of the derivative function:

a $f(x) = (x-1)^{\frac{1}{3}}$

b $f(x) = x^{\frac{1}{5}}$

c $f(x) = x^{\frac{2}{3}}$

d $f(x) = (x+2)^{\frac{2}{5}}$

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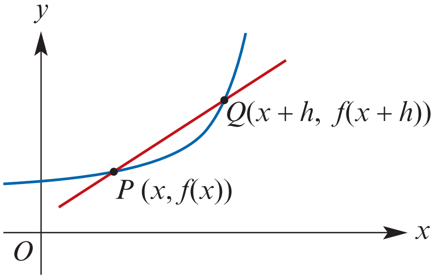
Chapter summary

Spreadsheet AS Nrich

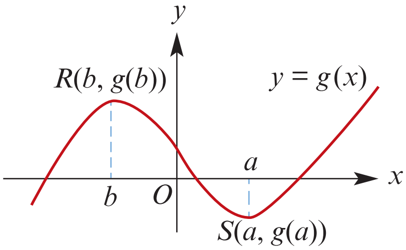
The derivative

- The notation for the limit as h approaches 0 is $\lim\limits_{h\to 0}$.
- For the graph of $y = f(x)$:
  - The gradient of the secant PQ is given by
    $$\frac{f(x+h)-f(x)}{h}$$
  - The gradient of the tangent to the graph at the point P is given by
    $$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$



- The **derivative** of the function f is denoted f' and is defined by
  $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
- At a point $(a, g(a))$ on the curve $y = g(x)$, the gradient is $g'(a)$.

  For the graph shown:
  - $g'(x) > 0$ for $x < b$ and for $x > a$
  - $g'(x) < 0$ for $b < x < a$
  - $g'(x) = 0$ for $x = b$ and for $x = a$.



Basic derivatives

$f(x)$	$f'(x)$	
c	0	where c is a constant
x^n	nx^{n-1}	where n is a non-zero integer
x^a	ax^{a-1}	where $a \in \mathbb{R} \setminus \{0\}$
e^{kx}	ke^{kx}	
$\log_e(kx)$	$\frac{1}{x}$	
$\sin(kx)$	$k\cos(kx)$	
$\cos(kx)$	$-k\sin(kx)$	
$\tan(kx)$	$k\sec^2(kx)$	

Rules for differentiation

- For $f(x) = k\,g(x)$, where k is a constant, $f'(x) = k\,g'(x)$.
  That is, the derivative of a number multiple is the multiple of the derivative.
- For $f(x) = g(x) + h(x)$, $f'(x) = g'(x) + h'(x)$.
  That is, the derivative of a sum is the sum of the derivatives.

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- The chain rule
  - If $q(x) = f(g(x))$, then $q'(x) = f'(g(x))\,g'(x)$
  - $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$
- The product rule
  - If $F(x) = f(x) \cdot g(x)$, then $F'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$
  - If $y = uv$, then $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
- The quotient rule
  - If $F(x) = \dfrac{f(x)}{g(x)}$, then $F'(x) = \dfrac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}$
  - If $y = \dfrac{u}{v}$, then $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

Algebra of limits

- $\lim\limits_{x\to a}(f(x) + g(x)) = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x)$

  That is, the limit of the sum is the sum of the limits.
- $\lim\limits_{x\to a} kf(x) = k\lim\limits_{x\to a} f(x)$, where k is a real number.
- $\lim\limits_{x\to a}(f(x)\,g(x)) = \lim\limits_{x\to a} f(x)\ \lim\limits_{x\to a} g(x)$

  That is, the limit of the product is the product of the limits.
- $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$, provided $\lim\limits_{x\to a} g(x) \neq 0$.

  That is, the limit of the quotient is the quotient of the limits.

Continuity and differentiability

- A function f is **continuous** at the point $x = a$ if:
  - $f(x)$ is defined at $x = a$
  - $\lim\limits_{x\to a} f(x) = f(a)$
- A function is **discontinuous** at a point if it is not continuous at that point.
- A function f is **differentiable** at the point $x = a$ if $\lim\limits_{h\to 0}\dfrac{f(a+h) - f(a)}{h}$ exists.

Technology-free questions

1 For $y = x^2 + 1$:

a Find the average rate of change of y with respect to x over the interval $[3, 5]$.

b Find the instantaneous rate of change of y with respect to x at the point where $x = -4$.

2 Differentiate each of the following with respect to x:

a $x + \sqrt{1 - x^2}$ **b** $\dfrac{4x + 1}{x^2 + 3}$ **c** $\sqrt{1 + 3x}$ **d** $\dfrac{2 + \sqrt{x}}{x}$

e $(x - 9)\sqrt{x - 3}$ **f** $x\sqrt{1 + x^2}$ **g** $\dfrac{x^2 - 1}{x^2 + 1}$ **h** $\dfrac{x}{x^2 + 1}$

i $(2 + 5x^2)^{\frac{1}{3}}$ **j** $\dfrac{2x + 1}{x^2 + 2}$ **k** $(3x^2 + 2)^{\frac{2}{3}}$

3 For each of the following functions, find the gradient of the tangent to the curve at the point corresponding to the given x-value:

a $y = 3x^2 - 4$ at $x = -1$ **b** $y = \dfrac{x - 1}{x^2 + 1}$ at $x = 0$

c $y = (x - 2)^5$ at $x = 1$ **d** $y = (2x + 2)^{\frac{1}{3}}$ at $x = 3$

4 Differentiate each of the following with respect to x:

a $\log_e(x + 2)$ **b** $\sin(3x + 2)$ **c** $\cos\left(\dfrac{x}{2}\right)$ **d** $e^{x^2 - 2x}$

e $\log_e(3 - x)$ **f** $\sin(2\pi x)$ **g** $\sin^2(3x + 1)$ **h** $\sqrt{\log_e x}, \ x > 1$

i $\dfrac{2\log_e(2x)}{x}$ **j** $x^2 \sin(2\pi x)$

5 Differentiate each of the following with respect to x:

a $e^x \sin(2x)$ **b** $2x^2 \log_e x$ **c** $\dfrac{\log_e x}{x^3}$ **d** $\sin(2x)\cos(3x)$

e $\dfrac{\sin(2x)}{\cos(2x)}$ **f** $\cos^3(3x + 2)$ **g** $x^2 \sin^2(3x)$

6 Find the gradient of each of the following curves at the stated value of x:

a $y = e^{2x} + 1, \ x = 1$ **b** $y = e^{x^2+1}, \ x = 0$

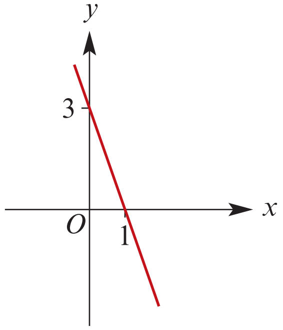
c $y = 5e^{3x} + x^2, \ x = 1$ **d** $y = 5 - e^{-x}, \ x = 0$

7 Differentiate each of the following with respect to x:

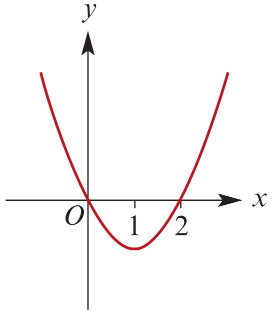
a e^{ax} **b** e^{ax+b} **c** e^{a-bx} **d** $be^{ax} - ae^{bx}$ **e** $\dfrac{e^{ax}}{e^{bx}}$

8 Sketch the graph of the derivative function for each of the following functions:

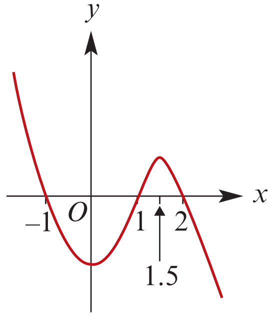
a



b



c



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9 Find the derivative of $\left(4x + \frac{9}{x}\right)^2$ and find the values of x at which the derivative is zero.

10 **a** For $y = \frac{2x - 3}{x^2 + 4}$, show that $\frac{dy}{dx} = \frac{8 + 6x - 2x^2}{(x^2 + 4)^2}$.

b Find the values of x for which both y and $\frac{dy}{dx}$ are positive.

11 Find the derivative of each of the following, given that the function f is differentiable for all real numbers:

a $xf(x)$ **b** $\frac{1}{f(x)}$ **c** $\frac{x}{f(x)}$ **d** $\frac{x^2}{[f(x)]^2}$

12 Let $f(x) = 2x^3 - 1$ and $g(x) = \cos x$.

a Find the rule for $f \circ g$. **b** Find the rule for $g \circ f$. **c** Find the rule for $g' \circ f$.

d Find the rule for $(g \circ f)'$. **e** Find $f'\left(g\left(\frac{\pi}{3}\right)\right)$. **f** Find $(f \circ g)'\left(\frac{\pi}{3}\right)$.

Multiple-choice questions

1 The average rate of change of the function with rule $f(x) = e^x + x^3$ for $x \in [0, 1]$ is

A e **B** $e^3 + 1$ **C** $\frac{e^3 + 1}{2}$ **D** $e + 1$ **E** $e^x + 3x^2$

2 If $f(x) = \frac{4x^4 - 12x^2}{3x}$, then $f'(x)$ is equal to

A $\frac{16x^3 - 24x}{3}$ **B** $4x^2 - 4$ **C** $\frac{16x^3 - 24x}{3x}$ **D** $4x^2 - 8x$ **E** $\frac{8x^2 - 16x}{3x}$

3 If $f: \mathbb{R} \setminus \{7\} \to \mathbb{R}$ where $f(x) = 5 + \frac{5}{(7 - x)^2}$, then $f'(x) > 0$ for

A $x \in \mathbb{R} \setminus \{7\}$ **B** $x \in \mathbb{R}$ **C** $x < 7$ **D** $x > 7$ **E** $x > 5$

4 Let $y = f(g(x))$ where $g(x) = 2x^4$. Then $\frac{dy}{dx}$ is equal to

A $8x^3 f'(2x^4)$ **B** $8x^2 f(4x^3)$ **C** $8x^4 f(x) f'(x^3)$

D $2f(x) f'(x^3)$ **E** $8x^3$

5 Which of the following is *not true* for the curve of $y = f(x)$ where $f(x) = x^{\frac{1}{3}}$?

A The gradient is defined for all real numbers.

B The curve passes through the origin.

C The curve passes through the points with coordinates $(1, 1)$ and $(-1, -1)$.

D For $x > 0$, the gradient is positive.

E For $x > 0$, the gradient is decreasing.

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6 The graph of the function with rule $y = \dfrac{k}{2(x^3 + 1)}$ has gradient 1 when $x = 1$. The value of k is

A 1 **B** $\dfrac{-8}{3}$ **C** $\dfrac{-1}{2}$ **D** -4 **E** $\dfrac{-1}{4}$

7 For the graph shown, the gradient is positive for

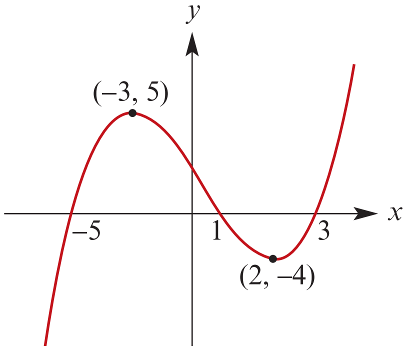
A $-3 < x < 2$

B $-3 \le x \le 2$

C $x < -3$ or $x > 2$

D $x \le -3$ or $x \ge 2$

E $-3 \le x \le 3$



8 For the function $f(x) = 4x(2 - 3x)$, $f'(x) < 0$ for

A $x < \dfrac{1}{3}$ **B** $0 < x < \dfrac{2}{3}$ **C** $x = \dfrac{1}{3}$ **D** $x > \dfrac{1}{3}$ **E** $x = 0, \dfrac{2}{3}$

9 If $y = \sqrt{3 - 2f(x)}$, then $\dfrac{dy}{dx}$ is equal to

A $\dfrac{2f'(x)}{\sqrt{3 - 2f(x)}}$ **B** $\dfrac{-1}{2\sqrt{3 - 2f(x)}}$ **C** $\dfrac{1}{2}\sqrt{3 - 2f'(x)}$

D $\dfrac{3}{2(3 - 2f'(x))}$ **E** $\dfrac{-f'(x)}{\sqrt{3 - 2f(x)}}$

10 The point on the curve defined by the equation $y = (x + 3)(x - 2)$ at which the gradient is -7 has coordinates

A $(-4, 6)$ **B** $(-4, 0)$ **C** $(-3, 0)$ **D** $(-3, -5)$ **E** $(-2, 0)$

11 The function $y = ax^2 - bx$ has zero gradient only for $x = 2$. The x-axis intercepts of the graph of this function are

A $\frac{1}{2}, -\frac{1}{2}$ **B** $0, 4$ **C** $0, -4$ **D** $0, \frac{1}{2}$ **E** $0, -\frac{1}{2}$

12 The derivative of $e^{-2ax}\cos(ax)$ with respect to x is

A $-ae^{-2ax}\cos(ax) - 2ae^{-2ax}\sin(ax)$ **B** $ae^{-2ax}\cos(ax) - 2ae^{-2ax}\sin(ax)$

C $-2ae^{-2ax}\cos(ax) - ae^{-2ax}\sin(ax)$ **D** $2ae^{-2ax}\cos(ax) + 2ae^{-2ax}\sin(ax)$

E $-ae^{-2ax}\cos(ax) - 2ae^{2ax}\sin(ax)$

13 If $f(x) = \dfrac{\cos x}{x - a}$, where a is a constant, then $f'(x)$ is

A $\dfrac{\sin x}{x - a} + \dfrac{\cos x}{(x - a)^2}$ **B** $-\dfrac{\sin x}{x - a} - \dfrac{\cos x}{(x - a)^2}$ **C** $\dfrac{\sin x}{x - a} - \dfrac{\cos x}{(x - a)^2}$

D $\dfrac{x\sin x}{x - a} + \dfrac{x\cos x}{(x - a)^2}$ **E** $\dfrac{\sin x}{x} - \dfrac{\cos x}{x}$

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Extended-response questions

1 **a** For functions f and g, which are defined and differentiable for all real numbers, it is known that:

- $f(1) = 6,\ g(1) = -1,\ g(6) = 7$ and $f(-1) = 8$
- $f'(1) = 6,\ g'(1) = -2,\ f'(-1) = 2$ and $g'(6) = -1$

Find:

i $(f \circ g)'(1)$ **ii** $(g \circ f)'(1)$ **iii** $(fg)'(1)$ **iv** $(gf)'(1)$ **v** $\left(\frac{f}{g}\right)'(1)$ **vi** $\left(\frac{g}{f}\right)'(1)$

b It is known that f is a cubic function with rule $f(x) = ax^3 + bx^2 + cx + d$. Find the values of a, b, c and d.

2 For a function f, which is differentiable for $\mathbb{R}$, it is known that:

- $f'(x) = 0$ for $x = 1$ and $x = 5$
- $f'(x) > 0$ for $x > 5$ and $x < 1$
- $f'(x) < 0$ for $1 < x < 5$
- $f(1) = 6$ and $f(5) = 1$

a For $y = f(x + 2)$, find the values of x for which:

i $\frac{dy}{dx} = 0$ **ii** $\frac{dy}{dx} > 0$

b Find the coordinates of the points on the graph of $y = f(x - 2)$ where $\frac{dy}{dx} = 0$.

c Find the coordinates of the points on the graph of $y = f(2x)$ where $\frac{dy}{dx} = 0$.

d Find the coordinates of the points on the graph of $y = f\left(\frac{x}{2}\right)$ where $\frac{dy}{dx} = 0$.

e Find the coordinates of the points on the graph of $y = 3f\left(\frac{x}{2}\right)$ where $\frac{dy}{dx} = 0$.

3 Let $f(x) = (x - \alpha)^n(x - \beta)^m$, where m and n are positive integers with $m > n$ and $\beta > \alpha$.

a Solve the equation $f(x) = 0$ for x.

b Find $f'(x)$.

c Solve the equation $f'(x) = 0$ for x.

d **i** If m and n are odd, find the set of values for which $f'(x) > 0$.

ii If m is odd and n is even, find the set of values for which $f'(x) > 0$.

4 Consider the function with rule $f(x) = \frac{x^n}{1 + x^n}$, where n is an even positive integer.

a Show that $f(x) = 1 - \frac{1}{x^n + 1}$.

b Find $f'(x)$.

c Show that $0 \le f(x) < 1$ for all x.

d State the set of values for which $f'(x) = 0$.

e State the set of values for which $f'(x) > 0$.

f Show that f is an even function.

10

Applications of differentiation

Objectives

- To be able to find the equations of the **tangent** and the **normal** at a given point on a curve.
- To be able to find the **stationary points** on the curves of certain polynomial functions and state the nature of such points.
- To use differentiation techniques to **sketch graphs**.
- To solve **maximum and minimum problems**.
- To use the derivative of a function in **rates of change** problems.

In this chapter we continue our study of differential calculus. There are two main aspects of this chapter. One is to apply our knowledge of the derivative to sketching graphs and solving maximum and minimum problems. The other is to see that the derivative can be used to define instantaneous rate of change.

The new techniques for sketching graphs of polynomial functions are a useful addition to the skills that were introduced in Chapter 4. At that stage, rather frustratingly, we were only able to determine the coordinates of turning points of cubic and quartic functions using technology. The new techniques are also used for determining maximum or minimum values for problems set in a 'real world' context.

The use of the derivative to determine instantaneous rates of change is a very important application of calculus. One of the first areas of applied mathematics to be studied in the seventeenth century was motion in a straight line. The problems of kinematics were the motivation for Newton's work on calculus.

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10A Tangents and normals

The derivative of a function is a new function that gives the measure of the gradient of the tangent at each point on the curve. Having the gradient, we can find the equation of the tangent line at a given point on the curve.

Suppose that (x_1, y_1) is a point on the curve $y = f(x)$. Then, if f is differentiable at $x = x_1$, the equation of the tangent at (x_1, y_1) is given by

$$y - y_1 = f'(x_1)(x - x_1)$$

Example 1

Find the equation of the tangent to the curve $y = x^3 + \frac{1}{2}x^2$ at the point $x = 1$.

Solution

When $x = 1$, $y = \frac{3}{2}$, and so $\left(1, \frac{3}{2}\right)$ is a point on the tangent.

Since $\frac{dy}{dx} = 3x^2 + x$, the gradient of the tangent at $x = 1$ is 4.

Hence the equation of the tangent is

$$y - \frac{3}{2} = 4(x - 1)$$

i.e. $$y = 4x - \frac{5}{2}$$

The **normal** to a curve at a point on the curve is the line that passes through the point and is perpendicular to the tangent at that point.

Recall from Chapter 2 that two lines with gradients m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$.

Thus, if a tangent has gradient m, the normal has gradient $-\frac{1}{m}$.

Example 2

Find the equation of the normal to the curve with equation $y = x^3 - 2x^2$ at the point $(1, -1)$.

Solution

The point $(1, -1)$ is on the normal.

Since $\frac{dy}{dx} = 3x^2 - 4x$, the gradient of the normal at $x = 1$ is $\frac{-1}{-1} = 1$.

Hence the equation of the normal is

$$y - (-1) = 1(x - 1)$$

i.e. $$y = x - 2$$

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Example 3

Find the equation of the tangent to the curve with equation $y = x^{\frac{3}{2}} - 4x^{\frac{1}{2}}$ at the point on the graph where $x = 4$.

Solution

Let $y = x^{\frac{3}{2}} - 4x^{\frac{1}{2}}$. Then $\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} - 2x^{-\frac{1}{2}}$.

When $x = 4$,

$$y = 4^{\frac{3}{2}} - 4 \times 4^{\frac{1}{2}} = 0$$

and $$\frac{dy}{dx} = \frac{3}{2} \times 4^{\frac{1}{2}} - 2 \times 4^{-\frac{1}{2}} = 2$$

Hence the equation of the tangent is

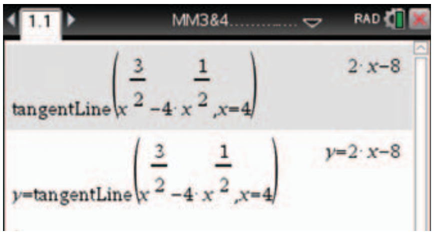
$$y - 0 = 2(x - 4)$$

i.e. $$y = 2x - 8$$

Using the TI-Nspire

Use (menu) > **Calculus** > **Tangent Line** and complete as shown.

Note: The equation of the tangent can also be found in a **Graphs** application.

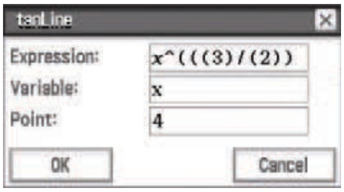


$\text{tangentLine}\left(x^{\frac{3}{2}} - 4\cdot x^{\frac{1}{2}}, x=4\right)$ $2\cdot x-8$

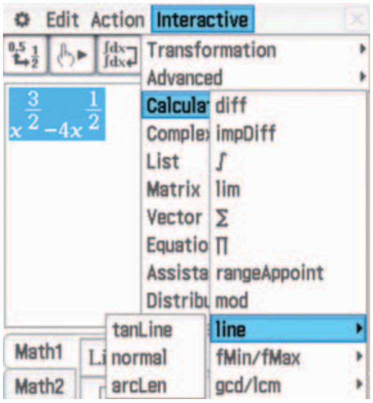
$y\text{=tangentLine}\left(x^{\frac{3}{2}} - 4\cdot x^{\frac{1}{2}}, x=4\right)$ $y=2\cdot x-8$

Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, enter and highlight the expression $x^{\frac{3}{2}} - 4x^{\frac{1}{2}}$.
- Go to **Interactive** > **Calculation** > **line** > **tanLine**.
- Enter the x-value 4 in the tanLine window and tap OK.


tanLine
OK
Cancel

Expression: x^(((3)/(2))
Variable: x
Point: 4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Equatio Π
Assista rangeAppoint
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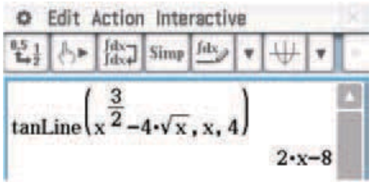
tanLine line
Math1 Li normal fMin/fMax
Math2 arcLen gcd/lcm
$x^{\frac{3}{2}}-4x^{\frac{1}{2}}$

- Write your answer as an equation: $y = 2x - 8$.

Note: You can also obtain the tangent line by sketching the graph and using **Analysis** > **Sketch** > **Tangent**.


Edit Action Interactive
Simp

$\text{tanLine}\left(x^{\frac{3}{2}} - 4\cdot\sqrt{x}, x, 4\right)$

$2\cdot x-8$

Example 4

Find the equation of the tangent to the graph of $y = \sin x$ at the point where $x = \frac{\pi}{3}$.

Solution

Let $y = \sin x$. Then $\frac{dy}{dx} = \cos x$. When $x = \frac{\pi}{3}$, $y = \frac{\sqrt{3}}{2}$ and $\frac{dy}{dx} = \frac{1}{2}$.

Therefore the equation of the tangent is

$$y - \frac{\sqrt{3}}{2} = \frac{1}{2}\left(x - \frac{\pi}{3}\right)$$

i.e. $$y = \frac{x}{2} - \frac{\pi}{6} + \frac{\sqrt{3}}{2}$$

Example 5

Find the equations of the tangent and normal to the graph of $y = -\cos x$ at the point $\left(\frac{\pi}{2}, 0\right)$.

Solution

First find the gradient of the curve at this point:

$\frac{dy}{dx} = \sin x$ and so, when $x = \frac{\pi}{2}$, $\frac{dy}{dx} = 1$.

The equation of the tangent is

$$y - 0 = 1\left(x - \frac{\pi}{2}\right)$$

i.e. $$y = x - \frac{\pi}{2}$$

The gradient of the normal is -1 and therefore the equation of the normal is

$$y - 0 = -1\left(x - \frac{\pi}{2}\right)$$

i.e. $$y = -x + \frac{\pi}{2}$$

The following example shows two situations in which we can view a graph as having a 'vertical tangent line' at a point where the derivative is not defined.

Example 6

Find the equation of the tangent to:

a $f(x) = x^{\frac{1}{3}}$ where $x = 0$ **b** $f(x) = x^{\frac{2}{3}}$ where $x = 0$.

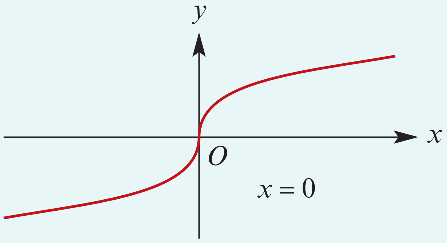
Solution

a The derivative of f is not defined at $x = 0$.

For $x \in \mathbb{R} \setminus \{0\}$, $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$.

It is clear that f is continuous at $x = 0$ and that $f'(x) \to \infty$ as $x \to 0$.

The graph has a **vertical tangent** at $x = 0$.



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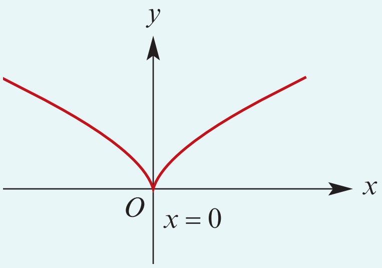
b $f(x) = x^{\frac{2}{3}}$

The derivative of f is not defined at $x = 0$.

For $x \in \mathbb{R} \setminus \{0\}$, $f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$.

It is clear that f is continuous at $x = 0$ and that $f'(x) \to \infty$ as $x \to 0^+$ and $f'(x) \to -\infty$ as $x \to 0^-$.

There is a **cusp** at $x = 0$, and the graph of $y = f(x)$ has a **vertical tangent** at $x = 0$.



Section summary

- **Equation of a tangent** Suppose (x_1, y_1) is a point on the curve $y = f(x)$. Then, if f is differentiable at $x = x_1$, the equation of the tangent to the curve at (x_1, y_1) is given by $y - y_1 = f'(x_1)(x - x_1)$.
- **Gradient of normal** If a tangent has gradient m, the normal has gradient $-\frac{1}{m}$.

Exercise 10A

Example 1 **1** Find the equation of the tangent to the curve $y = x^2 - 1$ at the point $(2, 3)$.

Example 2 **2** Find the equation of the normal to the curve $y = x^2 + 3x - 1$ at the point where the curve cuts the y-axis.

3 Find the equations of the normals to the curve $y = x^2 - 5x + 6$ at the points where it cuts the x-axis.

4 Find the equations of the tangent and the normal to the curve $y = (2x + 1)^9$ at the point $(0, 1)$.

5 Find the coordinates of the point on $y = x^2 - 5$ at which the curve has gradient 3. Hence find the value of c for which the line $y = 3x + c$ is tangent to $y = x^2 - 5$.

6 Find the equations of **i** the tangent and **ii** the normal at the point corresponding to the given x-value on each of the following curves:

a $y = x^2 - 2;\ x = 1$

b $y = x^2 - 3x - 1;\ x = 0$

c $y = \frac{1}{x};\ x = -1$

d $y = (x - 2)(x^2 + 1);\ x = -1$

e $y = \sqrt{3x + 1};\ x = 0$

f $y = \sqrt{x};\ x = 1$

g $y = x^{\frac{2}{3}} + 1;\ x = 1$

h $y = x^3 - 8x;\ x = 2$

i $y = x^3 - 3x^2 + 2;\ x = 2$

j $y = 2x^3 + x^2 - 4x + 1;\ x = 1$

Example 3 **7** Use a CAS calculator to find the equation of the tangent to the curve with equation $y = 4x^{\frac{5}{2}} - 8x^{\frac{3}{2}}$ at the point on the graph where $x = 4$.

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8 Find the equation of the tangent at the point corresponding to the given x-value on each of the following curves:

a $y = \dfrac{x^2 - 1}{x^2 + 1};\ x = 0$ **b** $y = \sqrt{3x^2 + 1};\ x = 1$

c $y = \dfrac{1}{2x - 1};\ x = 0$ **d** $y = \dfrac{1}{(2x - 1)^2};\ x = 1$

Example 4, 5

9 Find the equation of the tangent to each of the following curves at the given x-value:

a $y = \sin(2x);\ x = 0$ **b** $y = \cos(2x);\ x = \dfrac{\pi}{2}$ **c** $y = \tan x;\ x = \dfrac{\pi}{4}$

d $y = \tan(2x);\ x = 0$ **e** $y = \sin x + x\sin(2x);\ x = 0$ **f** $y = x - \tan x;\ x = \dfrac{\pi}{4}$

10 For each function, find the equation of the tangent to the graph at the given value of x:

a $f(x) = e^x + e^{-x};\ x = 0$ **b** $f(x) = \dfrac{e^x - e^{-x}}{2};\ x = 0$

c $f(x) = x^2e^{2x};\ x = 1$ **d** $f(x) = e^{\sqrt{x}};\ x = 1$

e $f(x) = xe^{x^2};\ x = 1$ **f** $f(x) = x^2e^{-x};\ x = 2$

11 **a** Find the equation of the tangent and the normal to the graph of $f(x) = \log_e x$ at the point $(1, 0)$.

b Find the equation of the tangent to the graph of $f(x) = \log_e(2x)$ at the point $\left(\dfrac{1}{2}, 0\right)$.

c Find the equation of the tangent to the graph of $f(x) = \log_e(kx)$ at the point $\left(\dfrac{1}{k}, 0\right)$, where $k \in \mathbb{R}^+$.

Example 6

12 Find the equation of the tangent at the point where $y = 0$ for each of the following curves:

a $y = x^{\frac{1}{5}}$ **b** $y = x^{\frac{3}{5}}$ **c** $y = (x - 4)^{\frac{1}{3}}$

d $y = (x + 5)^{\frac{2}{3}}$ **e** $y = (2x + 1)^{\frac{1}{3}}$ **f** $y = (x + 5)^{\frac{4}{5}}$

13 The tangent to the curve with equation $y = \tan(2x)$ at the point where $x = \dfrac{\pi}{8}$ meets the y-axis at the point A. Find the distance OA, where O is the origin.

14 The tangent to the curve with equation $y = 2e^x$ at the point $(a, 2e^a)$ passes through the origin. Find the value of a.

15 The tangent to the curve with equation $y = \log_e x$ at the point $(a, \log_e a)$ passes through the origin. Find the value of a.

16 The tangent to the curve with equation $y = x^2 + 2x$ at the point $(a, a^2 + 2a)$ passes through the origin. Find the value of a.

17 The tangent to the curve with equation $y = x^3 + x$ at the point $(a, a^3 + a)$ passes through the point $(1, 1)$. Find the value of a.

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10B Rates of change

The derivative was defined geometrically in the previous chapter. However, the process of differentiation may also be used to tackle many kinds of problems involving rates of change.

For the function with rule $f(x)$:

- The **average rate of change** for $x \in [a, b]$ is given by $\dfrac{f(b) - f(a)}{b - a}$.
- The **instantaneous rate of change** of f with respect to x when $x = a$ is given by $f'(a)$.

The derivative $\dfrac{dy}{dx}$ gives the instantaneous rate of change of y with respect to x.

- If $\dfrac{dy}{dx} > 0$, then y is increasing as x increases.
- If $\dfrac{dy}{dx} < 0$, then y is decreasing as x increases.

Example 7

For the function with rule $f(x) = x^2 + 2x$, find:

a the average rate of change for $x \in [2, 3]$

b the average rate of change for the interval $[2, 2 + h]$

c the instantaneous rate of change of f with respect to x when $x = 2$.

Solution

a Average rate of change $= \dfrac{f(3) - f(2)}{3 - 2} = 15 - 8 = 7$

b
$$\begin{aligned}\text{Average rate of change} &= \frac{f(2+h) - f(2)}{2 + h - 2}\\ &= \frac{(2+h)^2 + 2(2+h) - 8}{h}\\ &= \frac{4 + 4h + h^2 + 4 + 2h - 8}{h}\\ &= \frac{6h + h^2}{h} = 6 + h\end{aligned}$$

c The derivative is $f'(x) = 2x + 2$. When $x = 2$, the instantaneous rate of change is $f'(2) = 6$. This can also be seen from the result of part **b**.

Example 8

A balloon develops a microscopic leak and gradually decreases in volume. Its volume, V cm^3, at time t seconds is $V = 600 - 10t - \dfrac{1}{100}t^2$, $t \geq 0$.

a Find the rate of change of volume after:

i 10 seconds **ii** 20 seconds

b For how long could the model be valid?

Solution

a $\dfrac{dV}{dt} = -10 - \dfrac{t}{50}$

i When $t = 10$, $\dfrac{dV}{dt} = -10\frac{1}{5}$

i.e. the volume is decreasing at a rate of $10\frac{1}{5}$ cm^3 per second.

ii When $t = 20$, $\dfrac{dV}{dt} = -10\frac{2}{5}$

i.e. the volume is decreasing at a rate of $10\frac{2}{5}$ cm^3 per second.

b The model will not be meaningful when $V < 0$. Consider $V = 0$.

$$600 - 10t - \frac{1}{100}t^2 = 0$$

$$\therefore \quad t = 100(\sqrt{31} - 5) \quad \text{or} \quad t = -100(\sqrt{31} + 5)$$

The model may be suitable for $0 \le t \le 100(\sqrt{31} - 5)$.

Example 9

A pot of liquid is put on the stove. When the temperature of the liquid reaches 80°C, the pot is taken off the stove and placed on the kitchen bench. The temperature in the kitchen is 20°C. The temperature of the liquid, T°C, at time t minutes is given by

$$T = 20 + 60e^{-0.3t}$$

a Find the rate of change of temperature with respect to time in terms of T.

b Find the rate of change of temperature with respect to time when:

i $T = 80$ **ii** $T = 30$

Solution

a By rearranging $T = 20 + 60e^{-0.3t}$, we see that $e^{-0.3t} = \dfrac{T - 20}{60}$.

$$\text{Now} \quad T = 20 + 60e^{-0.3t}$$

$$\therefore \quad \frac{dT}{dt} = -18e^{-0.3t}$$

$$\begin{aligned} \text{Hence} \quad \frac{dT}{dt} &= -18\left(\frac{T - 20}{60}\right) \\ &= -3\left(\frac{T - 20}{10}\right) \\ &= 0.3(20 - T) \end{aligned}$$

b **i** When $T = 80$, $\dfrac{dT}{dt} = 0.3(20 - 80) = -18$

The liquid is cooling at a rate of 18°C per minute.

ii When $T = 30$, $\dfrac{dT}{dt} = 0.3(20 - 30) = -3$

The liquid is cooling at a rate of 3°C per minute.

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Motion in a straight line

Position, velocity and acceleration were introduced for an object moving in a straight line in Mathematical Methods Units 1 & 2.

Position (x m) is specified with respect to a reference point O on the line. Velocity (v m/s) and acceleration (a m/s^2) are given by:

$$\text{velocity} \quad v = \frac{dx}{dt} \qquad \text{acceleration} \quad a = \frac{dv}{dt}$$

Example 10

A particle moves along a straight line such that its position, x m, relative to a point O at time t seconds is given by the formula $x = t^3 - 6t^2 + 9t$. Find:

a at what times and in what positions the particle will have zero velocity

b its acceleration at those instants

c its velocity when its acceleration is zero.

Solution

Velocity $v = \dfrac{dx}{dt} = 3t^2 - 12t + 9$

a When $v = 0$,

$$\begin{aligned} 3(t^2 - 4t + 3) &= 0 \\ (t-1)(t-3) &= 0 \\ t = 1 \text{ or } t &= 3 \end{aligned}$$

i.e. the velocity is zero when $t = 1$ and $t = 3$ and where $x = 4$ and $x = 0$.

b Acceleration $a = \dfrac{dv}{dt} = 6t - 12$

When $t = 1$, $a = -6$ m/s^2
When $t = 3$, $a = 6$ m/s^2

c The acceleration is zero when $6t - 12 = 0$, i.e. when $t = 2$.
When $t = 2$, the velocity $v = 3 \times 4 - 24 + 9$

$$= -3 \text{ m/s}$$

Section summary

For the function with rule $f(x)$:

- The average rate of change for $x \in [a, b]$ is given by $\dfrac{f(b) - f(a)}{b - a}$.
- The instantaneous rate of change of f with respect to x when $x = a$ is given by $f'(a)$.

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Exercise 10B

Skillsheet

Example 7

1 For the function with rule $f(x) = 3x^2 + 6x$, find:

a the average rate of change for $x \in [2, 3]$

b the average rate of change for the interval $[2, 2 + h]$

c the instantaneous rate of change of f with respect to x when $x = 2$.

2 Express each of the following in symbols:

a the rate of change of volume (V) with respect to time (t)

b the rate of change of surface area (S) of a sphere with respect to radius (r)

c the rate of change of volume (V) of a cube with respect to edge length (x)

d the rate of change of area (A) with respect to time (t)

e the rate of change of volume (V) of water in a glass with respect to depth of water (h)

Example 8

3 If your interest (I) in Mathematical Methods can be expressed as

$$I = \frac{4}{(t+1)^2}$$

where t is the time in days measured from the first day of Term 1, how fast is your interest waning when $t = 10$?

4 A reservoir is being emptied and the quantity of water, V m^3, remaining in the reservoir t days after it starts to empty is given by

$$V(t) = 10^3(90 - t)^3$$

a At what rate is the reservoir being emptied at time t?

b How long does it take to empty the reservoir?

c What is the volume of water in the reservoir when $t = 0$?

d After what time is the reservoir being emptied at 3×10^5 m^3/day?

e Sketch the graph of $V(t)$ against t.

f Sketch the graph of $V'(t)$ against t.

5 A coffee percolator allows 1000 mL of water to flow into a filter in 20 minutes. The volume which has flowed into the filter at time t minutes is given by

$$V(t) = \frac{1}{160}\left(5t^4 - \frac{t^5}{5}\right), \quad 0 \le t \le 20$$

a At what rate is the water flowing into the filter at time t minutes?

b Sketch the graph of $\frac{dV}{dt}$ against t for $0 \le t \le 20$.

c When is the rate of flow greatest?

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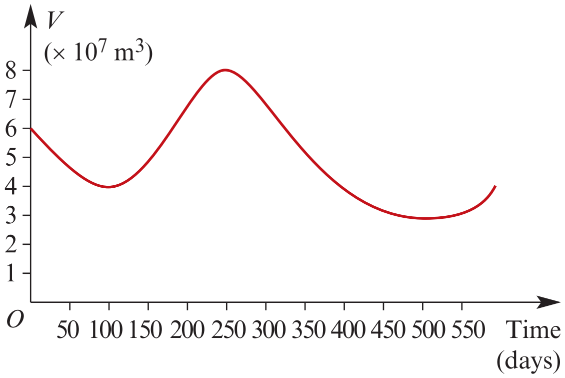
6 The graph shows the volume, V m^3, of water in a reservoir at time t days.



- **a** At what times is the rate of flow from the reservoir 0 m^3/day?
- **b** Find an estimate for the rate of flow at $t = 200$.
- **c** Find the average rate of flow for the interval $[100, 250]$.
- **d** State the times for which there is net flow into the reservoir.

7 A car tyre is inflated to a pressure of 30 units. Eight hours later it is found to have deflated to a pressure of 10 units. The pressure, P, at time t hours is given by

$$P = P_0 e^{-\lambda t}$$

- **a** Find the values of P_0 and λ.
- **b** At what time would the pressure be 8 units?
- **c** Find the rate of loss of pressure at:
  - **i** time $t = 0$
  - **ii** time $t = 8$

Example 9

8 A liquid is heated to a temperature of 90°C and then allowed to cool in a room in which the temperature is 15°C. While the liquid is cooling, its temperature, T°C, at time t minutes is given by $T = 15 + 75e^{-0.3t}$.

- **a** Find the rate of change of temperature with respect to time in terms of T.
- **b** Find the rate of change of temperature with respect to time when:
  - **i** $T = 90$
  - **ii** $T = 60$
  - **iii** $T = 30$

9 If $y = 3x + 2\cos x$, find $\dfrac{dy}{dx}$ and hence show that y increases as x increases.

10 The volume of water in a reservoir at time t is given by $V(t) = 3 + 2\sin\left(\dfrac{t}{4}\right)$.

- **a** Find the volume in the reservoir at time $t = 10$.
- **b** Find the rate of change of the volume of water in the reservoir at time $t = 10$.

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Example 10

11 A particle moves along a straight line such that its position, x cm, relative to a point O at time t seconds is given by $x = 2t^3 - 9t^2 + 12t$.

a Find the velocity, v, as a function of t.

b At what times and in what positions will the particle have zero velocity?

c Find its acceleration at those instants.

d Find its velocity when its acceleration is zero.

12 A particle moves in a straight line such that its position, x cm, relative to a point O at time t seconds is given by the equation $x = 8 + 2t - t^2$. Find:

a its initial position

b its initial velocity

c when and where the velocity is zero

d its acceleration at time t.

13 A particle is moving in a straight line such that its position, x cm, relative to a point O at time t seconds is given by $x = \sqrt{2t^2 + 2}$. Find:

a the velocity as a function of t

b the acceleration as a function of t

c the velocity and acceleration when $t = 1$.

14 A vehicle is travelling in a straight line away from point O. Its distance from O after t seconds is $0.4e^t$ metres. Find the velocity of the vehicle when $t = 0$, $t = 1$, $t = 2$.

15 A manufacturing company has a daily output on day t of a production run given by $y = 600(1 - e^{-0.5t})$, where the first day of the production run is $t = 0$.

a Sketch the graph of y against t. (Assume a continuous model.)

b Find the instantaneous rate of change of output y with respect to t on the 10th day.

16 For each of the following, find $\dfrac{dy}{dx}$ in terms of y:

a $y = e^{-2x}$

b $y = Ae^{kx}$

17 The mass, m kg, of radioactive lead remaining in a sample t hours after observations began is given by $m = 2e^{-0.2t}$.

a Find the mass left after 12 hours.

b Find how long it takes for the mass to fall to half of its value at $t = 0$.

c Find how long it takes for the mass to fall to **i** one-quarter and **ii** one-eighth of its value at $t = 0$.

d Express the rate of decay as a function of m.

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10C Stationary points

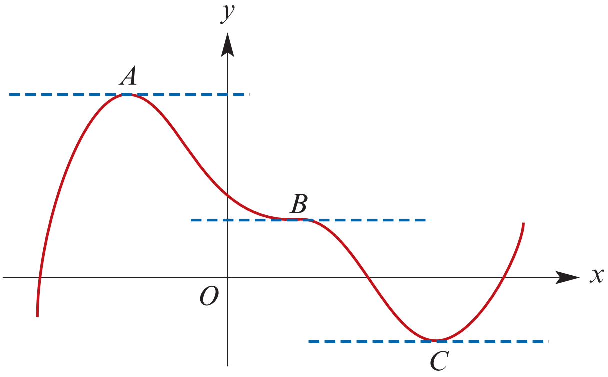
In the previous chapter, we have seen that the gradient of the tangent at a point $(a, f(a))$ on the curve with rule $y = f(x)$ is given by $f'(a)$.

A point $(a, f(a))$ on a curve $y = f(x)$ is said to be a **stationary point** if $f'(a) = 0$.

Equivalently, a point $(a, f(a))$ on $y = f(x)$ is a stationary point if $\frac{dy}{dx} = 0$ when $x = a$.

In the graph shown, there are stationary points at A, B and C.

At such points, the tangents are parallel to the x-axis (illustrated as dashed lines).



The reason for the name *stationary point* becomes clear if we look at an application to the motion of a particle.

Example 11

A particle is moving in a straight line. Its position, x metres, relative to a point O on the line at time t seconds is given by

$$x = 9t - \frac{1}{3}t^3, \quad 0 \le t \le 4$$

Find the particle's maximum distance from O. (Here the particle is always on the right of O and so its distance from O is its position.)

Solution

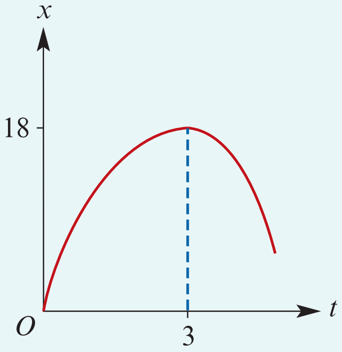
$$\frac{dx}{dt} = 9 - t^2$$

The maximum distance from O occurs when $\frac{dx}{dt} = 0$.

So $t = 3$ or $t = -3$. But $t = -3$ lies outside the domain.

At $t = 3$, $x = 18$.

Thus the stationary point is $(3, 18)$ and the maximum distance from O is 18 metres.



Note: The stationary point occurs when the rate of change of position with respect to time (the velocity) is zero. At this moment, the particle is stationary.

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Example 12

Find the stationary points of the following functions:

a $y = 9 + 12x - 2x^2$ **b** $y = 4 + 3x - x^3$ **c** $p = 2t^3 - 5t^2 - 4t + 13,\ t > 0$

Solution

a $y = 9 + 12x - 2x^2$

$$\frac{dy}{dx} = 12 - 4x$$

A stationary point occurs when $\frac{dy}{dx} = 0$, i.e. when $12 - 4x = 0$.

Hence $x = 3$ and $y = 9 + 12 \times 3 - 2 \times 3^2$

$$= 27$$

The stationary point is $(3, 27)$.

b $y = 4 + 3x - x^3$

$$\frac{dy}{dx} = 3 - 3x^2$$

$$\frac{dy}{dx} = 0 \text{ implies } 3(1 - x^2) = 0$$

$$\therefore \quad x = \pm 1$$

The stationary points are $(1, 6)$ and $(-1, 2)$.

c $p = 2t^3 - 5t^2 - 4t + 13$

$$\frac{dp}{dt} = 6t^2 - 10t - 4, \quad t > 0$$

$$\begin{aligned} \frac{dp}{dt} = 0 \text{ implies } 2(3t^2 - 5t - 2) &= 0 \\ (3t + 1)(t - 2) &= 0 \\ \therefore \quad t = -\frac{1}{3} \text{ or } t &= 2 \end{aligned}$$

But $t > 0$, and so the only acceptable solution is $t = 2$. The corresponding stationary point is $(2, 1)$.

Example 13

Find the stationary points of the following functions:

a $y = \sin(2x),\ x \in [0, 2\pi]$ **b** $y = e^{2x} - x$ **c** $y = x \log_e(2x),\ x \in (0, \infty)$

Solution

a $y = \sin(2x)$

$$\frac{dy}{dx} = 2\cos(2x)$$

$$\begin{aligned} \text{So } \frac{dy}{dx} = 0 \text{ implies } 2\cos(2x) &= 0 \\ \cos(2x) &= 0 \\ 2x &= \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \text{ or } \frac{7\pi}{2} \\ \therefore \quad x &= \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4} \text{ or } \frac{7\pi}{4} \end{aligned}$$

The stationary points are $\left(\frac{\pi}{4}, 1\right), \left(\frac{3\pi}{4}, -1\right), \left(\frac{5\pi}{4}, 1\right)$ and $\left(\frac{7\pi}{4}, -1\right)$.

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b $y = e^{2x} - x$

$$\frac{dy}{dx} = 2e^{2x} - 1$$

So $\frac{dy}{dx} = 0$ implies

$$\begin{aligned} 2e^{2x} - 1 &= 0 \\ e^{2x} &= \frac{1}{2} \\ \therefore \quad x &= \frac{1}{2}\log_e\left(\frac{1}{2}\right) \\ &= -\frac{1}{2}\log_e 2 \end{aligned}$$

When $x = -\frac{1}{2}\log_e 2$,

$$\begin{aligned} y &= e^{2\times\frac{1}{2}\log_e\left(\frac{1}{2}\right)} + \frac{1}{2}\log_e 2 \\ &= \frac{1}{2} + \frac{1}{2}\log_e 2 \end{aligned}$$

The coordinates of the stationary point are $(-\frac{1}{2}\log_e 2, \frac{1}{2} + \frac{1}{2}\log_e 2)$.

c $y = x\log_e(2x)$

$$\frac{dy}{dx} = \log_e(2x) + 1$$

So $\frac{dy}{dx} = 0$ implies

$$\begin{aligned} \log_e(2x) + 1 &= 0 \\ \log_e(2x) &= -1 \\ 2x &= e^{-1} \\ \therefore \quad x &= \frac{1}{2e} \end{aligned}$$

When $x = \frac{1}{2e}$, $y = \frac{1}{2e}\log_e\left(\frac{2}{2e}\right)$

$$= \frac{-1}{2e}$$

The coordinates of the stationary point are $\left(\frac{1}{2e}, \frac{-1}{2e}\right)$.

Example 14

The curve with equation $y = x^3 + ax^2 + bx + c$ passes through the point $(0, 5)$ and has a stationary point at $(2, 7)$. Find a, b and c.

Solution

When $x = 0$, $y = 5$. Thus $5 = c$.

$\frac{dy}{dx} = 3x^2 + 2ax + b$ and at $x = 2$, $\frac{dy}{dx} = 0$. Therefore

$$12 + 4a + b = 0 \qquad (1)$$

The point $(2, 7)$ is on the curve and so

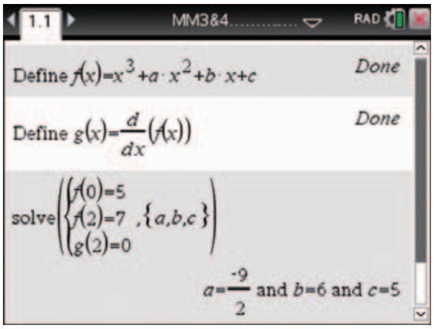
$$\begin{aligned} 8 + 4a + 2b + 5 &= 7 \\ \therefore \quad 6 + 4a + 2b &= 0 \qquad (2) \end{aligned}$$

Subtracting (1) from (2) gives $-6 + b = 0$. Thus $b = 6$. Substitute in (1):

$$\begin{aligned} 12 + 4a + 6 &= 0 \\ 4a &= -18 \end{aligned}$$

Hence $a = -\frac{9}{2}$, $b = 6$ and $c = 5$.

Using the TI-Nspire



Define $f(x)=x^3+a\cdot x^2+b\cdot x+c$ Done

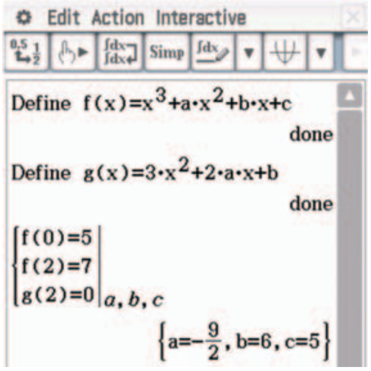
Define $g(x)=\frac{d}{dx}(f(x))$ Done

$\text{solve}\left(\begin{cases} f(0)=5 \\ f(2)=7 \\ g(2)=0 \end{cases},\{a,b,c\}\right)$

$a=\frac{-9}{2}$ and $b=6$ and $c=5$

- Use menu > **Actions** > **Define** to define $f(x) = x^3 + ax^2 + bx + c$.
- Define $g(x)$ to be the derivative (menu > **Calculus** > **Derivative**) of $f(x)$ as shown.
- Use the simultaneous equations solver (menu > **Algebra** > **Solve System of Equations** > **Solve System of Equations**) to find a, b and c given that $f(0) = 5$, $f(2) = 7$ and $g(2) = 0$.

Using the Casio ClassPad



Define f(x)=x^3+a·x^2+b·x+c
done
Define g(x)=3·x^2+2·a·x+b
done

$\begin{cases} f(0)=5 \\ f(2)=7 \\ g(2)=0 \end{cases}\Big|_{a,b,c}$

$\left\{a=-\frac{9}{2}, b=6, c=5\right\}$

- Use **Interactive** > **Define** to define the functions $f(x) = x^3 + ax^2 + bx + c$ and $g(x) = 3x^2 + 2ax + b$.
- In Math1, tap {■ twice.
- Enter the equations and variables as shown and tap EXE.

Section summary

- A point $(a, f(a))$ on a curve $y = f(x)$ is said to be a **stationary point** if $f'(a) = 0$.
- Equivalently, a point $(a, f(a))$ on $y = f(x)$ is a stationary point if $\frac{dy}{dx} = 0$ when $x = a$.

Exercise 10C

Example 12

1 Find the stationary points for each of the following:

a $f(x) = x^3 - 12x$
b $g(x) = 2x^2 - 4x$
c $h(x) = 5x^4 - 4x^5$
d $f(t) = 8t + 5t^2 - t^3$ for $t > 0$
e $g(z) = 8z^2 - 3z^4$
f $f(x) = 5 - 2x + 3x^2$
g $h(x) = x^3 - 4x^2 - 3x + 20, x > 0$
h $f(x) = 3x^4 - 16x^3 + 24x^2 - 10$

Example 13

2 Find the stationary points of the following functions:

a $y = e^{2x} - 2x$
b $y = x\log_e(3x), x \in (0, \infty)$
c $y = \cos(2x), x \in [-\pi, \pi]$
d $y = xe^x$
e $y = x^2e^{-x}$
f $y = 2x\log_e x, x \in (0, \infty)$

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3 **a** The curve with rule $f(x) = x^2 - ax + 9$ has a stationary point when $x = 3$. Find a.

b The curve with rule $h(x) = x^3 - bx^2 - 9x + 7$ has a stationary point when $x = -1$. Find b.

Example 14

4 The curve with equation $y = x^3 + bx^2 + cx + d$ passes through the point $(0, 3)$ and has a stationary point at $(1, 3)$. Find b, c and d.

5 The tangent to the curve of $y = ax^2 + bx + c$ at the point where $x = 2$ is parallel to the line $y = 4x$. There is a stationary point at $(1, -3)$. Find the values of a, b and c.

6 The graph of $y = ax^3 + bx^2 + cx + d$ touches the line $2y + 6x = 15$ at the point $A(0, 7\frac{1}{2})$ and has a stationary point at $B(3, -6)$. Find the values of a, b, c and d.

7 The curve with equation $y = ax + \dfrac{b}{2x - 1}$ has a stationary point at $(2, 7)$. Find:

a the values of a and b **b** the coordinates of the other stationary point.

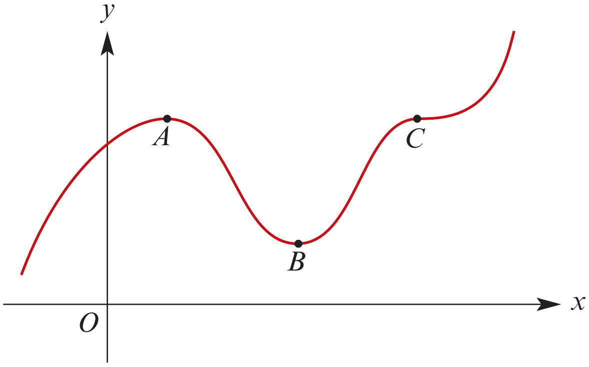
8 Find the x-coordinates, in terms of n, of the stationary points of the curve with equation $y = (2x - 1)^n(x + 2)$, where n is a natural number.

9 Find the x-coordinates of the stationary points of the curve with equation $y = (x^2 - 1)^n$ where n is an integer greater than 1.

10 Find the coordinates of the stationary points of the curve with equation $y = \dfrac{x}{x^2 + 1}$.

10D Types of stationary points

The graph of $y = f(x)$ shown has three stationary points A, B, C.



A Point A is called a **local maximum** point.
Notice that immediately to the left of A the gradient is positive, and immediately to right the gradient is negative.

gradient	+	0	−
shape of f	╱	—	╲

B Point B is called a **local minimum** point.
Notice that immediately to the left of B the gradient is negative, and immediately to the right the gradient is positive.

gradient	−	0	+
shape of f	╲	—	╱

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C Point C is called a **stationary point of inflection**. The gradient is positive immediately to the left and right of C.

gradient	+	0	+
shape of f	/	—	/

Clearly it is also possible to have stationary points of inflection such that the gradient is negative immediately to the left and right.

gradient	–	0	–
shape of f	\	—	\

Stationary points of types A and B are referred to as **turning points**.

Example 15

For the function $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 3x^3 - 4x + 1$:

a Find the stationary points and state their nature.

b Sketch the graph.

Solution

a The derivative is $f'(x) = 9x^2 - 4$.

The stationary points occur where $f'(x) = 0$:

$$9x^2 - 4 = 0$$

$$\therefore \quad x = \pm\frac{2}{3}$$

There are stationary points at $(-\frac{2}{3}, f(-\frac{2}{3}))$ and $(\frac{2}{3}, f(\frac{2}{3}))$, that is, at $(-\frac{2}{3}, 2\frac{7}{9})$ and $(\frac{2}{3}, -\frac{7}{9})$.
So $f'(x)$ is of constant sign for each of

$$\{\, x : x < -\tfrac{2}{3} \,\}, \quad \{\, x : -\tfrac{2}{3} < x < \tfrac{2}{3} \,\} \quad \text{and} \quad \{\, x : x > \tfrac{2}{3} \,\}$$

To calculate the sign of $f'(x)$ for each of these sets, simply choose a representative number in the set.

Thus $f'(-1) = 9 - 4 = 5 > 0$

$f'(0) = 0 - 4 = -4 < 0$

$f'(1) = 9 - 4 = 5 > 0$

We can now put together the table shown on the right.

x		$-\frac{2}{3}$		$\frac{2}{3}$	
$f'(x)$	+	0	–	0	+
shape of f	/	—	\	—	/

There is a local maximum at $(-\frac{2}{3}, 2\frac{7}{9})$ and a local minimum at $(\frac{2}{3}, -\frac{7}{9})$.

b To sketch the graph of this function we need to find the axis intercepts and investigate the behaviour of the graph for $x > \frac{2}{3}$ and $x < -\frac{2}{3}$.

The y-axis intercept is $f(0) = 1$.

To find the x-axis intercepts, consider $f(x) = 0$, which implies $3x^3 - 4x + 1 = 0$.
Using the factor theorem, we find that $x - 1$ is a factor of $3x^3 - 4x + 1$.
By division:

$$3x^3 - 4x + 1 = (x - 1)(3x^2 + 3x - 1)$$

Now $f(x) = (x - 1)(3x^2 + 3x - 1) = 0$ implies that $x = 1$ or $3x^2 + 3x - 1 = 0$.
We have

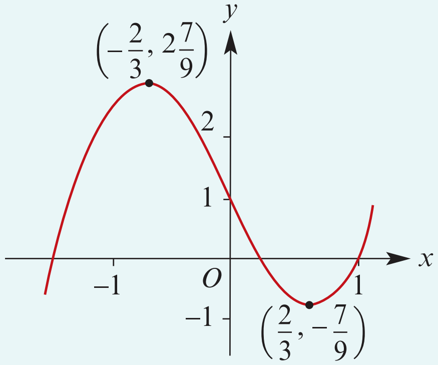
$$\begin{aligned} 3x^2 + 3x - 1 &= 3\left[\left(x + \frac{1}{2}\right)^2 - \frac{1}{4} - \frac{1}{3}\right] \\ &= 3\left[\left(x + \frac{1}{2}\right)^2 - \frac{21}{36}\right] \\ &= 3\left(x + \frac{1}{2} - \frac{\sqrt{21}}{6}\right)\left(x + \frac{1}{2} + \frac{\sqrt{21}}{6}\right) \end{aligned}$$

Thus the x-axis intercepts are at

$$x = -\frac{1}{2} + \frac{\sqrt{21}}{6}, \quad x = -\frac{1}{2} - \frac{\sqrt{21}}{6}, \quad x = 1$$

For $x > \frac{2}{3}$, $f(x)$ becomes larger.

For $x < \frac{2}{3}$, $f(x)$ becomes smaller.



A CAS calculator can be used to plot the graph of a function and determine its key features, including:

- the value of the function at any point
- the value of its derivative at any point
- the axis intercepts
- the local maximum and local minimum points.

Example 16

Plot the graph of $y = x^3 - 19x + 20$ and determine:

a the value of y when $x = -4$

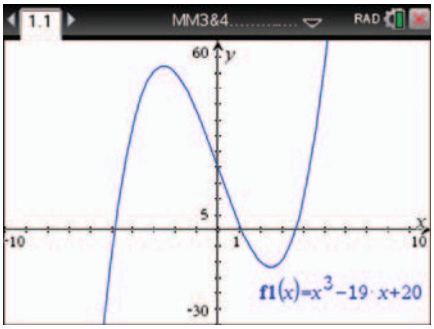
b the values of x when $y = 0$

c the value of $\frac{dy}{dx}$ when $x = -1$

d the coordinates of the local maximum.

Using the TI-Nspire

Graph $y = x^3 - 19x + 20$ in an appropriate window (menu > **Window/Zoom** > **Window Settings**).



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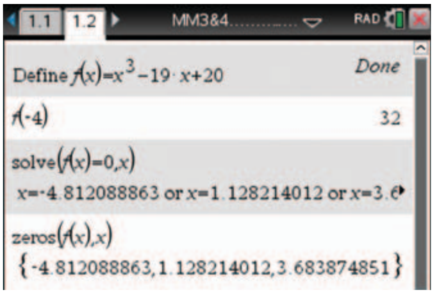
Define $f(x) = x^3 - 19x + 20$.

a $f(-4) = 32$

b Use solve($f(x) = 0,\ x$).

Note: Alternatively, menu > **Algebra** > **Zeros** can be used to solve equations equal to zero as shown.



Define $f(x)=x^3-19\cdot x+20$ Done

$f(-4)$ 32

solve($f(x)=0,x$)

$x=-4.812088863$ or $x=1.128214012$ or $x=3.6$▸

zeros($f(x),x$)

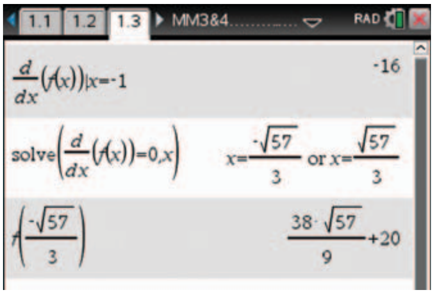
$\{-4.812088863, 1.128214012, 3.683874851\}$

c Find the derivative of $f(x)$ at $x = -1$ as shown.

d To find the stationary points, use

$$\text{solve}\left(\frac{d}{dx}(f(x)) = 0,\ x\right)$$

and then substitute to find the y-coordinate.



$\frac{d}{dx}(f(x))|x=-1$ -16

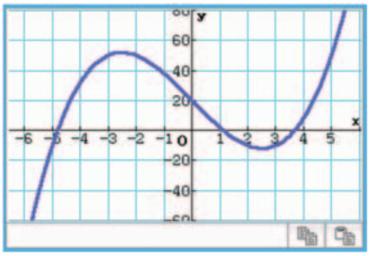
solve$\left(\frac{d}{dx}(f(x))=0,x\right)$ $x=\frac{-\sqrt{57}}{3}$ or $x=\frac{\sqrt{57}}{3}$

$f\left(\frac{-\sqrt{57}}{3}\right)$ $\frac{38\cdot\sqrt{57}}{9}+20$

Note: Since the function was also defined in the **Graphs** application as $f1$, the name $f1$ could have been used in place of f in these calculations.

Using the Casio ClassPad

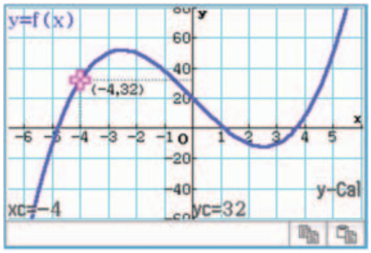
- Define $f(x) = x^3 - 19x + 20$.
- Tap ⊞ to open the graph window.
- Drag $f(x)$ into the graph window.
- Adjust the window using ⊡.



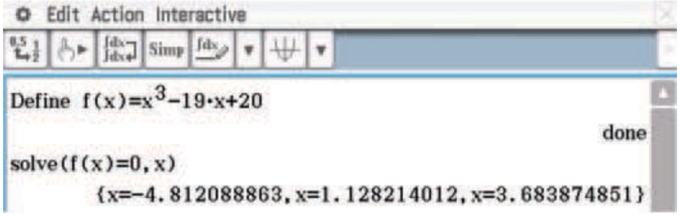
a To find $f(-4) = 32$, there are two methods:

1 In Main $\sqrt{\alpha}$, type $f(-4)$ and tap EXE.

2 In ⊞, go to **Analysis** > **G-Solve** > **x-Cal/y-Cal** > **y-Cal** and type -4.



b
- In Main $\sqrt{\alpha}$, enter and highlight $f(x) = 0$.
- Go to **Interactive** > **Equation/Inequality** > **solve**.
- Rotate the screen and press ► to view all solutions.



Define $f(x)=x^3-19\cdot x+20$

done

solve($f(x)=0,x$)

$\{x=-4.812088863, x=1.128214012, x=3.683874851\}$

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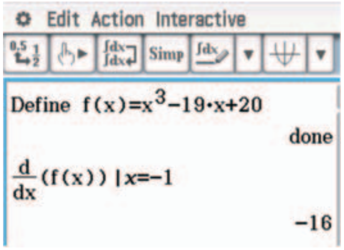
c To find $\frac{dy}{dx}$ when $x = -1$:

- In Main $\sqrt{\alpha}$, enter and highlight $f(x)$.
- Go to **Interactive** > **Calculation** > **diff** and then tap OK.
- Select | from Math3 and type $x = -1$ as shown.
- Tap EXE.



Define f(x)=x^3−19·x+20
done
$\frac{d}{dx}$(f(x))|x=−1
−16

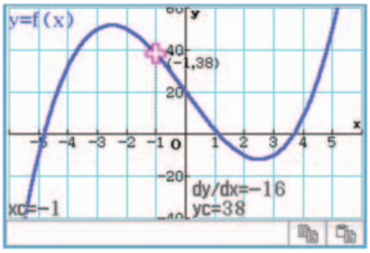
Using the graph window

To view the derivative at any point on a graph, first ensure that the Derivative/Slope setting is activated:

- Go to settings ⚙, select **Graph Format**, tick **Derivative/Slope** and tap Set.

Now in ⊞:

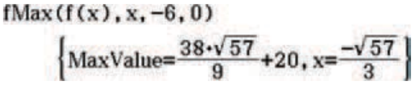
- Go to **Analysis** > **Trace**, type −1 and tap OK.



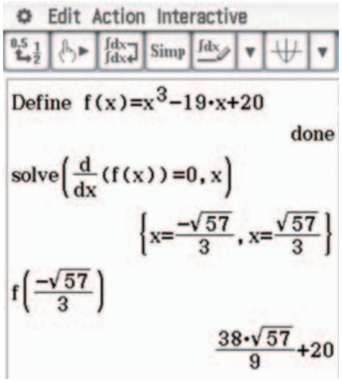
d To find the local maximum:

- In Main $\sqrt{\alpha}$, solve $\frac{d}{dx}(f(x)) = 0$ as shown.
- Substitute to find the y-coordinate.

Alternatively, find **fMax** using an appropriate domain:



fMax(f(x), x, −6, 0)
$\left\{\text{MaxValue}=\frac{38\cdot\sqrt{57}}{9}+20, x=\frac{-\sqrt{57}}{3}\right\}$



Define f(x)=x^3−19·x+20
done
solve$\left(\frac{d}{dx}(f(x))=0, x\right)$
$\left\{x=\frac{-\sqrt{57}}{3}, x=\frac{\sqrt{57}}{3}\right\}$
f$\left(\frac{-\sqrt{57}}{3}\right)$
$\frac{38\cdot\sqrt{57}}{9}+20$

Example 17

Sketch the graph of $f: \mathbb{R} \to \mathbb{R}, f(x) = e^{x^3}$.

Solution

As $x \to -\infty$, $f(x) \to 0$.

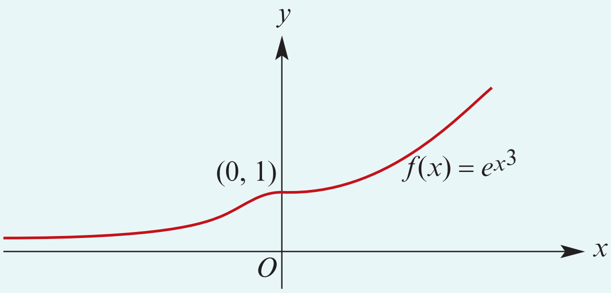
Axis intercepts

When $x = 0$, $f(x) = 1$.

Stationary points

$f'(x) = 3x^2e^{x^3}$

So $f'(x) = 0$ implies $x = 0$.



The gradient of f is always greater than or equal to 0, which means that (0, 1) is a stationary point of inflection.

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Example 18

For $f\colon (0, \infty) \to \mathbb{R}$, $f(x) = x \log_e x$:

a Find $f'(x)$.

b Solve the equation $f(x) = 0$.

c Solve the equation $f'(x) = 0$.

d Sketch the graph of $y = f(x)$.

Solution

a $$f'(x) = x \times \frac{1}{x} + \log_e x \qquad \text{(product rule)}$$
$$= 1 + \log_e x$$

b $f(x) = x \log_e x$

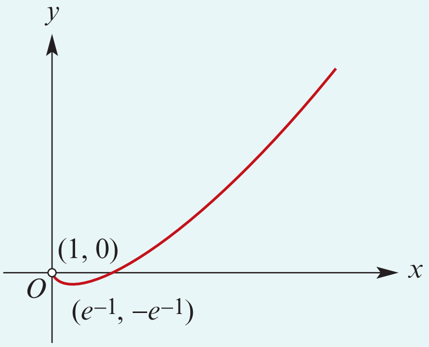
Thus $f(x) = 0$ implies $x = 0$ or $\log_e x = 0$.

Since $x \in (0, \infty)$, the only solution is $x = 1$.

c $f'(x) = 0$ implies $1 + \log_e x = 0$.

Therefore $\log_e x = -1$ and so $x = e^{-1}$.

d When $x = e^{-1}$, $y = e^{-1} \log_e(e^{-1})$
$$= e^{-1} \times (-1) = -e^{-1}$$



Example 19

Find the local maximum and local minimum points of $f(x) = 2 \sin x + 1 - 2 \sin^2 x$, where $0 < x < 2\pi$.

Solution

Find $f'(x)$ and solve $f'(x) = 0$:

$$f(x) = 2 \sin x + 1 - 2 \sin^2 x$$
$$\therefore \quad f'(x) = 2 \cos x - 4 \sin x \cos x$$
$$= 2 \cos x \cdot (1 - 2 \sin x)$$

Thus $f'(x) = 0$ implies

$$\cos x = 0 \quad \text{or} \quad 1 - 2 \sin x = 0$$
$$\text{i.e.} \quad \cos x = 0 \quad \text{or} \quad \sin x = \tfrac{1}{2}$$
$$\text{i.e.} \quad x = \frac{\pi}{2}, \frac{3\pi}{2} \quad \text{or} \quad x = \frac{\pi}{6}, \frac{5\pi}{6}$$

We have $f\left(\frac{\pi}{2}\right) = 1$, $f\left(\frac{3\pi}{2}\right) = -3$, $f\left(\frac{\pi}{6}\right) = \frac{3}{2}$ and $f\left(\frac{5\pi}{6}\right) = \frac{3}{2}$

x		$\frac{\pi}{6}$		$\frac{\pi}{2}$		$\frac{5\pi}{6}$		$\frac{3\pi}{2}$	
$f'(x)$	+	0	−	0	+	0	−	0	+
shape of f	/	‾	\	_	/	‾	\	_	/

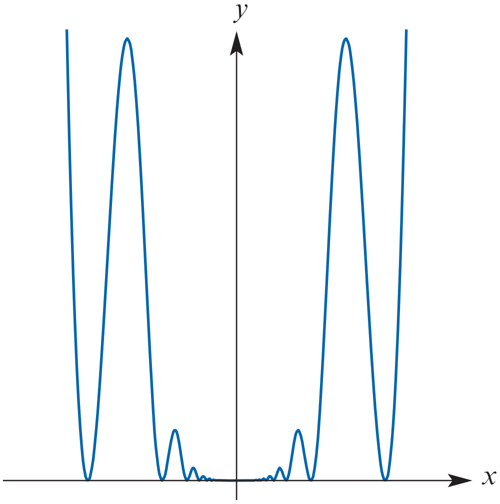
Local maxima at $\left(\frac{\pi}{6}, \frac{3}{2}\right)$ and $\left(\frac{5\pi}{6}, \frac{3}{2}\right)$. Local minima at $\left(\frac{\pi}{2}, 1\right)$ and $\left(\frac{3\pi}{2}, -3\right)$.

Bad behaviour? In this course, and in school courses around the world, we deal with functions that are 'conveniently behaved'. This avoids some complications.

For an example of a function which is not in this category, consider

$$f(x) = \begin{cases} x^4 \sin^2\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

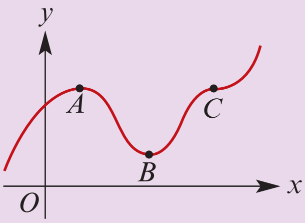
The derivative of this function is defined for all $x \in \mathbb{R}$. In any open interval around $x = 0$, the graph of this function has infinitely many stationary points, no matter how small the interval.



Section summary

A point $(a, f(a))$ on a curve $y = f(x)$ is said to be a **stationary point** if $f'(a) = 0$.

Types of stationary points



A Point A is a **local maximum**:

- $f'(x) > 0$ immediately to the left of A
- $f'(x) < 0$ immediately to the right of A.

B Point B is a **local minimum**:

- $f'(x) < 0$ immediately to the left of B
- $f'(x) > 0$ immediately to the right of B.

C Point C is a **stationary point of inflection**.

Stationary points of types A and B are called **turning points**.

Exercise 10D

Skillsheet
Example 15

1 For each of the following derivative functions, write down the values of x at which the derivative is zero and prepare a gradient table (as in Example 15) showing whether the corresponding points on the graph of $y = f(x)$ are local maxima, local minima or stationary points of inflection:

a $f'(x) = 4x^2$
b $f'(x) = (x-2)(x+5)$
c $f'(x) = (x+1)(2x-1)$
d $f'(x) = -x^2 + x + 12$
e $f'(x) = x^2 - x - 12$
f $f'(x) = 5x^4 - 27x^3$
g $f'(x) = (x-1)(x-3)$
h $f'(x) = -(x-1)(x-3)$

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2 Find the stationary points on each of the following curves and state their nature:

a $y = x(x^2 - 12)$ **b** $y = x^2(3 - x)$

c $y = x^3 - 5x^2 + 3x + 2$ **d** $y = 3 - x^3$

e $y = 3x^4 + 16x^3 + 24x^2 + 3$ **f** $y = x(x^2 - 1)$

3 Sketch the graph of each of the following, finding **i** axis intercepts and **ii** stationary points:

a $y = 4x^3 - 3x^4$ **b** $y = x^3 - 6x^2$ **c** $y = 3x^2 - x^3$

d $y = x^3 + 6x^2 + 9x + 4$ **e** $y = (x^2 - 1)^5$ **f** $y = (x^2 - 1)^4$

4 **a** Find the stationary points of the graph of $y = 2x^3 + 3x^2 - 12x + 7$, stating the nature of each.

b Show that the graph passes through $(1, 0)$.

c Find the other axis intercepts.

d Sketch the graph.

5 **a** Show that the polynomial $P(x) = x^3 + ax^2 + b$ has a stationary point at $x = 0$ for all a and b.

b Given that $P(x)$ has a second stationary point at $(-2, 6)$, find the values of a and b and the nature of both stationary points.

6 Sketch the graph of $f(x) = (2x - 1)^5(2x - 4)^4$.

a State the coordinates of the axis intercepts.

b State the coordinates and nature of each stationary point.

7 **a** Sketch the graphs of $f(x) = (4x^2 - 1)^6$ and $g(x) = (4x^2 - 1)^5$ on the one set of axes.

b **i** Find $\{ x : (4x^2 - 1)^6 > (4x^2 - 1)^5 \}$. **ii** Find $\{ x : f'(x) > g'(x) \}$.

8 Sketch the graph of each of the following. State the axis intercepts and the coordinates of stationary points.

a $y = x^3 + x^2 - 8x - 12$ **b** $y = 4x^3 - 18x^2 + 48x - 290$

9 For each of the following, find the coordinates of the stationary points and determine their nature:

a $f(x) = 3x^4 + 4x^3$ **b** $f(x) = x^4 + 2x^3 - 1$ **c** $f(x) = 3x^3 - 3x^2 + 12x + 9$

10 Consider the function f defined by $f(x) = \frac{1}{8}(x - 1)^3(8 - 3x) + 1$.

a Show that $f(0) = 0$ and $f(3) = 0$.

b Show that $f'(x) = \frac{3}{8}(x - 1)^2(9 - 4x)$ and specify the values of x for which $f'(x) \geq 0$.

c Sketch the graph of $y = f(x)$.

11 Sketch the graph of $y = 3x^4 - 44x^3 + 144x^2$, finding the coordinates of all turning points.

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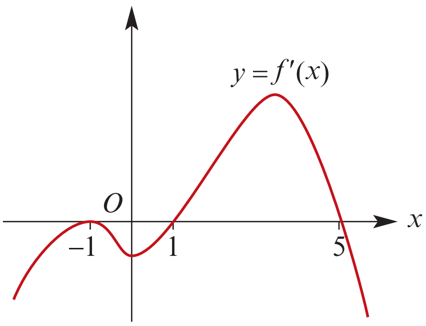
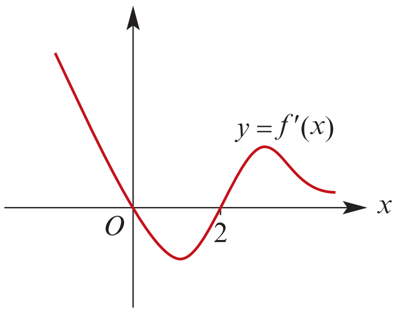
12 Each graph below shows the graph of f' for a function f. Find the values of x for which the graph of $y = f(x)$ has a stationary point and state the nature of each stationary point.

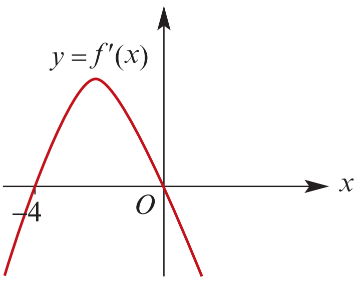
a



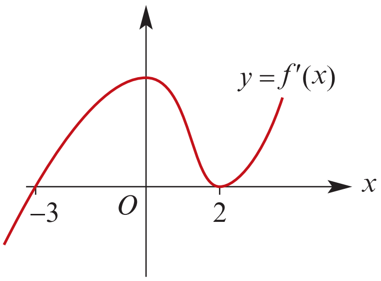
b



c



d



13 Find the coordinates of the stationary points, and state the nature of each, for the curve with equation:

a $y = x^4 - 16x^2$

b $y = x^{2m} - 16x^{2m-2}$, where m is a natural number greater than or equal to 2.

Example 17 **14** Sketch the graph of $f(x) = e^{-\frac{x^2}{2}}$.

15 Let $f(x) = x^2e^x$. Find $\{ x : f'(x) < 0 \}$.

16 Find the values of x for which $100e^{-x^2+2x-5}$ increases as x increases and hence find the maximum value of $100e^{-x^2+2x-5}$.

17 Let $f(x) = e^x - 1 - x$.

a Find the minimum value of $f(x)$. **b** Hence show $e^x \geq 1 + x$ for all real x.

18 For $f(x) = x + e^{-x}$:

a Find the position and nature of any stationary points.

b Find, if they exist, the equations of any asymptotes.

c Sketch the graph of $y = f(x)$.

19 The curve $y = e^x(px^2 + qx + r)$ is such that the tangents at $x = 1$ and $x = 3$ are parallel to the x-axis. The point with coordinates $(0, 9)$ is on the curve. Find p, q and r.

20 **a** Let $y = e^{4x^2-8x}$. Find $\dfrac{dy}{dx}$.

b Find the coordinates of the stationary point on the curve of $y = e^{4x^2-8x}$ and state its nature.

c Sketch the graph of $y = e^{4x^2-8x}$.

d Find the equation of the normal to the curve of $y = e^{4x^2-8x}$ at the point where $x = 2$.

21 On the same set of axes, sketch the graphs of $y = \log_e x$ and $y = \log_e(5x)$, and use them to explain why $\dfrac{d}{dx}\big(\log_e x\big) = \dfrac{d}{dx}\big(\log_e(5x)\big)$.

Example 18 **22** For the function $f\colon (0,\infty) \to \mathbb{R}$, $f(x) = x^2 \log_e x$:

a Find $f'(x)$.

b Solve the equation $f(x) = 0$.

c Solve the equation $f'(x) = 0$.

d Sketch the graph of $y = f(x)$.

23 Let $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 - 3x^2 - 9x + 11$. Sketch the graph of:

a $y = f(x)$ **b** $y = 2f(x)$ **c** $y = f(x+2)$ **d** $y = f(x-2)$ **e** $y = -f(x)$

24 Let $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 2 + 3x - x^3$. Sketch the graph of:

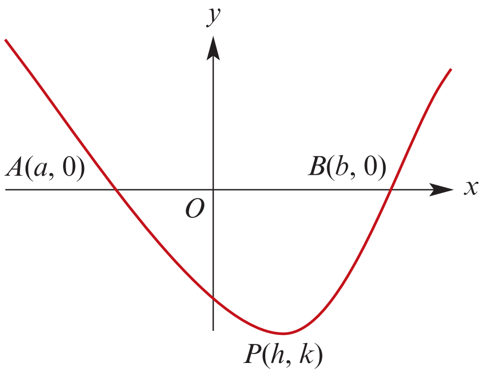
a $y = f(x)$ **b** $y = -2f(x)$ **c** $y = 2f(x-1)$ **d** $y = f(x) - 3$ **e** $y = 3f(x+1)$

25 The graph shown opposite has equation $y = f(x)$. Suppose a dilation of factor p from the x-axis followed by a translation of ℓ units in the positive direction of the x-axis is applied to the graph.

For the graph of the image, state:

a the axis intercepts

b the coordinates of the turning point.



Example 19 **26** Find the values of x for which the graph of $y = f(x)$ has a stationary point and state the nature of each stationary point. Consider $0 \le x \le 2\pi$ only.

a $f(x) = 2\cos x - (2\cos^2 x - 1)$

b $f(x) = 2\cos x + 2\sin x\cos x$

c $f(x) = 2\sin x - (2\cos^2 x - 1)$

d $f(x) = 2\sin x + 2\sin x\cos x$

27 The graph of a quartic function passes through the points with coordinates $(1, 21)$, $(2, 96)$, $(5, 645)$, $(6, 816)$ and $(7, 861)$.

a Find the rule of the quartic and plot the graph. Determine the turning points and axis intercepts.

b Plot the graph of the derivative on the same screen.

c Find the value of the function when $x = 10$.

d For what value(s) of x is the value of the function 500?

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10E Absolute maximum and minimum values

Local maximum and minimum values were discussed in the previous section. These are often not the actual maximum and minimum values of the function.

For a function defined on an interval:

- the actual maximum value of the function is called the **absolute maximum**
- the actual minimum value of the function is called the **absolute minimum**.

The corresponding points on the graph of the function are not necessarily stationary points.

More precisely, for a continuous function f defined on an interval $[a, b]$:

- if M is a value of the function such that $f(x) \leq M$ for all $x \in [a, b]$, then M is the absolute maximum value of the function
- if N is a value of the function such that $f(x) \geq N$ for all $x \in [a, b]$, then N is the absolute minimum value of the function.

Example 20

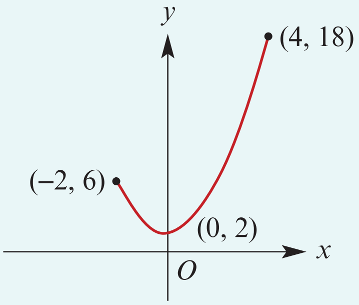
Let $f\colon [-2, 4] \to \mathbb{R}$, $f(x) = x^2 + 2$. Find the absolute maximum value and the absolute minimum value of the function.

Solution

The maximum value is 18 and occurs when $x = 4$.

The minimum value is 2 and occurs when $x = 0$.

(Note that the absolute minimum occurs at a stationary point of the graph. The absolute maximum occurs at an endpoint, not at a stationary point.)



Example 21

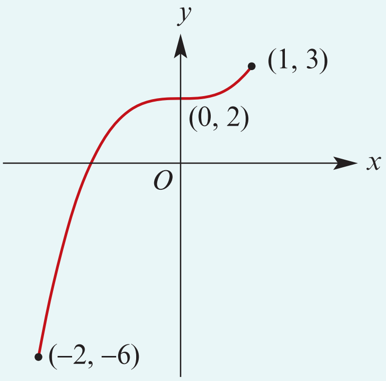
Let $f\colon [-2, 1] \to \mathbb{R}$, $f(x) = x^3 + 2$. Find the maximum and minimum values of the function.

Solution

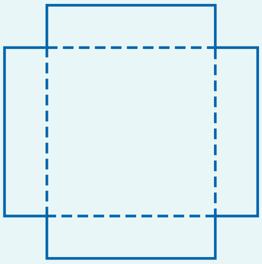
The maximum value is 3 and occurs when $x = 1$.

The minimum value is -6 and occurs when $x = -2$.

(Note that the absolute maximum and minimum values do not occur at stationary points.)



Example 22



From a square piece of metal of side length 2 m, four squares are removed as shown in the diagram. The metal is then folded along the dashed lines to form an open box with height x m.

a Show that the volume of the box, V m^3, is given by $V = 4x^3 - 8x^2 + 4x$.

b Find the value of x that gives the box its maximum volume and show that the volume is a maximum for this value.

c Sketch the graph of V against x for a suitable domain.

d If the height of the box must be less than 0.3 m, i.e. $x \le 0.3$, what will be the maximum volume of the box?

Solution

a The box has length and width $2 - 2x$ metres, and has height x metres. Thus

$$\begin{aligned} V &= (2 - 2x)^2 x \\ &= (4 - 8x + 4x^2)x \\ &= 4x^3 - 8x^2 + 4x \end{aligned}$$

b Let $V(x) = 4x^3 - 8x^2 + 4x$. A local maximum will occur when $V'(x) = 0$.
We have $V'(x) = 12x^2 - 16x + 4$, and so $V'(x) = 0$ implies that

$$\begin{aligned} 12x^2 - 16x + 4 &= 0 \\ 3x^2 - 4x + 1 &= 0 \\ (3x - 1)(x - 1) &= 0 \\ \therefore \quad x &= \frac{1}{3} \text{ or } x = 1 \end{aligned}$$

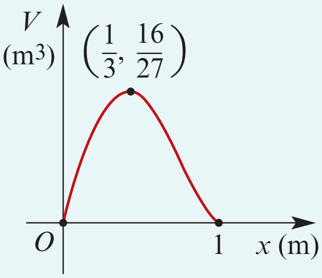
But, when $x = 1$, the length of the box is $2 - 2x = 0$. Therefore the only value to be considered is $x = \frac{1}{3}$. We show the entire chart for completeness.

x		$\frac{1}{3}$		1
$V'(x)$	+	0	−	0
shape of V	╱	—	╲	—

The maximum occurs when $x = \frac{1}{3}$.

$$\begin{aligned} \therefore \text{ Maximum volume} &= \left(2 - 2 \times \frac{1}{3}\right)^2 \times \frac{1}{3} \\ &= \frac{16}{27} \text{ m}^3 \end{aligned}$$

c



d The local maximum of $V(x)$ defined on $[0, 1]$ is at $\left(\frac{1}{3}, \frac{16}{27}\right)$.

But $\frac{1}{3}$ is not in the interval $[0, 0.3]$.

Since $V'(x) > 0$ for all $x \in [0, 0.3]$, the maximum volume for this situation occurs when $x = 0.3$ and is 0.588 m^3.

Section summary

For a continuous function f defined on an interval $[a, b]$:

- if M is a value of the function such that $f(x) \leq M$ for all $x \in [a, b]$, then M is the **absolute maximum** value of the function
- if N is a value of the function such that $f(x) \geq N$ for all $x \in [a, b]$, then N is the **absolute minimum** value of the function.

Exercise 10E

Skillsheet
Example 20

1 Let $f\colon [-3, 3] \to \mathbb{R}$, $f(x) = 2 - 8x^2$. Find the absolute maximum value and the absolute minimum value of the function.

Example 21

2 Let $f\colon [-3, 2] \to \mathbb{R}$, $f(x) = x^3 + 2x + 3$. Find the absolute maximum value and the absolute minimum value of the function for its domain.

3 Let $f\colon [-1.5, 2.5] \to \mathbb{R}$, $f(x) = 2x^3 - 6x^2$. Find the absolute maximum and absolute minimum values of the function.

4 Let $f\colon [-2, 6] \to \mathbb{R}$, $f(x) = 2x^4 - 8x^2$. Find the absolute maximum and absolute minimum values of the function.

Example 22

5 A rectangular block is such that the sides of its base are of length x cm and $3x$ cm. The sum of the lengths of all its edges is 20 cm.

- **a** Show that the volume, V cm^3, of the block is given by $V = 15x^2 - 12x^3$.
- **b** Find $\dfrac{dV}{dx}$.
- **c** Find the coordinates of the local maximum of the graph of V against x for $x \in [0, 1.25]$.
- **d** If $x \in [0, 0.8]$, find the absolute maximum value of V and the value of x for which this occurs.
- **e** If $x \in [0, 1]$, find the absolute maximum value of V and the value of x for which this occurs.

6 Variables x, y and z are such that $x + y = 30$ and $z = xy$.

- **a** If $x \in [2, 5]$, find the possible values of y.
- **b** Find the absolute maximum and absolute minimum values of z.

7 Consider the function $f\colon [2, 3] \to \mathbb{R}$, $f(x) = \dfrac{1}{x-1} + \dfrac{1}{4-x}$.

- **a** Find $f'(x)$.
- **b** Find the coordinates of the stationary point of the graph of $y = f(x)$.
- **c** Find the absolute maximum and absolute minimum of the function.

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8 A piece of string 10 metres long is cut into two pieces to form two squares.

- **a** If one piece of string has length x metres, show that the combined area of the two squares is given by $A = \frac{1}{8}(x^2 - 10x + 50)$.
- **b** Find $\frac{dA}{dx}$.
- **c** Find the value of x that makes A a minimum.
- **d** If two squares are formed but $x \in [0, 1]$, find the maximum possible combined area of the two squares.

9 Find the absolute maximum and minimum values of the function $g\colon [2.1, 8] \to \mathbb{R}$, $g(x) = x + \frac{1}{x - 2}$.

10 Consider the function $f\colon [0, 3] \to \mathbb{R}$, $f(x) = \frac{1}{x + 1} + \frac{1}{4 - x}$.

- **a** Find $f'(x)$.
- **b** Find the coordinates of the stationary point of the graph of $y = f(x)$.
- **c** Find the absolute maximum and absolute minimum of the function.

11 For the function $f\colon \left[-\frac{\pi}{2}, \frac{\pi}{8}\right] \to \mathbb{R}$, $f(x) = \sin(2x)$, state the absolute maximum and minimum values of the function.

12 For the function $f\colon \left[0, \frac{\pi}{8}\right] \to \mathbb{R}$, $f(x) = \cos(2x)$, state the absolute maximum and minimum values of the function.

13 For the function $f\colon [-1, 8] \to \mathbb{R}$, $f(x) = 2 - x^{\frac{2}{3}}$, sketch the graph and state the absolute maximum and minimum values of the function.

14 For the function $f\colon [-1, 2] \to \mathbb{R}$, $f(x) = 2e^x + e^{-x}$, sketch the graph and state the absolute maximum and minimum values of the function.

15 For the function $f\colon [-2, 2] \to \mathbb{R}$, $f(x) = 2e^{(x-1)^2}$, sketch the graph and state the absolute maximum and minimum values of the function.

16 For the function $f\colon [6, 10] \to \mathbb{R}$, $f(x) = (x - 5)\log_e\left(\frac{x - 5}{10}\right)$, sketch the graph and state the absolute maximum and minimum values of the function.

10F Maximum and minimum problems

Many practical problem require that some quantity (for example, cost of manufacture or fuel consumption) be **minimised**, that is, be made as small as possible. Other problems require that some quantity (for example, profit on sales or attendance at a concert) be **maximised**, that is, be made as large as possible. We can use differential calculus to solve many of these problems.

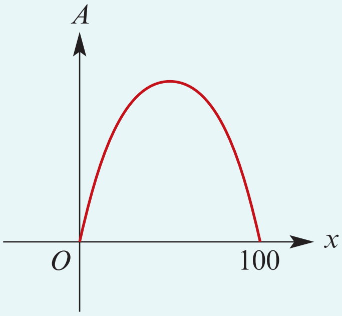
Example 23

A farmer has sufficient fencing to make a rectangular pen of perimeter 200 metres. What dimensions will give an enclosure of maximum area?

Solution

Let the length of the rectangle be x metres. Then the width is $100 - x$ metres and the area is A m^2, where

$$\begin{aligned} A &= x(100 - x) \\ &= 100x - x^2 \end{aligned}$$



The maximum value of A occurs when $\dfrac{dA}{dx} = 0$.

$$\frac{dA}{dx} = 100 - 2x$$

$$\therefore \quad \frac{dA}{dx} = 0 \text{ implies } x = 50$$

From the gradient chart, the maximum area occurs when $x = 50$.

x		50	
$\dfrac{dA}{dx}$	+	0	−
shape of A	╱	—	╲

The pen with maximum area has dimensions 50 m by 50 m, and so has area 2500 m^2.

Example 24

Two variables x and y are such that $x^4y = 8$. A third variable z is defined by $z = x + y$. Find the values of x and y that give z a stationary value and show that this value of z is a minimum.

Solution

Obtain y in terms of x from the equation $x^4y = 8$:

$$y = 8x^{-4}$$

Substitute in the equation $z = x + y$:

$$z = x + 8x^{-4} \qquad (1)$$

Now z is expressed in terms of one variable, x. Differentiate with respect to x:

$$\frac{dz}{dx} = 1 - 32x^{-5}$$

A stationary point occurs where $\dfrac{dz}{dx} = 0$:

$$\begin{aligned} 1 - 32x^{-5} &= 0 \\ 32x^{-5} &= 1 \\ x^5 &= 32 \\ \therefore \quad x &= 2 \end{aligned}$$

There is a stationary point at $x = 2$. The corresponding value of y is $8 \times 2^{-4} = \frac{1}{2}$.

Now substitute in equation (1) to find z:

$$z = 2 + \frac{8}{16} = 2\tfrac{1}{2}$$

Determine the nature of the stationary point using a gradient chart.

x		2	
$\frac{dz}{dx}$	−	0	+
shape of z	\	—	/

The minimum value of z is $2\frac{1}{2}$ and occurs when $x = 2$ and $y = \frac{1}{2}$.

Example 25

A cylindrical tin canister closed at both ends has a surface area of 100 cm^2. Find, correct to two decimal places, the greatest volume it can have. If the radius of the canister can be at most 2 cm, find the greatest volume it can have.

Solution

Let the radius of the circular end of the tin be r cm, let the height of the tin be h cm and let the volume of the tin be V cm^3.

Obtain equations for the surface area and the volume.

$$\text{Surface area:} \quad 100 = 2\pi r^2 + 2\pi rh \qquad (1)$$

$$\text{Volume:} \quad V = \pi r^2 h \qquad (2)$$

The process we follow now is very similar to Example 23. Obtain h in terms of r from equation (1):

$$h = \frac{1}{2\pi r}(100 - 2\pi r^2)$$

Substitute in equation (2):

$$V = \pi r^2 \times \frac{1}{2\pi r}(100 - 2\pi r^2)$$

$$\therefore \quad V = 50r - \pi r^3 \qquad (3)$$

A stationary point of the graph of $V = 50r - \pi r^3$ occurs when $\frac{dV}{dr} = 0$.

$$\frac{dV}{dr} = 0 \text{ implies } 50 - 3\pi r^2 = 0$$

$$\therefore \quad r = \pm\sqrt{\frac{50}{3\pi}} \approx \pm 2.3$$

But $r = -2.3$ does not fit the practical situation.

Substitute $r = 2.3$ in equation (3) to find V:

$$V \approx 76.78$$

So there is a stationary point at $(2.3, 76.8)$.

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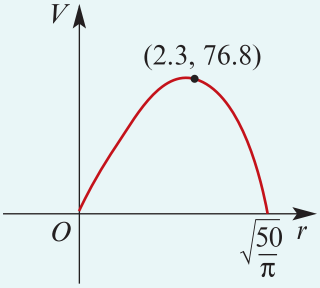
Use a gradient chart to determine the nature of this stationary point.

r		2.3	
$\dfrac{dV}{dr}$	$+$	0	$-$
shape of V	/	—	\

The maximum volume is 76.78 cm^3 correct to two decimal places.

It can be observed that the volume is given by a function f with rule $f(r) = 50r - \pi r^3$ and domain $\left[0, \sqrt{\frac{50}{\pi}}\right]$, giving the graph on the right.



If the greatest radius the canister can have is 2 cm, then the function f has domain $[0, 2]$. It has been seen that $f'(r) > 0$ for all $r \in [0, 2]$. The maximum value occurs when $r = 2$. The maximum volume in this case is $f(2) = 100 - 8\pi \approx 74.87$ cm^3.

In some situations the variables may not be continuous. For instance, one of them may only take integer values. In such cases it is not strictly valid to use techniques of differentiation to solve the problem. However, in some problems we may model the non-continuous case with a continuous function so that the techniques of differential calculus may be used. Examples 26 and 27 illustrate this.

Example 26

A TV cable company has 1000 subscribers who are paying \$5 per month. It can get 100 more subscribers for each \$0.10 decrease in the monthly fee. What monthly fee will yield the maximum revenue and what will this revenue be?

Solution

Let x denote the monthly fee. Then the number of subscribers is $1000 + 100\left(\dfrac{5-x}{0.1}\right)$.

(Note that we are treating a discrete situation with a continuous function.)

Let R denote the revenue. Then

$$\begin{aligned} R &= x(1000 + 1000(5-x)) \\ &= 1000(6x - x^2) \\ \therefore \quad \frac{dR}{dx} &= 1000(6 - 2x) \end{aligned}$$

Thus $\dfrac{dR}{dx} = 0$ implies $6 - 2x = 0$ and hence $x = 3$.

The gradient chart is shown.

x		3	
$\dfrac{dR}{dx}$	$+$	0	$-$
shape of R	/	—	\

For maximum revenue, the monthly fee should be \$3 and this gives a total revenue of \$9000.

Example 27

A manufacturer annually produces and sells 10 000 shirts. Sales are uniformly distributed throughout the year. The production cost of each shirt is \$23 and the carrying costs (storage, insurance, interest) depend on the total number of shirts in a production run. (A production run is the number, x, of shirts which are under production at a given time.)

The set-up costs for a production run are \$40. The annual carrying costs are $\$x^{\frac{3}{2}}$. Find the size of a production run that minimises the total set-up and carrying costs for a year.

Solution

Number of production runs per year $= \dfrac{10\,000}{x}$

Set-up costs for these production runs $= 40\left(\dfrac{10\,000}{x}\right)$

Let C be the total set-up and carrying costs. Then

$$\begin{aligned} C &= x^{\frac{3}{2}} + \frac{400\,000}{x} \\ &= x^{\frac{3}{2}} + 400\,000x^{-1}, \qquad x > 0 \end{aligned}$$

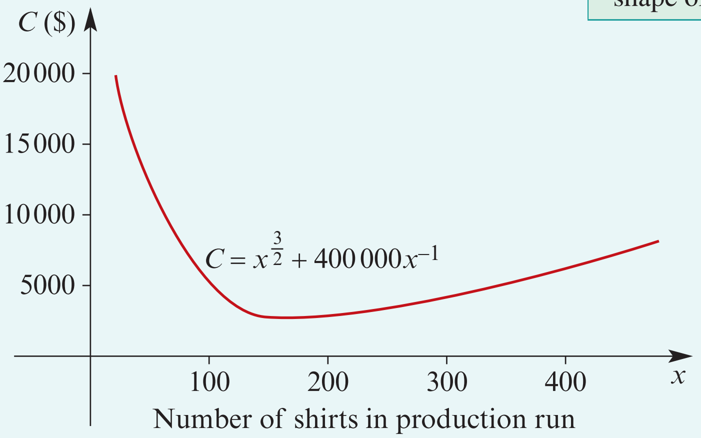
$$\therefore \quad \frac{dC}{dx} = \frac{3}{2}x^{\frac{1}{2}} - \frac{400\,000}{x^2}$$

Thus $\dfrac{dC}{dx} = 0$ implies

$$\begin{aligned} \frac{3}{2}x^{\frac{1}{2}} &= \frac{400\,000}{x^2} \\ x^{\frac{5}{2}} &= \frac{400\,000 \times 2}{3} \\ \therefore \quad x &\approx 148.04 \end{aligned}$$

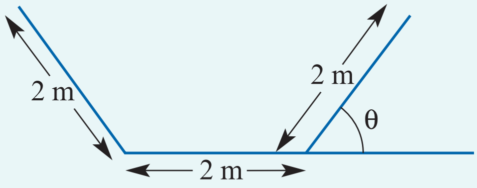
Each production run should be 148 shirts.

x		148.04	
$\dfrac{dC}{dx}$	−	0	+
shape of C	╲	—	╱



Example 28

The cross-section of a drain is to be an isosceles trapezium, with three sides of length 2 metres, as shown. Find the angle θ that maximises the cross-sectional area, and find this maximum area.



Solution

Let A m^2 be the area of the trapezium. Then

$$\begin{aligned} A &= \frac{1}{2} \times 2\sin\theta \times (2 + 2 + 4 \times \cos\theta) \\ &= \sin\theta \cdot (4 + 4\cos\theta) \end{aligned}$$

and
$$\begin{aligned} A'(\theta) &= \cos\theta \cdot (4 + 4\cos\theta) - 4\sin^2\theta \\ &= 4\cos\theta + 4\cos^2\theta - 4(1 - \cos^2\theta) \\ &= 4\cos\theta + 8\cos^2\theta - 4 \end{aligned}$$

The maximum will occur when $A'(\theta) = 0$:

$$\begin{aligned} 8\cos^2\theta + 4\cos\theta - 4 &= 0 \\ 2\cos^2\theta + \cos\theta - 1 &= 0 \\ (2\cos\theta - 1)(\cos\theta + 1) &= 0 \\ \therefore \quad \cos\theta = \frac{1}{2} \ \text{ or } \ \cos\theta &= -1 \end{aligned}$$

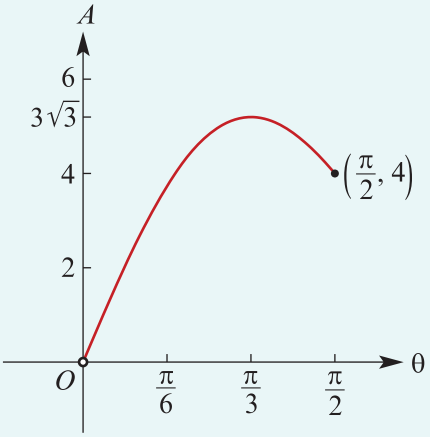
The practical restriction on θ is that $0 < \theta \le \dfrac{\pi}{2}$.

Therefore the only possible solution is $\theta = \dfrac{\pi}{3}$, and a gradient chart confirms that $\dfrac{\pi}{3}$ gives a maximum.

θ		$\frac{\pi}{3}$	
$A'(\theta)$	$+$	0	$-$
shape of A	/	—	\

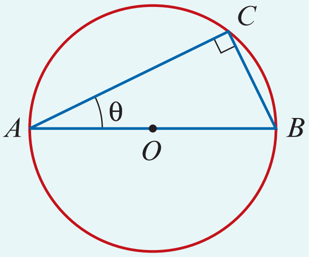
When $\theta = \dfrac{\pi}{3}$, $A = \dfrac{\sqrt{3}}{2}(4 + 2) = 3\sqrt{3}$,

i.e. the maximum cross-sectional area is $3\sqrt{3}$ m^2.



Example 29

The figure shows a circular lake, centre O, of radius 2 km. A man swims across the lake from A to C at 3 km/h and then walks around the edge of the lake from C to B at 4 km/h.



a If $\angle BAC = \theta$ radians and the total time taken is T hours, show that

$$T = \frac{1}{3}(4\cos\theta + 3\theta)$$

b Find the value of θ for which $\frac{dT}{d\theta} = 0$ and determine whether this gives a maximum or minimum value of T $(0^\circ < \theta^\circ < 90^\circ)$.

Solution

a Time taken $= \dfrac{\text{distance travelled}}{\text{speed}}$

Therefore the swim takes $\dfrac{4\cos\theta}{3}$ hours and the walk takes $\dfrac{4\theta}{4}$ hours.

Thus the total time taken is given by $T = \frac{1}{3}(4\cos\theta + 3\theta)$.

b $\dfrac{dT}{d\theta} = \dfrac{1}{3}(-4\sin\theta + 3)$

The stationary point occurs where $\dfrac{dT}{d\theta} = 0$, and $\dfrac{1}{3}(-4\sin\theta + 3) = 0$ implies $\sin\theta = \dfrac{3}{4}$.

Therefore $\theta = 48.59^\circ$ to two decimal places.

From the gradient chart, the value of T is a maximum when $\theta = 48.59^\circ$.

θ		48.59°	
$\frac{dT}{d\theta}$	+	0	−
shape of T	/	—	\

Notes:

- The maximum time taken is 1.73 hours.
- If the man swims straight across the lake, it takes $1\frac{1}{3}$ hours.
- If he walks around all the way around the edge, it takes approximately 1.57 hours.

Example 30

Assume that the number of bacteria present in a culture at time t is given by $N(t)$, where $N(t) = 36te^{-0.1t}$. At what time will the population be at a maximum? Find the maximum population.

Solution

$$\begin{aligned} N(t) &= 36te^{-0.1t} \\ \therefore\quad N'(t) &= 36e^{-0.1t} - 3.6te^{-0.1t} \\ &= e^{-0.1t}(36 - 3.6t) \end{aligned}$$

Thus $N'(t) = 0$ implies $t = 10$.

The maximum population is $N(10) = 360e^{-1} \approx 132$.

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Maximum rates of increase and decrease

We know that when we take the derivative of a function we obtain a new function, the derivative, which gives the instantaneous rate of change. We can apply the same technique to the new function to find the maximum rate of increase or decrease.

Remember:

- If $\dfrac{dy}{dx} > 0$, then y is increasing as x increases.
- If $\dfrac{dy}{dx} < 0$, then y is decreasing as x increases.

We illustrate this technique by revisiting Example 30.

Example 31

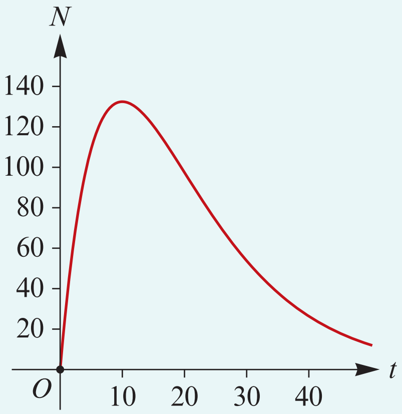
Assume that the number of bacteria present in a culture at time t is given by $N(t)$, where $N(t) = 36te^{-0.1t}$.

a Sketch the graphs of $N(t)$ against t and $N'(t)$ against t.

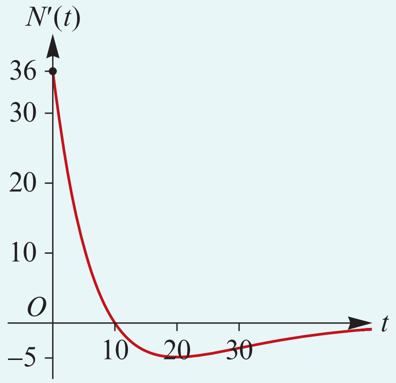
b Find the maximum rates of increase and decrease of the population and the times at which these occur.

Solution

a $N(t) = 36te^{-0.1t}$



$N'(t) = 36e^{-0.1t} - 3.6te^{-0.1t}$



b Let $R(t) = N'(t) = e^{-0.1t}(36 - 3.6t)$ be the rate of change of the population.
From the graph, the maximum value of $R(t)$ occurs at $t = 0$. Thus the maximum rate of increase of the population is $R(0) = 36$ bacteria per unit of time.

We now calculate

$$\begin{aligned} R'(t) &= -7.2e^{-0.1t} + 0.36te^{-0.1t} \\ &= e^{-0.1t}(-7.2 + 0.36t) \end{aligned}$$

Thus $R'(t) = 0$ implies $t = 20$.

The minimum value of $R(t)$ occurs at $t = 20$. Since $R(20) = -36e^{-2} \approx -4.9$, the maximum rate of decrease of the population is 4.9 bacteria per unit of time.

The second derivative and points of inflection*

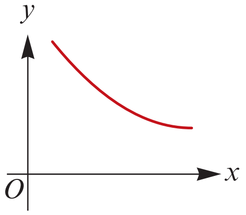
In Example 31, we used the derivative of the derivative, called the **second derivative**, to find the maximum rate of change. The second derivative can also be used in graph sketching.

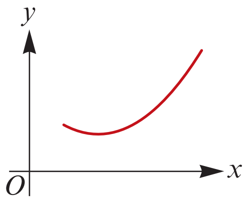
For a function f with $y = f(x)$, the second derivative of f is denoted by f'' or by $\dfrac{d^2y}{dx^2}$.

Concave up and concave down

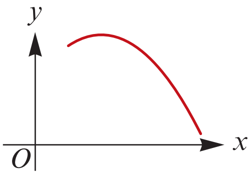
Let f be a function defined on an interval (a, b), and assume that both $f'(x)$ and $f''(x)$ exist for all $x \in (a, b)$.

If $f''(x) > 0$ for all $x \in (a, b)$, then the gradient of the curve $y = f(x)$ is increasing in the interval (a, b). The curve is **concave up**.





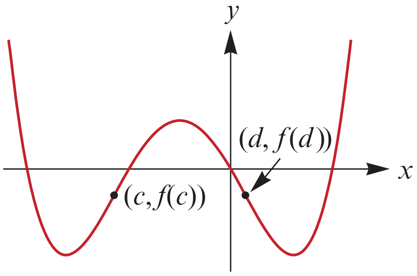
If $f''(x) < 0$ for all $x \in (a, b)$, then the gradient of the curve $y = f(x)$ is decreasing in the interval (a, b). The curve is **concave down**.



Inflection points

A point where a curve changes from concave up to concave down or from concave down to concave up is called a **point of inflection**.

In the graph on the right, there are points of inflection at $x = c$ and $x = d$.

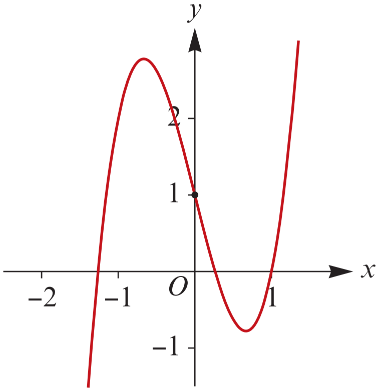


At a point of inflection of a twice differentiable function f, we must have $f''(x) = 0$. However, this condition does not necessarily guarantee a point of inflection. At a point of inflection, there must also be a change of concavity.

We return to the graph of Example 15. The function is $f(x) = 3x^3 - 4x + 1$. Here $f'(x) = 9x^2 - 4$ and $f''(x) = 18x$. We can observe:

- $f''(0) = 0$
- $f''(x) > 0$ for $x > 0$
- $f''(x) < 0$ for $x < 0$

Hence the curve is concave down to the left of 0, and the curve is concave up to the right of 0.



You may like to use this technique when you are sketching graphs in the future.

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* This material is not included in the study design of Mathematical Methods Units 3 & 4.

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Section summary

Here are some steps for solving maximum and minimum problems:

- Where possible, draw a diagram to illustrate the problem. Label the diagram and designate your variables and constants. Note the values that the variables can take.
- Write an expression for the quantity that is going to be maximised or minimised. Form an equation for this quantity in terms of a single independent variable. This may require some algebraic manipulation.
- If $y = f(x)$ is the quantity to be maximised or minimised, find the values of x for which $f'(x) = 0$.
- Test each point for which $f'(x) = 0$ to determine whether it is a local maximum, a local minimum or neither.
- If the function $y = f(x)$ is defined on an interval, such as $[a, b]$ or $[0, \infty)$, check the values of the function at the endpoints.

Exercise 10F

Example 23

1 Find the maximum area of a rectangular field that can be enclosed by 100 m of fencing.

Example 24

2 Find two positive numbers that sum to 4 and such that the sum of the cube of the first and the square of the second is as small as possible.

3 For $x + y = 100$, prove that the product $P = xy$ is a maximum when $x = y$ and find the maximum value of P.

4 A farmer has 4 km of fencing wire and wishes to fence a rectangular piece of land through which flows a straight river, which is to be utilised as one side of the enclosure. How can this be done to enclose as much land as possible?

5 Two positive quantities p and q vary in such a way that $p^3q = 9$. Another quantity z is defined by $z = 16p + 3q$. Find values of p and q that make z a minimum.

Example 25

6 A cuboid has a total surface area of 150 cm^2 with a square base of side length x cm.

a Show that the height, h cm, of the cuboid is given by $h = \dfrac{75 - x^2}{2x}$.

b Express the volume of the cuboid in terms of x.

c Hence determine its maximum volume as x varies.

Example 26, 27

7 A manufacturer finds that the daily profit, \$$P$, from selling n articles is given by $P = 100n - 0.4n^2 - 160$.

a **i** Find the value of n which maximises the daily profit.

ii Find the maximum daily profit.

b Sketch the graph of P against n. (Use a continuous graph.)

c State the allowable values of n for a profit to be made.

d Find the value of n which maximises the profit per article.

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8 The number of salmon swimming upstream in a river to spawn is approximated by $s(x) = -x^3 + 3x^2 + 360x + 5000$ with x representing the temperature of the water in degrees (°C). (This function is valid only if $6 \le x \le 20$.) Find the water temperature that produces the maximum number of salmon swimming upstream.

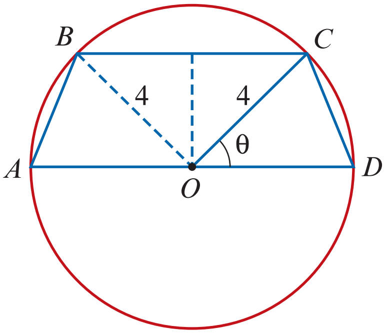
9 The number of mosquitos, $M(x)$ in millions, in a certain area depends on the average daily rainfall, x mm, during September and is approximated by

$$M(x) = \frac{1}{30}(50 - 32x + 14x^2 - x^3) \qquad \text{for } 0 \le x \le 10$$

Find the rainfall that will produce the maximum and the minimum number of mosquitos.

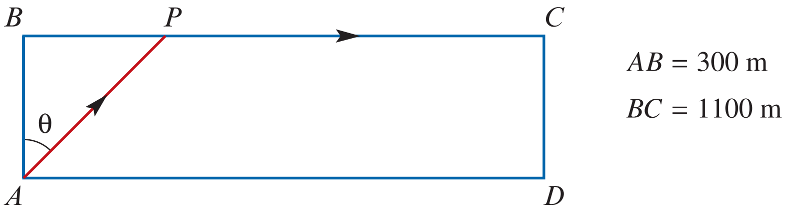
Example 28 **10** $ABCD$ is a trapezium with $AB = CD$. The vertices are on a circle with centre O and radius 4 units. The line segment AD is a diameter of the circle.

- **a** Find BC in terms of θ.
- **b** Find the area of the trapezium in terms of θ and hence find the maximum area.



11 Find the point on the parabola $y = x^2$ that is closest to the point $(3, 0)$.

Example 29 **12** The figure shows a rectangular field in which $AB = 300$ m and $BC = 1100$ m.



- **a** An athlete runs across the field from A to P at 4 m/s. Find the time taken to run from A to P in terms of θ.
- **b** The athlete, on reaching P, immediately runs to C at 5 m/s. Find the time taken to run from P to C in terms of θ.
- **c** Use the results from **a** and **b** to show that the total time taken, T seconds, is given by $T = 220 + \dfrac{75 - 60\sin\theta}{\cos\theta}$.
- **d** Find $\dfrac{dT}{d\theta}$.
- **e** Find the value of θ for which $\dfrac{dT}{d\theta} = 0$ and show that this is the value of θ for which T is a minimum.
- **f** Find the minimum value of T and find the distance of point P from B that will minimise the athlete's running time.

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Example 30 **13** The number $N(t)$ of insects in a population at time t is given by $N(t) = 50te^{-0.1t}$. At what time will the population be at a maximum? Find the maximum population.

Example 31 **14** The number $N(t)$ of insects in a population at time t is given by $N(t) = 50te^{-0.1t}$.

a Sketch the graphs of $N(t)$ against t and $N'(t)$ against t.

b Find the maximum rates of increase and decrease of the population and the times at which these occur.

15 Water is being poured into a flask. The volume, V mL, of water in the flask at time t seconds is given by

$$V(t) = \frac{3}{4}\left(10t^2 - \frac{t^3}{3}\right), \quad 0 \le t \le 20$$

a Find the volume of water in the flask when:

i $t = 0$ **ii** $t = 20$

b Find $V'(t)$, the rate of flow of water into the flask.

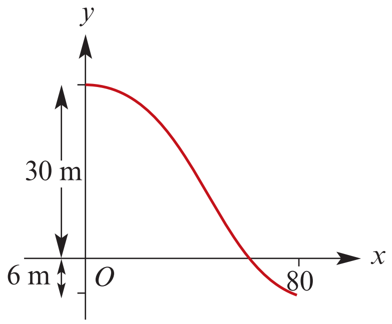
c Sketch the graph of $V(t)$ against t for $0 \le t \le 20$.

d Sketch the graph of $V'(t)$ against t for $0 \le t \le 20$.

e At what time is the flow greatest and what is the flow at this time?

16 A section of a roller coaster can be described by the rule

$$y = 18\cos\left(\frac{\pi x}{80}\right) + 12, \quad 0 \le x \le 80$$



a Find the gradient function, $\frac{dy}{dx}$.

b Sketch the graph of $\frac{dy}{dx}$ against x.

c State the coordinates of the point on the track for which the magnitude of the gradient is maximum.

17 The depth, $D(t)$ metres, of water at the entrance to a harbour at t hours after midnight on a particular day is given by

$$D(t) = 10 + 3\sin\left(\frac{\pi t}{6}\right), \quad 0 \le t \le 24$$

a Sketch the graph of $y = D(t)$ for $0 \le t \le 24$.

b Find the values of t for which $D(t) \ge 8.5$.

c Find the rate at which the depth is changing when:

i $t = 3$ **ii** $t = 6$ **iii** $t = 12$

d **i** At what times is the depth increasing most rapidly?

ii At what times is the depth decreasing most rapidly?

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10G Families of functions

Example 32

Consider the family of functions with rules of the form $f(x) = (x - a)^2(x - b)$, where a and b are positive constants with $b > a$.

a Find the derivative of $f(x)$ with respect to x.

b Find the coordinates of the stationary points of the graph of $y = f(x)$.

c Show that the stationary point at $(a, 0)$ is always a local maximum.

d Find the values of a and b if the stationary points occur where $x = 3$ and $x = 4$.

Solution

a Use a CAS calculator to find that $f'(x) = (x - a)(3x - a - 2b)$.

b The coordinates of the stationary points are $(a, 0)$ and $\left(\frac{a + 2b}{3}, \frac{4(a - b)^3}{27}\right)$.

c If $x < a$, then $f'(x) > 0$, and if $a < x < \frac{a + 2b}{3}$, then $f'(x) < 0$.

Therefore the stationary point at $(a, 0)$ is a local maximum.

d Since $a < b$, we must have $a = 3$ and $\frac{a + 2b}{3} = 4$. Therefore $b = \frac{9}{2}$.

Example 33

The graph of $y = x^3 - 3x^2$ is translated by a units in the positive direction of the x-axis and b units in the positive direction of the y-axis (where a and b are positive constants).

a Find the coordinates of the turning points of the graph of $y = x^3 - 3x^2$.

b Find the coordinates of the turning points of its image.

Solution

a The turning points have coordinates $(0, 0)$ and $(2, -4)$.

b The turning points of the image are (a, b) and $(2 + a, -4 + b)$.

Example 34

A cubic function with rule $f(x) = ax^3 + bx^2 + cx$ has a stationary point at $(1, 6)$.

a Find a and b in terms of c.

b Find the value of c for which the graph has a stationary point at $x = 2$.

Solution

a Since $f(1) = 6$, we obtain

$$a + b + c = 6 \qquad (1)$$

Since $f'(x) = 3ax^2 + 2bx + c$ and $f'(1) = 0$, we obtain

$$3a + 2b + c = 0 \qquad (2)$$

The solution of equations (1) and (2) is $a = c - 12$ and $b = 18 - 2c$.

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b The rule is

$$f(x) = (c - 12)x^3 + (18 - 2c)x^2 + cx$$
$$\therefore \quad f'(x) = 3(c - 12)x^2 + 2(18 - 2c)x + c$$

If $f'(2) = 0$, then

$$12(c - 12) + 4(18 - 2c) + c = 0$$
$$5c - 72 = 0$$
$$\therefore \quad c = \frac{72}{5}$$

Exercise 10G

Example 32

1 Consider the family of functions with rules $f(x) = (x - 1)^2(x - b)$, where $b > 1$.

a Find the derivative of $f(x)$ with respect to x.

b Find the coordinates of the stationary points of the graph of $y = f(x)$.

c Show that the stationary point at $(1, 0)$ is always a local maximum.

d Find the value of b if the stationary points occur where $x = 1$ and $x = 4$.

Example 33

2 The graph of the function $y = x^4 - 4x^2$ is translated by a units in the positive direction of the x-axis and b units in the positive direction of the y-axis (where a and b are positive constants).

a Find the coordinates of the turning points of the graph of $y = x^4 - 4x^2$.

b Find the coordinates of the turning points of its image.

Example 34

3 A cubic function f has rule $f(x) = ax^3 + bx^2 + cx$. The graph has a stationary point at $(1, 10)$.

a Find a and b in terms of c.

b Find the value of c for which the graph has a stationary point at $x = 3$.

4 Consider the function $f\colon [0, \infty) \to \mathbb{R}$ defined by $f(x) = x^2 - ax^3$, where a is a real number with $a > 0$.

a Determine the intervals on which f is a strictly decreasing function and the intervals on which f is a strictly increasing function.

b Find the equation of the tangent to the graph of f at the point $\left(\frac{1}{a}, 0\right)$.

c Find the equation of the normal to the graph of f at the point $\left(\frac{1}{a}, 0\right)$.

d What is the range of f?

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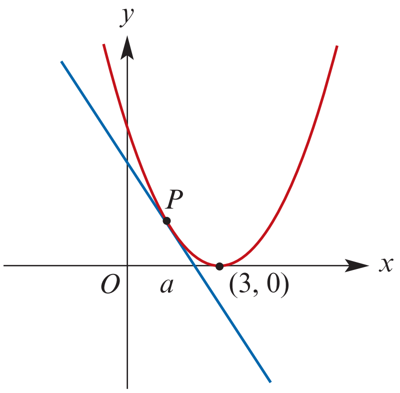
5 A line with equation $y = mx + c$ is a tangent to the curve $y = (x - 3)^2$ at a point $P(a, y)$ where $0 < a < 3$.



- **a** **i** Find the gradient of the curve at $x = a$ for $0 < a < 3$.
  - **ii** Hence express m in terms of a.
- **b** State the coordinates of the point P, expressing your answer in terms of a.
- **c** Find the equation of the tangent where $x = a$.
- **d** Find the x-axis intercept of the tangent.

6 **a** The graph of $f(x) = x^4$ is translated to the graph of $y = f(x + h)$. Find the possible values of h if $f(1 + h) = 16$.

- **b** The graph of $f(x) = x^3$ is transformed to the graph of $y = f(ax)$. Find the possible value of a if the graph of $y = f(ax)$ passes through the point with coordinates $(1, 8)$.
- **c** The quartic function with equation $y = ax^4 - bx^3$ has a turning point with coordinates $(1, 16)$. Find the values of a and b.

7 Consider the cubic function with rule $f(x) = (x - a)^2(x - 1)$ where $a > 1$.

- **a** Find the coordinates of the turning points of the graph of $y = f(x)$.
- **b** State the nature of each of the turning points.
- **c** Find the equation of the tangent to the curve at the point where:

  i $x = 1$ **ii** $x = a$ **iii** $x = \dfrac{a+1}{2}$

8 Consider the quartic function with rule $f(x) = (x - 1)^2(x - b)^2$ where $b > 1$.

- **a** Find the derivative of f.
- **b** Find the coordinates of the turning points of f.
- **c** Find the value of b such that the graph of $y = f(x)$ has a turning point at $(2, 1)$.

9 A cubic function has rule $y = ax^3 + bx^2 + cx + d$. It passes through the points $(1, 6)$ and $(10, 8)$ and has turning points where $x = -1$ and $x = 1$. Find the values of a, b, c and d.

10 A quartic function f has rule $f(x) = ax^4 + bx^3 + cx^2 + dx$. The graph has a stationary point at $(1, 1)$ and passes through the point $(-1, 4)$.

- **a** Find a, b and c in terms of d.
- **b** Find the value of d for which the graph has a stationary point at $x = 4$.

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Chapter summary

Spreadsheet AS Nrich

Tangents and normals

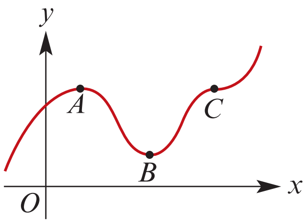
Let (x_1, y_1) be a point on the curve $y = f(x)$. If f is differentiable at $x = x_1$, then

- the equation of the **tangent** to the curve at (x_1, y_1) is given by $y - y_1 = f'(x_1)(x - x_1)$
- the equation of the **normal** to the curve at (x_1, y_1) is given by $y - y_1 = \dfrac{-1}{f'(x_1)}(x - x_1)$.

Stationary points

A point with coordinates $(a, f(a))$ on a curve $y = f(x)$ is a **stationary point** if $f'(a) = 0$.

The graph shown has three stationary points: A, B and C.



A Point A is a **local maximum** point. Notice that immediately to the left of A the gradient is positive, and immediately to the right the gradient is negative.

B Point B is a **local minimum** point. Notice that immediately to the left of B the gradient is negative, and immediately to the right the gradient is positive.

C Point C is a **stationary point of inflection**.

Stationary points of types A and B are referred to as **turning points**.

Maximum and minimum values

For a continuous function f defined on an interval $[a, b]$:

- if M is a value of the function such that $f(x) \leq M$ for all $x \in [a, b]$, then M is the **absolute maximum** value of the function
- if N is a value of the function such that $f(x) \geq N$ for all $x \in [a, b]$, then N is the **absolute minimum** value of the function.

Motion in a straight line

For an object moving in a straight line with position x at time t:

$$\text{velocity } v = \frac{dx}{dt} \qquad \text{acceleration } a = \frac{dv}{dt}$$

Technology-free questions

1 **a** Find the equation of the tangent to the curve $y = x^3 - 8x^2 + 15x$ at the point with coordinates $(4, -4)$.

b Find the coordinates of the point where the tangent meets the curve again.

2 Find the equation of the tangent to the curve $y = 3x^2$ at the point where $x = a$. If this tangent meets the y-axis at P, find the y-coordinate of P in terms of a.

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3 **a** Find the equation of the tangent to the curve with equation $y = x^3 - 7x^2 + 14x - 8$ at the point where $x = 1$.

b Find the x-coordinate of a second point on this curve at which the tangent is parallel to the tangent at $x = 1$.

4 Use the formula $A = \pi r^2$ for the area of a circle to find:

a the average rate at which the area of a circle changes with respect to the radius as the radius increases from $r = 2$ to $r = 3$

b the instantaneous rate at which the area changes with respect to r when $r = 3$.

5 For each of the following, find the stationary points of the graph and state their nature:

a $f(x) = 4x^3 - 3x^4$

b $g(x) = x^3 - 3x - 2$

c $h(x) = x^3 - 9x + 1$

6 Sketch the graph of $y = x^3 - 6x^2 + 9x$.

7 The derivative of the function $y = f(x)$ is $\frac{dy}{dx} = (x - 1)^2(x - 2)$. Find the x-coordinate and state the nature of each stationary point.

8 Find the equation of the tangent to the curve $y = x^3 - 3x^2 - 9x + 11$ at $x = 2$.

9 Let $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = 3 + 6x^2 - 2x^3$. Determine the values of x for which the graph of $y = f(x)$ has a positive gradient.

10 For what value(s) of x do the graphs of $y = x^3$ and $y = x^3 + x^2 + x - 2$ have the same gradient?

11 For the function with rule $f(x) = (x - 1)^{\frac{4}{5}}$:

a State the values for which the function is differentiable, and find the rule for f'.

b Find the equations of the tangents at the points $(2, 1)$ and $(0, 1)$.

c Find the coordinates of the point of intersection of the two tangents.

12 A spherical bubble, initially of radius 1 cm, expands steadily, its radius increases by 1 cm/s and it bursts after 5 seconds.

a Find the rate of increase of volume with respect to the radius when the radius is 4 cm.

b Find the rate of increase of volume with respect to time when the radius is 4 cm.

13 A vehicle is travelling in a straight line away from a point O. Its distance from O after t seconds is $0.25e^t$ metres. Find the velocity of the vehicle at $t = 0$, $t = 1$, $t = 2$, $t = 4$.

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14 The temperature, θ°C, of material inside a nuclear power station at time t seconds after a reaction begins is given by $\theta = \frac{1}{4}e^{100t}$.

a Find the rate of increase of temperature at time t.

b Find the rate of increase of temperature when $t = \frac{1}{20}$.

15 Find the equation of the tangent to $y = e^x$ at $(1, e)$.

16 The diameter of a tree (D cm) t years after 1 January 2010 is given by $D = 50e^{kt}$.

a Prove that $\frac{dD}{dt} = cD$ for some constant c.

b If $k = 0.2$, find the rate of increase of D when $D = 100$.

17 Find the minimum value of $e^{3x} + e^{-3x}$.

18 **a** Find the equation of the tangent to $y = \log_e x$ at the point $(e, 1)$.

b Find the equation of the tangent to $y = 2\sin\left(\frac{x}{2}\right)$ at the point $\left(\frac{\pi}{2}, \sqrt{2}\right)$.

c Find the equation of the tangent to $y = \cos x$ at the point $\left(\frac{3\pi}{2}, 0\right)$.

d Find the equation of the tangent to $y = \log_e(x^2)$ at the point $(-\sqrt{e}, 1)$.

Multiple-choice questions

1 The line with equation $y = 4x + c$ is a tangent to the curve with equation $y = x^2 - x - 5$. The value of c is

A $-\frac{45}{4}$ **B** $-1 + 2\sqrt{2}$ **C** 2 **D** $\frac{5}{2}$ **E** $-\frac{2}{5}$

2 The equation of the tangent to the curve with equation $y = x^4$ at the point where $x = 1$ is

A $y = -4x - 3$ **B** $y = \frac{1}{4}x - 3$ **C** $y = -4x$

D $y = \frac{1}{4}x + \frac{5}{4}$ **E** $y = 4x - 3$

3 For a polynomial function with rule $f(x)$, the derivative satisfies $f'(a) = f'(b) = 0$, $f'(x) > 0$ for $x \in (a, b)$, $f'(x) < 0$ for $x < a$ and $f'(x) > 0$ for $x > b$. The nature of the stationary points of the graph of $y = f(x)$ is

A local maximum at $(a, f(a))$ and local minimum at $(b, f(b))$

B local minimum at $(a, f(a))$ and local maximum at $(b, f(b))$

C stationary point of inflection at $(a, f(a))$ and local minimum at $(b, f(b))$

D stationary point of inflection at $(a, f(a))$ and local maximum at $(b, f(b))$

E local minimum at $(a, f(a))$ and stationary point of inflection at $(b, f(b))$

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4 The graph of a polynomial function with rule $y = f(x)$ has a local maximum at the point with coordinates $(a, f(a))$. The graph also has a local minimum at the origin, but no other stationary points. The graph of the function with rule $y = -2f\left(\frac{x}{2}\right) + k$, where k is a positive real number, has

A a local maximum at the point with coordinates $(2a, -2f(a) + k)$

B a local minimum at the point with coordinates $\left(\frac{a}{2}, 2f(a) + k\right)$

C a local maximum at the point with coordinates $\left(\frac{a}{2}, -2f(a) + k\right)$

D a local maximum at the point with coordinates $(2a, -2f(a) - k)$

E a local minimum at the point with coordinates $(2a, -2f(a) + k)$

5 For $f(x) = x^3 - x^2 - 1$, the values of x for which the graph of $y = f(x)$ has stationary points are

A $\frac{2}{3}$ only **B** 0 and $\frac{2}{3}$ **C** 0 and $-\frac{2}{3}$ **D** $-\frac{1}{3}$ and 1 **E** $\frac{1}{3}$ and -1

6 A function f is differentiable for all values of x in $[0, 6]$, and the graph with equation $y = f(x)$ has a local minimum point at $(2, 4)$. The equation of the tangent at the point with coordinates $(2, 4)$ is

A $y = 2x$ **B** $x = 2$ **C** $y = 4$ **D** $2x - 4y = 0$ **E** $4x - 2y = 0$

7 The volume, V cm^3, of a solid is given by the formula $V = -10x(2x^2 - 6)$ where x cm is a particular measurement. The value of x for which the volume is a maximum is

A 0 **B** 1 **C** $\sqrt{2}$ **D** $\sqrt{3}$ **E** 2

8 The equation of the normal to the curve with equation $y = x^2$ at the point where $x = a$ is

A $y = \frac{-1}{2a}x + 2 + a^2$ **B** $y = \frac{-1}{2a}x + \frac{1}{2} + a^2$ **C** $y = 2ax - a^2$

D $y = 2ax + 3a^2$ **E** $y = \frac{1}{2a}x + 2 + a^2$

9 For $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^x - ex$, the coordinates of the turning point of the graph of $y = f(x)$ are

A $\left(1, \frac{1}{e}\right)$ **B** $(1, e)$ **C** $(0, 1)$ **D** $(1, 0)$ **E** $(e, 1)$

10 The equation of the tangent to $y = e^{ax}$ at the point $\left(\frac{1}{a}, e\right)$ is

A $y = e^{ax-1} + 1$ **B** $y = ae^{ax}x$ **C** $y = 1 - ae^{ax}$ **D** $y = \frac{e^2x}{a}$ **E** $y = aex$

11 Under certain conditions, the number of bacteria, N, in a sample increases with time, t hours, according to the rule $N = 4000e^{0.2t}$. The rate, to the nearest whole number of bacteria per hour, that the bacteria are growing 3 hours from the start is

A 1458 **B** 7288 **C** 16 068 **D** 80 342 **E** 109 731

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12 The gradient of the tangent to the curve $y = x^2 \cos(5x)$ at the point where $x = \pi$ is

A $5\pi^2$ **B** $-5\pi^2$ **C** 5π **D** -5π **E** -2π

13 The equation of the tangent to the curve with equation $y = e^{-x} - 1$ at the point where the curve crosses the y-axis is

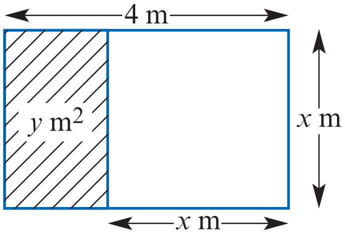
A $y = x$ **B** $y = -x$ **C** $y = \frac{1}{2}x$ **D** $y = -\frac{1}{2}x$ **E** $y = -2x$

14 For $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{ax} - \frac{ax}{e}$, the coordinates of the turning point of the graph of $y = f(x)$ are

A $\left(-\frac{1}{a}, 0\right)$ **B** $\left(\frac{1}{a}, \frac{1}{e}\right)$ **C** $\left(-\frac{1}{a}, \frac{2}{e}\right)$ **D** $\left(-1, \frac{1}{e}\right)$ **E** $(1, 0)$

Extended-response questions

1 The diagram shows a rectangle with sides 4 m and x m and a square with side x m. The area of the shaded region is y m^2.



a Find an expression for y in terms of x.

b Find the set of possible values for x.

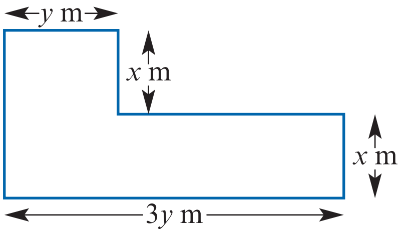
c Find the maximum value of y and the corresponding value of x.

d Explain briefly why this value of y is a maximum.

e Sketch the graph of y against x.

f State the set of possible values for y.

2 A flower bed is to be L-shaped, as shown in the figure, and its perimeter is 48 m.



a Write down an expression for the area, A m^2, in terms of y and x.

b Find y in terms of x.

c Write down an expression for A in terms of x.

d Find the values of x and y that give the maximum area.

e Find the maximum area.

3 It costs $(12 + 0.008x)$ dollars per kilometre to operate a truck at x kilometres per hour. In addition it costs \$14.40 per hour to pay the driver.

a What is the total cost per kilometre if the truck is driven at:

i 40 km/h **ii** 64 km/h?

b Write an expression for C, the total cost per kilometre, in terms of x.

c Sketch the graph of C against x for $0 < x < 120$.

d At what speed should the truck be driven to minimise the total cost per kilometre?

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4 A box is to be made from a 10 cm by 16 cm sheet of metal by cutting equal squares out of the corners and bending up the flaps to form the box. Let the lengths of the sides of the squares be x cm and let the volume of the box formed be V cm^3.

a Show that $V = 4(x^3 - 13x^2 + 40x)$.

b State the set of x-values for which the expression for V in terms of x is valid.

c Find the values of x such that $\frac{dV}{dx} = 0$.

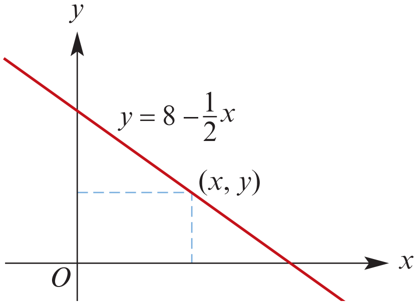
d Find the dimensions of the box if the volume is to be a maximum.

e Find the maximum volume of the box.

f Sketch the graph of V against x for the domain established in **b**.

5 A rectangle has one vertex at the origin, another on the positive x-axis, another on the positive y-axis and a fourth on the line $y = 8 - \frac{x}{2}$.

What is the greatest area the rectangle can have?



6 At a factory the time, T seconds, spent in producing a certain size metal component is related to its weight, w kg, by $T = k + 2w^2$, where k is a constant.

a If a 5 kg component takes 75 seconds to produce, find k.

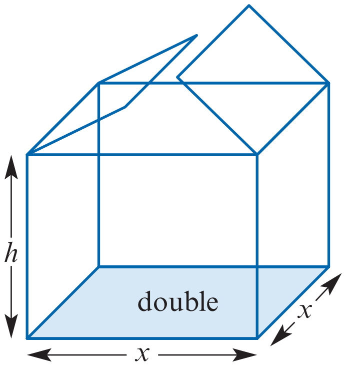
b Sketch the graph of T against w.

c Write down an expression for the average time A (in seconds per kilogram).

d **i** Find the weight that yields the minimum average machining time.

ii State the minimum average machining time.

7 A manufacturer produces cardboard boxes that have a square base. The top of each box consists of a double flap that opens as shown. The bottom of the box has a double layer of cardboard for strength. Each box must have a volume of 12 cubic metres.



a Show that the area of cardboard required is given by $C = 3x^2 + 4xh$.

b Express C as a function of x only.

c Sketch the graph of C against x for $x > 0$.

d **i** What dimensions of the box will minimise the amount of cardboard used?

ii What is the minimum area of cardboard used?

8 An open tank is to be constructed with a square base and vertical sides to contain 500 m^3 of water. What must be the area of sheet metal used in its construction if this area is to be a minimum?

9 A piece of wire of length 1 m is bent into the shape of a sector of a circle of radius a cm and sector angle θ. Let the area of the sector be A cm^2.

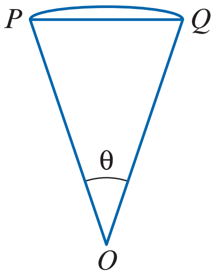
a Find A in terms of a and θ.

b Find A in terms of θ.

c Find the value of θ for which A is a maximum.

d Find the maximum area of the sector.

10 A piece of wire of fixed length, L cm, is bent to form the boundary $OPQO$ of a sector of a circle. The circle has centre O and radius r cm. The angle of the sector is θ radians.



a Show that the area, A cm^2, of the sector is given by

$$A = \frac{1}{2}rL - r^2$$

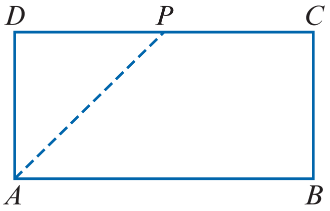
b **i** Find a relationship between r and L for which $\frac{dA}{dr} = 0$.

ii Find the corresponding value of θ.

iii Determine the nature of the stationary point found in **i**.

c Show that, for the value of θ found in **b ii**, the area of the triangle OPQ is approximately 45.5% of the area of sector OPQ.

11 A Queensland resort has a large swimming pool as illustrated, with $AB = 75$ m and $AD = 30$ m. A boy can swim at 1 m/s and run at $1\frac{2}{3}$ m/s. He starts at A, swims to a point P on DC, and runs from P to C. He takes 2 seconds to pull himself out of the pool. Let $DP = x$ m and the total time taken be T s.



a Show that $T = \sqrt{x^2 + 900} + \frac{3}{5}(75 - x) + 2$.

b Find $\frac{dT}{dx}$.

c **i** Find the value of x for which the time taken is a minimum.

ii Find the minimum time.

d Find the time taken if the boy runs from A to D and then from D to C.

12 **a** Find the equation of the tangent to the curve $y = e^x$ at the point $(1, e)$.

b Find the equation of the tangent to the curve $y = e^{2x}$ at the point $\left(\frac{1}{2}, e\right)$.

c Find the equation of the tangent to the curve $y = e^{kx}$ at the point $\left(\frac{1}{k}, e\right)$.

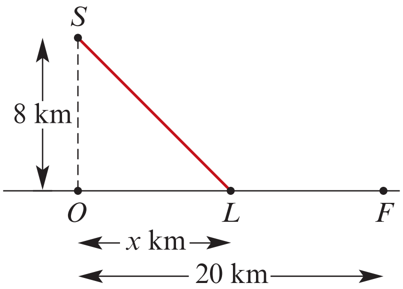
d Show that $y = xke$ is the only tangent to the curve $y = e^{kx}$ which passes through the origin.

e Hence determine for what values of k the equation $e^{kx} = x$ has:

i a unique real solution

ii no real solution.

13 The point S is 8 km offshore from the point O, which is located on the straight shore of a lake, as shown in the diagram. The point F is on the shore, 20 km from O. Contestants race from the start, S, to the finish, F, by rowing in a straight line to some point, L, on the shore and then running along the shore to F. A certain contestant rows at 5 km per hour and runs at 15 km per hour.



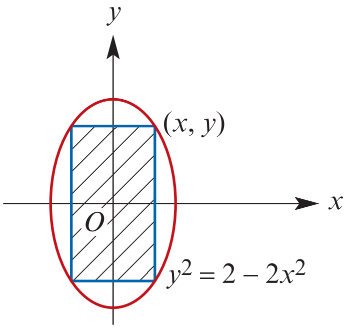
a Show that, if the distance OL is x km, the time taken by this contestant to complete the course is (in hours):

$$T(x) = \frac{\sqrt{64 + x^2}}{5} + \frac{20 - x}{15}$$

b Show that the time taken by this contestant to complete the course has its minimum value when $x = 2\sqrt{2}$. Find this time.

14 At noon the captain of a ship sees two fishing boats approaching. One of them is 10 km due east and travelling west at 8 km/h. The other is 6 km due north and travelling south at 6 km/h. At what time will the fishing boats be closest together and how far apart will they be?

15 A rectangular beam is to be cut from a non-circular tree trunk whose cross-sectional outline can be represented by the equation $y^2 = 2 - 2x^2$.



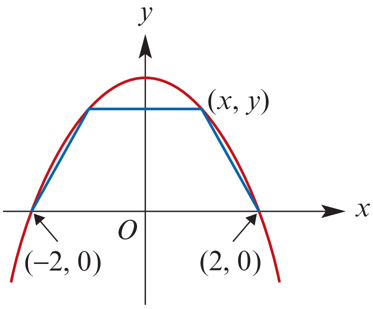
a Show that the area of the cross-section of the beam is given by $A = 4x\sqrt{2 - 2x^2}$ where x is the half-width of the beam.

b State the possible values for x.

c Find the value of x for which the cross-sectional area of the beam is a maximum and find the corresponding value of y.

d Find the maximum cross-sectional area of the beam.

16 An isosceles trapezium is inscribed in the parabola $y = 4 - x^2$ as illustrated.



a Show that the area of the trapezium is

$$\frac{1}{2}(4 - x^2)(2x + 4)$$

b Show that the trapezium has its greatest area when $x = \frac{2}{3}$.

c Repeat with the parabola $y = a^2 - x^2$:

- **i** Show that the area, A, of the trapezium is given by $(a^2 - x^2)(a + x)$.
- **ii** Use the product rule to find $\frac{dA}{dx}$.
- **iii** Show that a maximum occurs when $x = \frac{a}{3}$.

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17 Assume that the number of bacteria present in a culture at time t is given by $N(t)$ where $N(t) = 24te^{-0.2t}$. At what time will the population be at a maximum? Find the maximum population.

18 It is believed that, for some time after planting in ideal conditions, the area covered by a particular species of ground-cover plant has a rate of increase of y cm^2/week, given by $y = -t^3 + bt^2 + ct$ where t is the number of weeks after planting.

t	1	2
y	10	24

- **a** Find b and c using the table of observations on the right.
- **b** Assume that the model is accurate for the first 8 weeks after planting. When during this period is:
  - **i** the area covered by the plant a maximum
  - **ii** the rate of increase in area a maximum?
- **c** According to the model, if the plant covered 100 cm^2 when planted, what area will it cover after 4 weeks?
- **d** Discuss the implications for the future growth of the plant if the model remains accurate for longer than the first 4 weeks.

19 Let $f(x) = x^3 - 3x^2 + 6x - 10$.

- **a** Find the coordinates of the point on the graph of f for which $f'(x) = 3$.
- **b** Express $f'(x)$ in the form $a(x + p)^2 + q$.
- **c** Hence show that the gradient of f is greater than 3 for all points on the curve of f other than the point found in **a**.

20 A curve with equation of the form $y = ax^3 + bx^2 + cx + d$ has zero gradient at the point $\left(\frac{1}{3}, \frac{4}{27}\right)$ and also touches, but does not cross, the x-axis at the point $(1, 0)$.

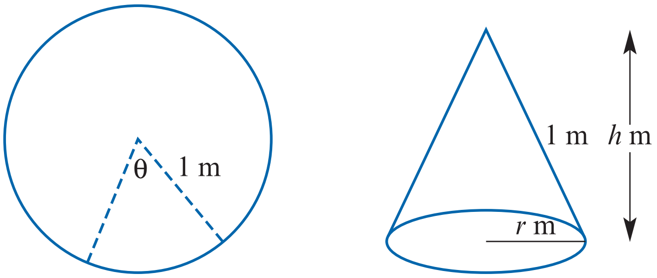
- **a** Find a, b, c and d.
- **b** Find the values of x for which the curve has a negative gradient.
- **c** Sketch the curve.

21 The volume of water, V m^3, in a reservoir when the depth indicator shows y metres is given by the formula

$$V = \frac{\pi}{3}[(y + 630)^3 - 630^3]$$

- **a** Find the volume of water in the reservoir when $y = 40$.
- **b** Find the rate of change of volume with respect to depth, y.
- **c** Sketch the graph of V against y for $0 \le y \le 60$.
- **d** If $y = 60$ m is the maximum depth of the reservoir, find the capacity (m^3) of the reservoir.
- **e** If $\frac{dV}{dt} = 20\,000 - 0.005\pi(y + 630)^2$, where t is the time in days from 1 January, sketch the graph of $\frac{dV}{dt}$ against y for $0 \le y \le 60$.

22 A cone is made by cutting out a sector with central angle θ from a circular piece of cardboard of radius 1 m and joining the two cut edges to form a cone of slant height 1 m as shown in the following diagrams.



The volume of a cone is given by the formula $V = \frac{1}{3}\pi r^2 h$.

a **i** Find r in terms of θ.

ii Find h in terms of θ.

iii Show that $V = \frac{1}{3}\pi\left(\frac{2\pi - \theta}{2\pi}\right)^2 \sqrt{1 - \left(\frac{2\pi - \theta}{2\pi}\right)^2}$.

b Find the value of V when $\theta = \frac{\pi}{4}$.

c Find the value(s) of θ for which the volume of the cone is 0.3 m^3.

d **i** Use a calculator to determine the value of θ that maximises the volume of the cone.

ii Find the maximum volume.

e Determine the maximum volume using calculus.

23 **a** For the function with rule $f(x) = x^3 + ax^2 + bx$, plot the graph of each of the following using a calculator. (Give axis intercepts, coordinates of stationary points and the nature of stationary points.)

i $a = 1,\ b = 1$

ii $a = -1,\ b = -1$

iii $a = 1,\ b = -1$

iv $a = -1,\ b = 1$

b **i** Find $f'(x)$.

ii Solve the equation $f'(x) = 0$ for x, giving your answer in terms of a and b.

c **i** Show that the graph of $y = f(x)$ has exactly one stationary point if $a^2 - 3b = 0$.

ii If $b = 3$, find the corresponding value(s) of a which satisfy $a^2 - 3b = 0$. Find the coordinates of the stationary points and state the nature of each.

iii Using a calculator, plot the graph(s) of $y = f(x)$ for these values of a and b.

iv Plot the graphs of the corresponding derivative functions on the same set of axes.

d State the relationship between a and b if no stationary points exist for the graph of $y = f(x)$.

24 For what value of x is $\frac{\log_e x}{x}$ a maximum? That is, when is the ratio of the logarithm of a number to the number a maximum?

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25 Consider the function with rule $f(x) = 6x^4 - x^3 + ax^2 - 6x + 8$.

a **i** If $x + 1$ is a factor of $f(x)$, find the value of a.

ii Using a calculator, plot the graph of $y = f(x)$ for this value of a.

b Let $g(x) = 6x^4 - x^3 + 21x^2 - 6x + 8$.

i Plot the graph of $y = g(x)$.

ii Find the minimum value of $g(x)$ and the value of x for which this occurs.

iii Find $g'(x)$.

iv Using a calculator, solve the equation $g'(x) = 0$ for x.

v Find $g'(0)$ and $g'(10)$.

vi Find the derivative of $g'(x)$.

vii Show that the graph of $y = g'(x)$ has no stationary points and thus deduce that $g'(x) = 0$ has only one solution.

26 For the quartic function f with rule $f(x) = (x - a)^2(x - b)^2$, where $a > 0$ and $b > 0$:

a Show that $f'(x) = 2(x - a)(x - b)[2x - (b + a)]$.

b **i** Solve the equation $f'(x) = 0$ for x. **ii** Solve the equation $f(x) = 0$ for x.

c Hence find the coordinates of the stationary points of the graph of $y = f(x)$.

d Plot the graph of $y = f(x)$ on a calculator for several values of a and b.

e **i** If $a = b$, then $f(x) = (x - a)^4$. Sketch the graph of $y = f(x)$.

ii If $a = -b$, find the coordinates of the stationary points.

iii Plot the graph of $y = f(x)$ for several values of a, given that $a = -b$.

27 For the quartic function f with rule $f(x) = (x - a)^3(x - b)$, where $a > 0$ and $b > 0$:

a Show that $f'(x) = (x - a)^2[4x - (3b + a)]$.

b **i** Solve the equation $f'(x) = 0$. **ii** Solve the equation $f(x) = 0$.

c Find the coordinates of the stationary points of the graph of $y = f(x)$ and state the nature of the stationary points.

d Using a calculator, plot the graph of $y = f(x)$ for several values of a and b.

e If $a = -b$, state the coordinates of the stationary points in terms of a.

f **i** State the relationship between b and a if there is a local minimum for $x = 0$.

ii Illustrate this for $b = 1$ and $a = -3$ on a calculator.

g Show that, if there is a turning point for $x = \frac{a + b}{2}$, then $b = a$ and $f(x) = (x - a)^4$.

28 A psychologist hypothesised that the ability of a mouse to memorise during the first 6 months of its life can be modelled by the function f given by $f\colon (0, 6) \to \mathbb{R}$, $f(x) = x \log_e x + 1$, i.e. the ability to memorise at age x months is $f(x)$.

a Find $f'(x)$.

b Find the value of x for which $f'(x) = 0$ and hence find when the mouse's ability to memorise is a minimum.

c Sketch the graph of f.

d When is the mouse's ability to memorise a maximum in this period?

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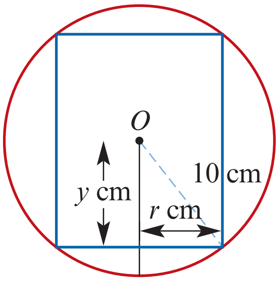
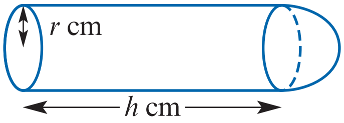
29 A cylinder is to be cut from a sphere. The cross-section through the centre of the sphere is as shown. The radius of the sphere is 10 cm. Let r cm be the radius of the cylinder.



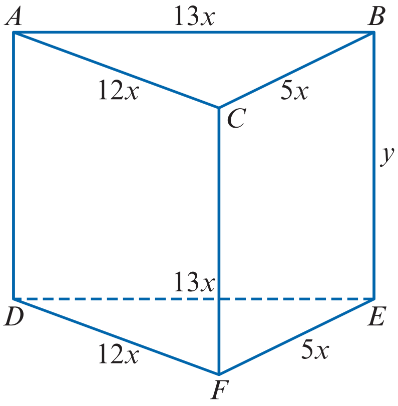
- **a** **i** Find y in terms of r and hence the height, h cm, of the cylinder.
  - **ii** The volume of a cylinder is given by $V = \pi r^2 h$. Find V in terms of r.
- **b** **i** Plot the graph of V against r using a calculator.
  - **ii** Find the maximum volume of the cylinder and the corresponding values of r and h. (Use a calculator.)
  - **iii** Find the two possible values of r if the volume is 2000 cm^3.
- **c** **i** Find $\dfrac{dV}{dr}$.
  - **ii** Hence find the exact value of the maximum volume and the volume of r for which this occurs.
- **d** **i** Plot the graph of the derivative function $\dfrac{dV}{dr}$ against r, using a calculator.
  - **ii** From the calculator, find the values of r for which $\dfrac{dV}{dr}$ is positive.
  - **iii** From the calculator, find the values of r for which $\dfrac{dV}{dr}$ is increasing.

30 A wooden peg consists of a cylinder of length h cm and a hemispherical cap of radius r cm, and so the volume, V cm^3, of the peg is given by $V = \pi r^2 h + \dfrac{2}{3}\pi r^3$. If the surface area of the peg is 100π cm:



- **a** Find h in terms of r.
- **b** Find V as a function of r.
- **c** Find the possible values of r (i.e. find the domain of the function defined in **b**).
- **d** Find $\dfrac{dV}{dr}$.
- **e** Sketch the graph of V against r.

31 A triangular prism has dimensions as shown in the diagram. All lengths are in centimetres. The volume of the prism is 3000 cm^3.



- **a** **i** Find y in terms of x.
  - **ii** Find the surface area of the prism, S cm^2, in terms of x.
- **b** **i** Find $\dfrac{dS}{dx}$.
  - **ii** Find the minimum surface area, correct to two decimal places.
- **c** Given that x is increasing at 0.5 cm/s find the rate at which the surface area is increasing when $x = 10$.

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32 The kangaroo population in a certain confined region is given by $f(x) = \dfrac{100\,000}{1 + 100e^{-0.3x}}$, where x is the time in years.

a Find $f'(x)$.

b Find the rate of growth of the kangaroo population when:

i $x = 0$ **ii** $x = 4$

33 Consider the function $f\colon \{x : x < a\} \to \mathbb{R}$, $f(x) = 8\log_e(6 - 0.2x)$ where a is the largest value for which f is defined.

a What is the value of a?

b Find the exact values for the coordinates of the points where the graph of $y = f(x)$ crosses each axis.

c Find the gradient of the tangent to the graph of $y = f(x)$ at the point where $x = 20$.

d Find the rule of the inverse function f^{-1}.

e State the domain of the inverse function f^{-1}.

f Sketch the graph of $y = f(x)$.

34 **a** Using a calculator, plot the graphs of $f(x) = \sin x$ and $g(x) = e^{\sin x}$ on the one screen.

b Find $g'(x)$ and hence find the coordinates of the stationary points of $y = g(x)$ for $x \in [0, 2\pi]$.

c Give the range of g. **d** State the period of g.

35 **a** Show that the tangent to the graph of $y = e^x$ for $x = 0$ has equation $y = x + 1$.

b Plot the graphs of $y = e^x$ and $y = x + 1$ on a calculator.

c Let $f(x) = e^x$ and $g(x) = x + 1$. Use a calculator to investigate functions of the form

$$h(x) = af(x - b) + c \quad \text{and} \quad k(x) = ag(x - b) + c$$

Comment on your observations.

d Use the chain rule and properties of transformations to prove that, if the tangent to the curve $y = f(x)$ at the point (x_1, y_1) has equation $y = mx + c$, then the tangent to the curve $y = af(bx)$ at the point $\left(\dfrac{x_1}{b}, y_1 a\right)$ has equation $y = a(mbx + c)$.

36 A certain chemical starts to dissolve in water at time $t = 0$. It is known that, if x is the number of grams not dissolved after t hours, then

$$x = \frac{60}{5e^{\lambda t} - 3}, \quad \text{where } \lambda = \frac{1}{2}\log_e\left(\frac{6}{5}\right)$$

a Find the amount of chemical present when:

i $t = 0$ **ii** $t = 5$

b Find $\dfrac{dx}{dt}$ in terms of t.

c **i** Show that $\dfrac{dx}{dt} = -\lambda x - \dfrac{\lambda x^2}{20}$.

ii Sketch the graph of $\dfrac{dx}{dt}$ against x for $x \geq 0$.

iii Write a short explanation of your result.

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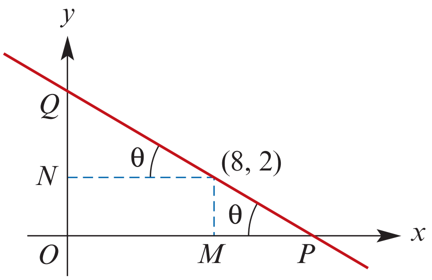
37 A straight line is drawn through the point $(8, 2)$ to intersect the positive y-axis at Q and the positive x-axis at P. (In this problem we will determine the minimum value of $OP + OQ$.)



a Show that the derivative of $\dfrac{1}{\tan\theta}$ is $-\operatorname{cosec}^2\theta$.

b Find MP in terms of θ.

c Find NQ in terms of θ.

d Hence find $OP + OQ$ in terms of θ. Denote $OP + OQ$ by x.

e Find $\dfrac{dx}{d\theta}$.

f Find the minimum value of x and the value of θ for which this occurs.

38 Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^x - e^{-x}$.

a Find $f'(x)$.

b Find $\{ x : f(x) = 0 \}$.

c Show that $f'(x) > 0$ for all x.

d Sketch the graph of f.

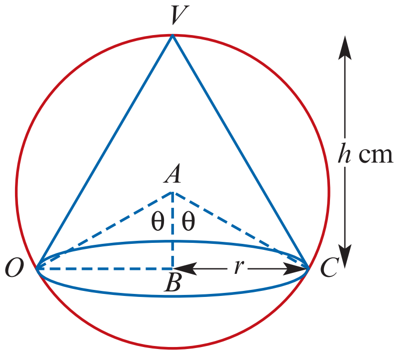
39 **a** Find all values of x for which $(\log_e x)^2 = 2\log_e x$.

b Find the gradient of each of the curves $y = 2\log_e x$ and $y = (\log_e x)^2$ at the point $(1, 0)$.

c Use these results to sketch, on one set of axes, the graphs of $y = 2\log_e x$ and $y = (\log_e x)^2$.

d Find $\{ x : 2\log_e x > (\log_e x)^2 \}$.

40 A cone is inscribed inside a sphere as illustrated. The radius of the sphere is a cm, and the magnitude of $\angle OAB$ = magnitude of $\angle CAB = \theta$. The height of the cone is h cm and the radius of the cone is r cm.



a Find h, the height of the cone, in terms of a and θ.

b Find r, the radius of the cone, in terms of a and θ.

The volume, V cm^3, of the cone is given by $V = \frac{1}{3}\pi r^2 h$.

c Use the results from **a** and **b** to show that

$$V = \frac{1}{3}\pi a^3 \sin^2\theta \cdot (1 + \cos\theta)$$

d Find $\dfrac{dV}{d\theta}$ (a is a constant) and hence find the value of θ for which the volume is a maximum.

e Find the maximum volume of the cone in terms of a.

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41 Some bacteria are introduced into a supply of fresh milk. After t hours there are y grams of bacteria present, where

$$y = \frac{Ae^{bt}}{1 + Ae^{bt}} \qquad (1)$$

and A and b are positive constants.

a Show that $0 < y < 1$ for all values of t.

b Find $\dfrac{dy}{dt}$ in terms of t.

c From equation (1), show that $Ae^{bt} = \dfrac{y}{1-y}$.

d **i** Show that $\dfrac{dy}{dt} = by(1-y)$.

ii Hence, or otherwise, show that the maximum value of $\dfrac{dy}{dt}$ occurs when $y = 0.5$.

e If $A = 0.01$ and $b = 0.7$, find when, to the nearest hour, the bacteria will be increasing at the fastest rate.

42 Let $f\colon \mathbb{R}^+ \to \mathbb{R}$, $f(x) = \dfrac{e^x}{x}$.

a Find $f'(x)$.

b Find $\{ x : f'(x) = 0 \}$.

c Find the coordinates of the one stationary point and state its nature.

d **i** Find $\dfrac{f'(x)}{f(x)}$. **ii** Find $\displaystyle\lim_{x\to\infty} \frac{f'(x)}{f(x)}$ and comment.

e Sketch the graph of f.

f Over a period of years, the number of birds (n) in an island colony decreased and increased with time (t years) according to the approximate formula

$$n = \frac{ae^{kt}}{t}$$

where t is measured from 1900 and a and k are constant. If during this period the population was the same in 1965 as it was in 1930, when was it least?

43 A culture contains 1000 bacteria and 5 hours later the number has increased to 10 000. The number, N, of bacteria present at any time, t hours, is given by $N = Ae^{kt}$.

a Find the values of A and k.

b Find the rate of growth at time t.

c Show that, at time t, the rate of growth is proportional to the number of bacteria present.

d Find this rate of growth when:

i $t = 4$ **ii** $t = 50$

44 The populations of two ant colonies, A and B, are increasing according to the rules:

A population $= 2 \times 10^4 e^{0.03t}$

B population $= 10^4 e^{0.05t}$

After how many years will their populations:

a be equal **b** be increasing at the same rate?

45 A particle on the end of a spring, which is hanging vertically, is oscillating such that its height, h metres, above the floor after t seconds is given by

$$y = 0.5 + 0.2\sin(3\pi t), \quad t \geq 0$$

a Find the greatest height above the floor and the time at which this height is first reached.

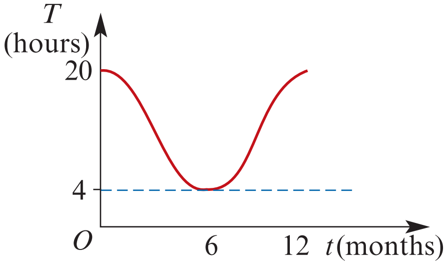
b Find the period of oscillation.

c Find the speed of the particle when $t = \frac{1}{3}, \frac{2}{3}, \frac{1}{6}$.

46 The length of night on Seal Island varies between 20 hours in midwinter and 4 hours in midsummer. The relationship between T, the number of hours of night, and t, the number of months past the longest night in 2010, is given by

$$T(t) = p + q\cos(\pi r t)$$

where p, q and r are constants.
Assume that the year consists of 12 months of equal length.
The graph of T against t is illustrated.



a Find the value of:

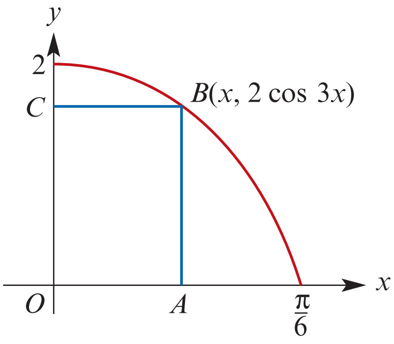
i r **ii** p and q

b Find $T'(3)$ and $T'(9)$ and find the rate of change of hours of night with respect to the number of months.

c Find the average rate of change of hours of night from $t = 0$ to $t = 6$.

d After how many months is the rate of change of hours of night a maximum?

47 A section of the graph of $y = 2\cos(3x)$ is shown in the diagram.



a Show that the area, A, of the rectangle $OABC$ in terms of x is $2x\cos(3x)$.

b **i** Find $\frac{dA}{dx}$.

ii Find $\frac{dA}{dx}$ when $x = 0$ and $x = \frac{\pi}{6}$.

c **i** On a calculator, plot the graph of $A = 2x\cos(3x)$ for $x \in \left[0, \frac{\pi}{6}\right]$.

ii Find the two values of x for which the area of the rectangle is 0.2 square units.

iii Find the maximum area of the rectangle and the value of x for which this occurs.

d **i** Show that $\frac{dA}{dx} = 0$ is equivalent to $\tan(3x) = \frac{1}{3x}$.

ii Using a calculator, plot the graphs of $y = \tan(3x)$ and $y = \frac{1}{3x}$ for $x \in \left(0, \frac{\pi}{6}\right)$ and find the coordinates of the point of intersection.

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48 **a** A population of insects grows according to the model

$$N(t) = 1000 - t + 2e^{\frac{t}{20}} \quad \text{for } t \geq 0$$

where t is the number of days after 1 January 2000.

- **i** Find the rate of growth of the population as a function of t.
- **ii** Find the minimum population size and value of t for which this occurs.
- **iii** Find $N(0)$.
- **iv** Find $N(100)$.
- **v** Sketch the graph of N against t for $0 \leq t \leq 100$.

b It is found that the population of another species is given by

$$N_2(t) = 1000 - t^{\frac{1}{2}} + 2e^{\frac{t^{\frac{1}{2}}}{20}}$$

- **i** Find $N_2(0)$.
- **ii** Find $N_2(100)$.
- **iii** Plot the graph of $y = N_2(t)$ for $t \in [0, 5000]$ on a calculator.
- **iv** Solve the equation $N_2'(t) = 0$ and hence give the minimum population of this species of insects.

c A third model is

$$N_3(t) = 1000 - t^{\frac{3}{2}} + 2e^{\frac{t}{20}}$$

Use a calculator to:

- **i** plot a graph for $0 \leq t \leq 200$
- **ii** find the minimum population and the time at which this occurs.

d
- **i** For N_3, find $N_3'(t)$.
- **ii** Show that $N_3'(t) = 0$ is equivalent to $t = 20\log_e(15\sqrt{t})$.

49 **a** Consider the curve with equation $y = (2x^2 - 5x)e^{ax}$. If the curve passes through the point with coordinates $(3, 10)$, find the value of a.

b
- **i** For the curve with equation $y = (2x^2 - 5x)e^{ax}$, find the x-axis intercepts.
- **ii** Use calculus to find the x-values for which there is a turning point, in terms of a.

11

Integration

Objectives

- To use **numerical methods** to estimate the area under the graph of a function.
- To be able to calculate **definite integrals**.
- To use the definite integral to find the **exact area** under the graph of a function.
- To integrate **polynomial functions**, **exponential functions** and **circular functions**.
- To use integration to determine **areas under curves**.
- To use integration to **solve problems**.

We have used the derivative to find the gradients of tangents to curves, and in turn this has been used in graph sketching. The derivative has also been used to define instantaneous rate of change and to solve problems involving motion in a straight line.

It comes as a surprise that a related idea can be used to determine areas. In this chapter we define an area function A for a given function f on an interval $[a, b]$, and show that the derivative of the area function is the original function f. Hence, you can go from the function f to its area function by a process which can loosely be described as 'undoing' the derivative. This result is so important that it carries the title **fundamental theorem of calculus**.

The result was developed over many centuries: a method for determining areas described in the last section of this chapter is due to Archimedes. The final result was brought together by both Leibniz and Newton in the seventeenth century. The wonder of it is that the two seemingly distinct ideas – calculation of areas and calculation of gradients – were shown to be so closely related.

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11A The area under a graph

Consider a function $f : \mathbb{R} \to \mathbb{R}$ and its graph $y = f(x)$. We want to find the area under the graph. For now we'll assume that the graph $y = f(x)$ is always *above the x-axis*, and we will determine the area between the graph $y = f(x)$ and the x-axis. We set left and right endpoints and determine the area between those endpoints.

Estimating the area under a graph

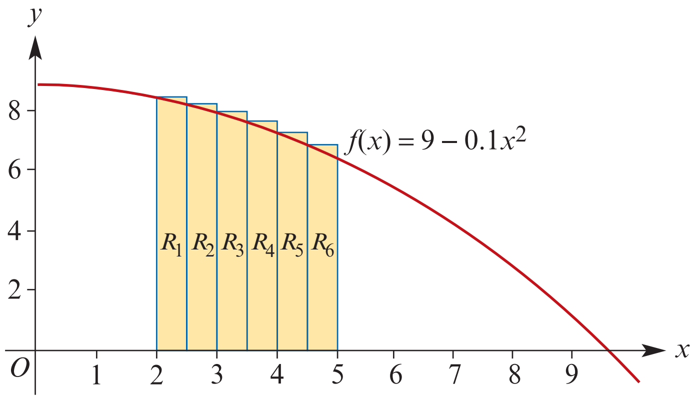
Below is the graph of

$$f(x) = 9 - 0.1x^2$$

We consider two methods for estimating the area under this graph between $x = 2$ and $x = 5$.

The left-endpoint estimate

We first find an approximation for the area under the graph between $x = 2$ and $x = 5$ by dividing the region into rectangles as illustrated. The width of each rectangle is 0.5.



Areas of rectangles:

- Area of $R_1 = 0.5 \times f(2.0) = 0.5 \times 8.60 = 4.30$ square units
- Area of $R_2 = 0.5 \times f(2.5) = 0.5 \times 8.38 = 4.19$ square units
- Area of $R_3 = 0.5 \times f(3.0) = 0.5 \times 8.10 = 4.05$ square units
- Area of $R_4 = 0.5 \times f(3.5) = 0.5 \times 7.78 = 3.89$ square units
- Area of $R_5 = 0.5 \times f(4.0) = 0.5 \times 7.40 = 3.70$ square units
- Area of $R_6 = 0.5 \times f(4.5) = 0.5 \times 6.98 = 3.49$ square units

The sum of the areas of the rectangles is 23.62 square units.

This is called the **left-endpoint estimate** for the area under the graph.

The left-endpoint estimate will be larger than the actual area for a graph that is decreasing over the interval, and smaller than the actual area for a graph that is increasing.

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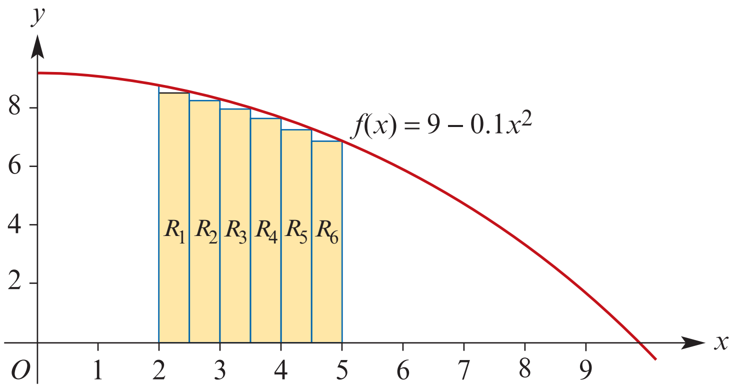
The right-endpoint estimate



Areas of rectangles:

- Area of $R_1 = 0.5 \times f(2.5) = 0.5 \times 8.38 = 4.19$ square units
- Area of $R_2 = 0.5 \times f(3.0) = 0.5 \times 8.10 = 4.05$ square units
- Area of $R_3 = 0.5 \times f(3.5) = 0.5 \times 7.78 = 3.89$ square units
- Area of $R_4 = 0.5 \times f(4.0) = 0.5 \times 7.40 = 3.70$ square units
- Area of $R_5 = 0.5 \times f(4.5) = 0.5 \times 6.98 = 3.49$ square units
- Area of $R_6 = 0.5 \times f(5.0) = 0.5 \times 6.50 = 3.25$ square units

The sum of the areas of the rectangles is 22.67 square units.

This is called the **right-endpoint estimate** for the area under the graph.

For f decreasing over $[a, b]$: left-endpoint estimate ≥ true area ≥ right-endpoint estimate

For f increasing over $[a, b]$: left-endpoint estimate ≤ true area ≤ right-endpoint estimate

It is clear that, if narrower strips are chosen, we obtain an estimate that is closer to the true value. This is time-consuming to do by hand, but a computer program or spreadsheet makes the process quite manageable.

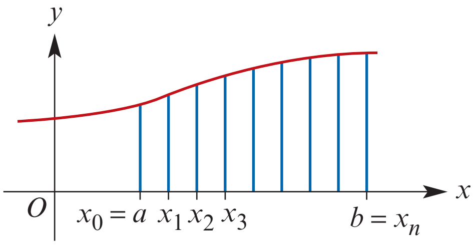
In general, to estimate the area under the graph of $y = f(x)$ between $x = a$ and $x = b$, we divide the interval $[a, b]$ on the x-axis into n equal subintervals $[a, x_1]$, $[x_1, x_2]$, $[x_2, x_3]$, . . . , $[x_{n-1}, b]$ as illustrated.

Left-endpoint estimate

$$L_n = \frac{b-a}{n}\Big[f(x_0) + f(x_1) + \cdots + f(x_{n-1})\Big]$$

Right-endpoint estimate

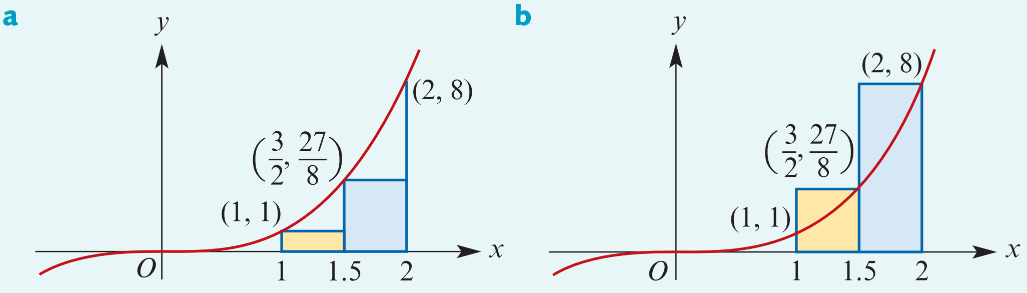
$$R_n = \frac{b-a}{n}\Big[f(x_1) + f(x_2) + \cdots + f(x_n)\Big]$$



These two methods are not limited to situations in which the graph is either increasing or decreasing for the whole interval. They may be used to determine the area under the curve for any continuous function on an interval $[a, b]$.

Example 1

Approximate the area under the curve $y = x^3$ between $x = 1$ and $x = 2$ by finding the sum of the areas of the shaded rectangles:



Solution

a Area $= \frac{1}{2} \times 1 + \frac{1}{2} \times \frac{27}{8}$

$= \frac{1}{2} + \frac{27}{16}$

$= 2\frac{3}{16}$ square units

(This is the left-endpoint method.)

b Area $= \frac{1}{2} \times \frac{27}{8} + \frac{1}{2} \times 8$

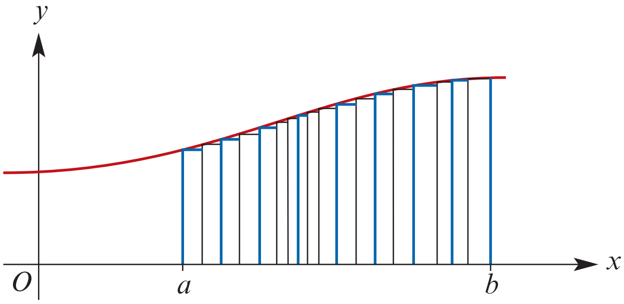
$= \frac{27}{16} + 4$

$= 5\frac{11}{16}$ square units

(This is the right-endpoint method.)

If f is a continuous function such that $f(x)$ is positive for all x in the interval $[a, b]$, and if the interval $[a, b]$ is partitioned into arbitrarily small subintervals, then the **area** under the curve between $x = a$ and $x = b$ can be defined by this limiting process.

The diagram on the right shows rectangles formed from a partition. The rectangles can be of varying width, but in the limit the width of all the rectangles must approach zero.



▶ The definite integral

Suppose that f is continuous function on a closed interval $[a, b]$ and that $f(x)$ is positive for all x in this interval. Then the area under the graph of $y = f(x)$ from $x = a$ to $x = b$ is called the **definite integral** of $f(x)$ from $x = a$ to $x = b$, and is denoted by

$$\int_a^b f(x)\, dx$$

The function f is called the integrand, and a and b are the lower and upper limits of the integral.

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By using summation notation (discussed in Appendix A), this limiting process can be expressed as

$$\int_a^b f(x)\,dx = \lim_{\delta x \to 0} \sum_{i=1}^{n} f(x_i^*)\,\delta x_i$$

where the interval $[a, b]$ is partitioned into n subintervals, with the ith subinterval of length δx_i and containing x_i^*, and $\delta x = \max\{\,\delta x_i : i = 1, 2, \dots, n\,\}$.

For a linear function or a piecewise-defined function with linear components, the area under the graph may be found using geometric techniques.

Example 2

Evaluate each of the following by using an area formula:

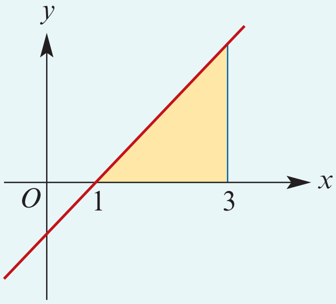
a $\int_1^3 x - 1\,dx$ **b** $\int_1^3 (x-1)\,dx + \int_{-1}^1 (1-x)\,dx$ **c** $\int_1^2 x + 1\,dx$

Solution

a Area of triangle $= \frac{1}{2} \times 2 \times 2$

$= 2$ square units

Therefore $\int_1^3 x - 1\,dx = 2$

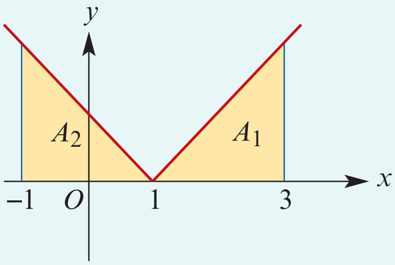


b Area $= A_1 + A_2$

$= 2 + \frac{1}{2} \times 2 \times 2$

$= 4$ square units

Therefore $\int_1^3 (x-1)\,dx + \int_{-1}^1 (1-x)\,dx = 4$

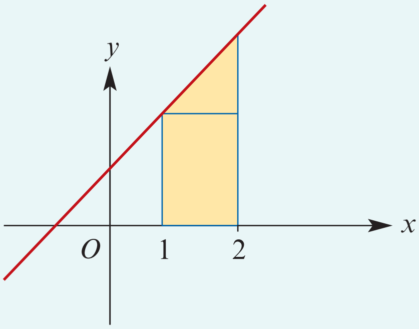


c The required region is a trapezium.

Area $= \frac{1}{2} \times 1 \times (2+3)$

$= \frac{5}{2}$ square units

Therefore $\int_1^2 x + 1\,dx = \frac{5}{2}$



A calculus method for determining areas will be introduced in Section 11E.

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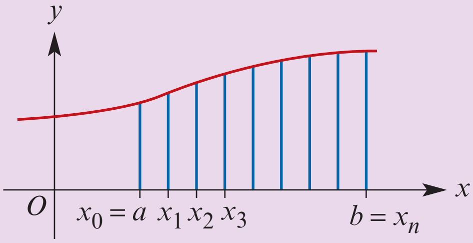
Section summary

- **Estimating area**
  Divide the interval $[a, b]$ on the x-axis into n equal subintervals $[a, x_1]$, $[x_1, x_2]$, $[x_2, x_3]$, $\dots$, $[x_{n-1}, b]$ as illustrated.
  Estimates for the area under the graph of $y = f(x)$ between $x = a$ and $x = b$:



  - Left-endpoint estimate

$$L_n = \frac{b-a}{n}\Big[f(x_0) + f(x_1) + \cdots + f(x_{n-1})\Big]$$

  - Right-endpoint estimate

$$R_n = \frac{b-a}{n}\Big[f(x_1) + f(x_2) + \cdots + f(x_n)\Big]$$

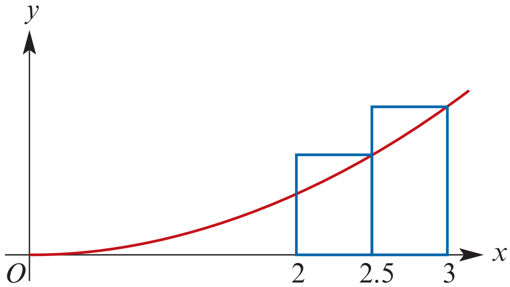
- **Exact area**
  Let f be a continuous function on a closed interval $[a, b]$ such that $f(x)$ is positive for all $x \in [a, b]$. The exact area under the graph of $y = f(x)$ from $x = a$ to $x = b$ is called the **definite integral** of $f(x)$ from $x = a$ to $x = b$, and is denoted by $\int_a^b f(x)\,dx$.

Exercise 11A

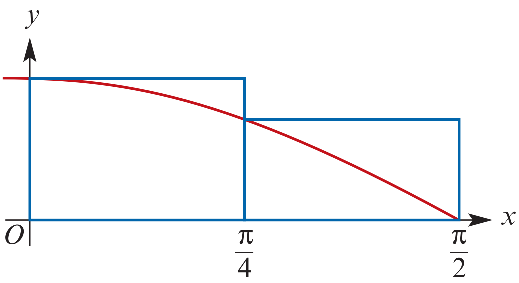
Example 1

1 Use two rectangles to approximate the area contained between the curve and the x-axis. Use the method indicated and give your answer correct to two decimal places.

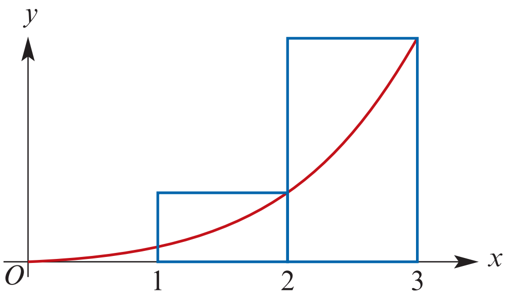
a $y = \frac{1}{2}x^2$ between $x = 2$ and $x = 3$ using the right-endpoint method



b $y = \cos x$ between $x = 0$ and $x = \frac{\pi}{2}$ using the left-endpoint method



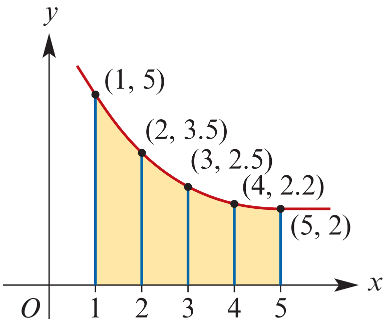
c $y = \frac{1}{2}x^3$ between $x = 1$ and $x = 3$ using the right-endpoint method



2 To approximate the area of the shaded region, use the subintervals shown to calculate:

a the left-endpoint estimate

b the right-endpoint estimate.



3 Calculate an approximation to the area under the graph of $y = x(4 - x)$ between $x = 0$ and $x = 4$ using:

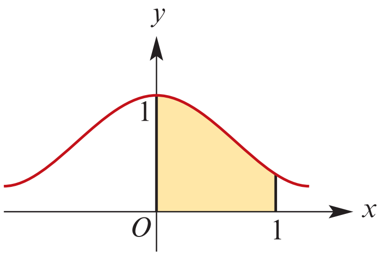
a 4 strips of width 1.0 (Do this by hand.)

b 20 strips of width 0.2 (You can use a calculator program if you wish.)

4 The graph is that of $y = \frac{1}{1 + x^2}$. It is known that the area of the shaded region is $\frac{\pi}{4}$.

a Apply the right-endpoint rule with strips of width 0.25 to estimate the area under the curve.

b Hence find an approximate value for π. How could you improve the approximation?



5 A table of values is given for the rule $y = f(x)$.

x	0	1	2	3	4	5	6	7	8	9	10
y	3	3.5	3.7	3.8	3.9	3.9	4.0	4.0	3.7	3.3	2.9

Find the area enclosed by the graph of $y = f(x)$, the lines $x = 0$ and $x = 10$, and the x-axis by using:

a the left-endpoint estimate **b** the right-endpoint estimate.

6 Calculate, by using the right-endpoint estimate, an approximation to the area under the graph of $y = 2^x$ between $x = 0$ and $x = 3$, using strips of width 0.5. Write your answer correct to two decimal places.

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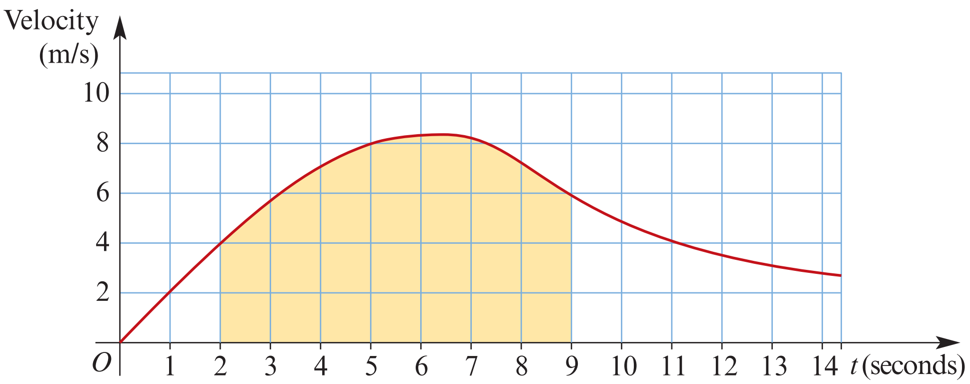
7 The graph shows the velocity (in m/s) of an object at time t seconds.

a Use the left-endpoint rule to estimate the area of the shaded region.

b What does this area represent?



Example 2

8 Evaluate each of the following by using an area formula:

a $\int_2^5 x - 2 \, dx$ **b** $\int_{-1}^2 (2 - x) \, dx + \int_2^5 (x - 2) \, dx$ **c** $\int_1^2 2x + 1 \, dx$

11B Antidifferentiation: indefinite integrals

Later in this chapter, we will see how to find the exact area under a graph using the technique of 'undoing' the derivative. In this section, we formalise the idea of 'undoing' a derivative.

The derivative of x^2 with respect to x is $2x$. Conversely, given that an unknown expression has derivative $2x$, it is clear that the unknown expression could be x^2. The process of finding a function from its derivative is called **antidifferentiation**.

Now consider the functions $f(x) = x^2 + 1$ and $g(x) = x^2 - 7$.

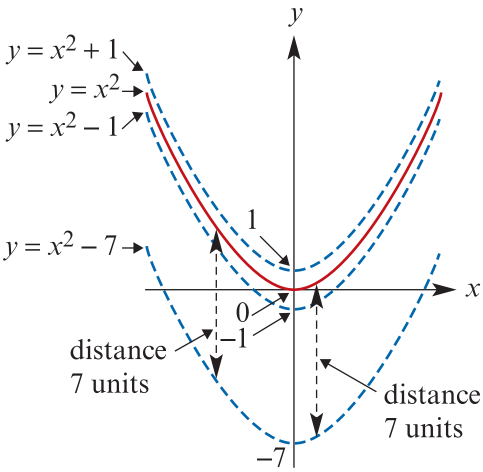
We have $f'(x) = 2x$ and $g'(x) = 2x$. So the two different functions have the same derivative function.

Both $x^2 + 1$ and $x^2 - 7$ are said to be **antiderivatives** of $2x$.

If two functions have the same derivative function, then they differ by a constant. So the graphs of the two functions can be obtained from each other by translation parallel to the y-axis.

The diagram shows several antiderivatives of $2x$.

Each of the graphs is a translation of $y = x^2$ parallel to the y-axis.



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Notation

The general antiderivative of $2x$ is $x^2 + c$, where c is an arbitrary real number. We use the notation of Leibniz to state this with symbols:

$$\int 2x\,dx = x^2 + c$$

This is read as 'the **general antiderivative** of $2x$ with respect to x is equal to $x^2 + c$' or as 'the **indefinite integral** of $2x$ with respect to x is $x^2 + c$'.

To be more precise, the indefinite integral is the set of all antiderivatives and to emphasise this we could write:

$$\int 2x\,dx = \{\, f(x) : f'(x) = 2x \,\} = \{\, x^2 + c : c \in \mathbb{R} \,\}$$

This set notation is not commonly used, but it should be clearly understood that there is not a unique antiderivative for a given function. We will not use this set notation, but it is advisable to keep it in mind when considering further results.

In general:

If $F'(x) = f(x)$, then $\int f(x)\,dx = F(x) + c$, where c is an arbitrary real number.

The reason why the symbol is the same as that used for the definite integral in Section 11A will become evident in Section 11E.

▶ The antiderivative of x^r where $r \neq -1$

We know that:

$f(x) = x^3$ implies $f'(x) = 3x^2$ $\qquad$ $f(x) = x^8$ implies $f'(x) = 8x^7$

$f(x) = x^{\frac{3}{2}}$ implies $f'(x) = \frac{3}{2}x^{\frac{1}{2}}$ $\qquad$ $f(x) = x^{-4}$ implies $f'(x) = -4x^{-5}$

Reversing this process gives:

$\int 3x^2\,dx = x^3 + c$ where c is an arbitrary constant

$\int 8x^7\,dx = x^8 + c$ where c is an arbitrary constant

$\int \frac{3}{2}x^{\frac{1}{2}}\,dx = x^{\frac{3}{2}} + c$ where c is an arbitrary constant

$\int -4x^{-5}\,dx = x^{-4} + c$ where c is an arbitrary constant

We also have:

$$\int x^2\,dx = \frac{1}{3}x^3 + c \qquad \int x^{\frac{1}{2}}\,dx = \frac{2}{3}x^{\frac{3}{2}} + c \qquad \int x^{-5}\,dx = -\frac{1}{4}x^{-4} + c$$

Generalising, it is seen that:

$$\int x^r\,dx = \frac{x^{r+1}}{r+1} + c, \qquad r \in \mathbb{Q} \setminus \{-1\}$$

Note: This result can only be applied for suitable values of x for a given value of r. For example, if $r = \frac{1}{2}$, then $x \in \mathbb{R}^+$ is a suitable restriction. If $r = -2$, we can take $x \in \mathbb{R} \setminus \{0\}$, and if $r = 3$, we can take $x \in \mathbb{R}$.

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We also record the following results, which follow immediately from the corresponding results for differentiation:

Sum $\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$

Difference $\int f(x) - g(x)\,dx = \int f(x)\,dx - \int g(x)\,dx$

Multiple $\int kf(x)\,dx = k\int f(x)\,dx$, where k is a real number

Example 3

Find the general antiderivative (indefinite integral) of each of the following:

a $3x^5$

b $3x^2 + 4x^{-2} + 3$

Solution

a
$$\begin{aligned}&\int 3x^5\,dx\\ &= 3\int x^5\,dx\\ &= 3\times\frac{x^6}{6} + c\\ &= \frac{x^6}{2} + c\end{aligned}$$

b
$$\begin{aligned}&\int 3x^2 + 4x^{-2} + 3\,dx\\ &= 3\int x^2\,dx + 4\int x^{-2}\,dx + 3\int 1\,dx\\ &= \frac{3x^3}{3} + \frac{4x^{-1}}{-1} + \frac{3x}{1} + c\\ &= x^3 - \frac{4}{x} + 3x + c\end{aligned}$$

Example 4

Find y in terms of x if:

a $\dfrac{dy}{dx} = \dfrac{1}{x^2}$

b $\dfrac{dy}{dx} = 3\sqrt{x}$

c $\dfrac{dy}{dx} = x^{\frac{3}{4}} + x^{-\frac{3}{4}}$

Solution

a
$$\begin{aligned}\int \frac{1}{x^2}\,dx &= \int x^{-2}\,dx\\ &= \frac{x^{-1}}{-1} + c\\ \therefore\quad y &= \frac{-1}{x} + c\end{aligned}$$

b
$$\begin{aligned}\int 3\sqrt{x}\,dx &= 3\int x^{\frac{1}{2}}\,dx\\ &= 3\times\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c\\ \therefore\quad y &= 2x^{\frac{3}{2}} + c\end{aligned}$$

c $y = \dfrac{4}{7}x^{\frac{7}{4}} + 4x^{\frac{1}{4}} + c$

Given extra information, we can find a unique antiderivative.

Example 5

It is known that $f'(x) = x^3 + 4x^2$ and $f(0) = 0$. Find $f(x)$.

Solution

$$\int x^3 + 4x^2\,dx = \frac{x^4}{4} + \frac{4x^3}{3} + c$$

$$\therefore\quad f(x) = \frac{x^4}{4} + \frac{4x^3}{3} + c$$

As $f(0) = 0$, we have $c = 0$. Hence $f(x) = \dfrac{x^4}{4} + \dfrac{4x^3}{3}$.

Example 6

If the gradient of the tangent at a point (x, y) on a curve is given by $2x$ and the curve passes through the point $(-1, 4)$, find the equation of the curve.

Solution

Let the curve have equation $y = f(x)$. Then $f'(x) = 2x$.

$$\int 2x\,dx = \frac{2x^2}{2} + c = x^2 + c$$

$$\therefore \quad f(x) = x^2 + c$$

But $f(-1) = 4$ and therefore $4 = (-1)^2 + c$.

Hence $c = 3$ and so $f(x) = x^2 + 3$.

Section summary

- Antiderivative of x^r, for $r \in \mathbb{Q} \setminus \{-1\}$:

$$\int x^r\,dx = \frac{x^{r+1}}{r+1} + c$$

- Properties of antidifferentiation:
  - $\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$
  - $\int f(x) - g(x)\,dx = \int f(x)\,dx - \int g(x)\,dx$
  - $\int kf(x)\,dx = k \int f(x)\,dx$, where k is a real number

Exercise 11B

Skillsheet

Example 3

1 Find:

a $\int \frac{1}{2}x^3\,dx$ **b** $\int 5x^3 - 2x\,dx$ **c** $\int \frac{4}{5}x^3 - 3x^2\,dx$ **d** $\int (2-z)(3z+1)\,dz$

Example 4

2 Find y in terms of x if:

a $\frac{dy}{dx} = \frac{1}{x^3}$ **b** $\frac{dy}{dx} = 4\sqrt[3]{x}$ **c** $\frac{dy}{dx} = x^{\frac{1}{4}} + x^{-\frac{3}{5}}$

3 Find:

a $\int 3x^{-2}\,dx$ **b** $\int 2x^{-4} + 6x\,dx$ **c** $\int 2x^{-2} + 6x^{-3}\,dx$

d $\int 3x^{\frac{1}{3}} - 5x^{\frac{5}{4}}\,dx$ **e** $\int 3x^{\frac{3}{4}} - 7x^{\frac{1}{2}}\,dx$ **f** $\int 4x^{\frac{3}{5}} + 12x^{\frac{5}{3}}\,dx$

Example 5

4 Find y in terms of x for each of the following:

a $\frac{dy}{dx} = 2x - 3$ and $y = 1$ when $x = 1$ **b** $\frac{dy}{dx} = x^3$ and $y = 6$ when $x = 0$

c $\frac{dy}{dx} = x^{\frac{1}{2}} + x$ and $y = 6$ when $x = 4$

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5 Find:

a $\int \sqrt{x}\,(2 + x)\,dx$

b $\int \dfrac{3z^4 + 2z}{z^3}\,dz$

c $\int \dfrac{5x^3 + 2x^2}{x}\,dx$

d $\int \sqrt{x}\,(2x + x^2)\,dx$

e $\int x^2(2 + 3x^2)\,dx$

f $\int \sqrt[3]{x}\,(x + x^4)\,dx$

Example 6

6 A curve with equation $y = f(x)$ passes through the point $(2, 0)$ and $f'(x) = 3x^2 - \dfrac{1}{x^2}$. Find $f(x)$.

7 Find s in terms of t if $\dfrac{ds}{dt} = 3t - \dfrac{8}{t^2}$ and $s = 1\frac{1}{2}$ when $t = 1$.

8 A curve $y = f(x)$ for which $f'(x) = 16x + k$, where k is a constant, has a stationary point at $(2, 1)$. Find:

a the value of k

b the value of $f(x)$ when $x = 7$.

11C The antiderivative of $(ax + b)^r$

Case 1: $r \neq -1$

For $f(x) = (ax + b)^{r+1}$, where $r \neq -1$, we can use the chain rule to find

$$f'(x) = a(r + 1)(ax + b)^r$$

Thus it follows that:

$$\int (ax + b)^r\,dx = \frac{1}{a(r + 1)}(ax + b)^{r+1} + c, \qquad r \neq -1$$

This result does not hold for $r = -1$.

Example 7

Find the general antiderivative of:

a $(3x + 1)^5$

b $(2x - 1)^{-2}$

Solution

a
$$\begin{aligned}\int (3x + 1)^5\,dx &= \frac{1}{3(5 + 1)}(3x + 1)^6 + c\\ &= \frac{1}{18}(3x + 1)^6 + c\end{aligned}$$

b
$$\begin{aligned}\int (2x - 1)^{-2}\,dx &= \frac{1}{2(-2 + 1)}(2x - 1)^{-1} + c\\ &= -\frac{1}{2}(2x - 1)^{-1} + c\end{aligned}$$

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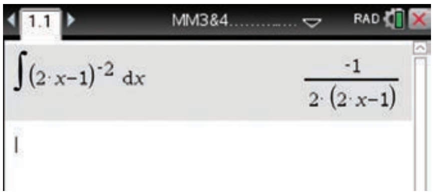
Using the TI-Nspire

- Use menu > **Calculus** > **Integral** to find the integral of $(2x - 1)^{-2}$.

Note: The integral template can also be accessed using the 2D-template palette or shift +.

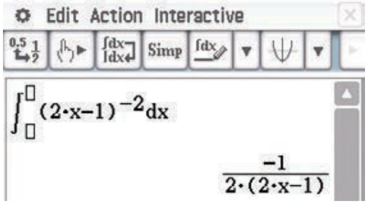


$\int (2 \cdot x-1)^{-2}\, dx \qquad \frac{-1}{2 \cdot (2 \cdot x-1)}$

Using the Casio ClassPad

- Enter and highlight the expression $(2x - 1)^{-2}$.
- Select **Interactive** > **Calculation** > **∫**.

Note: The two boxes on the integral allow for definite integrals to be evaluated. This is covered later in the chapter.



$\int_{\square}^{\square} (2 \cdot x-1)^{-2}\, dx$

$\frac{-1}{2 \cdot (2 \cdot x-1)}$

Case 2: $r = -1$

But what happens when $r = -1$? In other words, what is $\int \frac{1}{ax+b}\, dx$?

Remember that $\frac{d}{dx}(\log_e x) = \frac{1}{x}$. Thus $\int \frac{1}{x}\, dx = \log_e x + c$ provided that $x > 0$.

More generally:

For $ax + b > 0$,

$$\int \frac{1}{ax+b}\, dx = \frac{1}{a}\log_e(ax+b) + c$$

For $x < 0$, we have

$$\frac{d}{dx}\big(\log_e(-x)\big) = \frac{1}{-x} \times (-1) = \frac{1}{x}$$

and so $\int \frac{1}{x}\, dx = \log_e(-x)$.

More generally, for $ax + b < 0$, we have

$$\frac{d}{dx}\big(\log_e(-ax-b)\big) = \frac{1}{-ax-b} \times (-a) = \frac{a}{ax+b}$$

and so $\int \frac{1}{ax+b}\, dx = \frac{1}{a}\log_e(-ax-b)$.

We can summarise these results as:

$$\int \frac{1}{ax+b}\, dx = \begin{cases} \frac{1}{a}\log_e(ax+b) + c & \text{for } ax+b > 0 \\ \frac{1}{a}\log_e(-ax-b) + c & \text{for } ax+b < 0 \end{cases}$$

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Example 8

a Find the general antiderivative of $\frac{2}{3x-2}$ for $x > \frac{2}{3}$.

b Find the general antiderivative of $\frac{2}{3x-2}$ for $x < \frac{2}{3}$.

c Given $\frac{dy}{dx} = \frac{3}{x}$ for $x > 0$ and $y = 10$ when $x = 1$, find an expression for y in terms of x.

d Given $\frac{dy}{dx} = \frac{3}{x}$ for $x < 0$ and $y = 10$ when $x = -1$, find an expression for y in terms of x.

Solution

a For $x > \frac{2}{3}$,

$$\int \frac{2}{3x-2}\,dx = \frac{1}{3} \times 2\log_e(3x-2) + c$$
$$= \frac{2}{3}\log_e(3x-2) + c$$

b For $x < \frac{2}{3}$,

$$\int \frac{2}{3x-2}\,dx = \frac{1}{3} \times 2\log_e(2-3x) + c$$
$$= \frac{2}{3}\log_e(2-3x) + c$$

c $y = \int \frac{3}{x}\,dx = 3\log_e x + c$

When $x = 1$, $y = 10$ and so

$$10 = 3\log_e 1 + c$$
$$10 = 0 + c$$
$$\therefore \quad c = 10$$

Hence $y = 3\log_e x + 10$.

d $y = \int \frac{3}{x}\,dx = 3\log_e(-x) + c$

When $x = -1$, $y = 10$ and so

$$10 = 3\log_e 1 + c$$
$$10 = 0 + c$$
$$\therefore \quad c = 10$$

Hence $y = 3\log_e(-x) + 10$.

The situation is simplified by using the **absolute value function**, which is not explicitly in the syllabus of Mathematical Methods Units 3 & 4. It is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

For example, $|-2| = |2| = 2$.

For the type of example we are working with here, we have

$$|ax+b| = \begin{cases} ax+b & \text{if } ax+b \geq 0 \\ -ax-b & \text{if } ax+b < 0 \end{cases}$$

Thus:

For $ax + b \neq 0$,

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log_e|ax+b| + c$$

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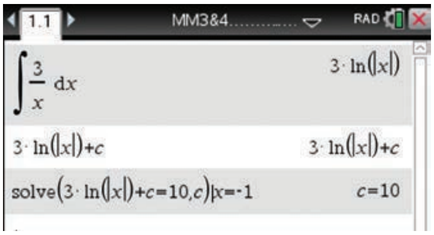
Notes:

- We can now deal with parts **a** and **b** of Example 8 simultaneously by writing $\int \frac{2}{3x-2}\,dx = \frac{2}{3}\log_e|3x-2| + c$. For parts **c** and **d**, write $y = 3\log_e|x| + c$.
- Using this notation is recommended as it avoids difficulties and is consistent with the calculators being used.

Using the TI-Nspire

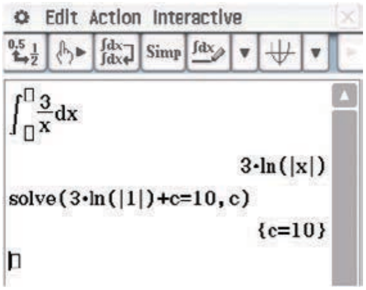
- Use (menu) > **Calculus** > **Integral** to find *an* antiderivative of $\frac{3}{x}$.
- Add c to find the *general* antiderivative of $\frac{3}{x}$.
- Use **solve( )** to determine the value of c as shown.



```
∫ 3/x dx                              3·ln(|x|)
3·ln(|x|)+c                         3·ln(|x|)+c
solve(3·ln(|x|)+c=10,c)|x=-1               c=10
```

Using the Casio ClassPad

- Enter and highlight the expression $\frac{3}{x}$.
- Select **Interactive** > **Calculation** > **∫**.
- Note that the ClassPad does not add c to the indefinite integral.
- Copy and paste the answer to the next line. Replace the x with 1 and add c to complete the equation $3\ln|1| + c = 10$.
- Select **Interactive** > **Equation/Inequality** > **solve** and ensure the variable is set to c.



```
∫ 3/x dx
                                     3·ln(|x|)
solve(3·ln(|1|)+c=10,c)
                                        {c=10}
```

Example 9

a Find the general antiderivative of $\frac{2}{2-3x}$.

b Given $\frac{dy}{dx} = \frac{2}{x}$ and $y = 10$ when $x = 1$, find an expression for y in terms of x.

c Given $\frac{dy}{dx} = \frac{2}{x}$ and $y = 10$ when $x = -1$, find an expression for y in terms of x.

Solution

a
$$\int \frac{2}{2-3x}\,dx = -\frac{1}{3} \times 2\log_e|2-3x| + c$$
$$= -\frac{2}{3}\log_e|2-3x| + c$$

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b $y = \int \frac{2}{x}\,dx = 2\log_e|x| + c$

When $x = 1$, $y = 10$ and so

$$\begin{aligned} 10 &= 2\log_e|1| + c \\ 10 &= 0 + c \\ \therefore\quad c &= 10 \end{aligned}$$

Hence $y = 2\log_e|x| + 10$.

c $y = \int \frac{2}{x}\,dx = 2\log_e|x| + c$

When $x = -1$, $y = 10$ and so

$$\begin{aligned} 10 &= 2\log_e|-1| + c \\ 10 &= 0 + c \\ \therefore\quad c &= 10 \end{aligned}$$

Hence $y = 2\log_e|x| + 10$.

The law of exponential change

We are now able to prove the following result from Chapter 5; we do not dwell on the proof as its place is in Specialist Mathematics.

If the rate at which a quantity increases or decreases is proportional to its current value, then the quantity obeys the **law of exponential change**.

Let A be the quantity at time t. Then

$$A = A_0 e^{kt}$$

where A_0 is the initial quantity and k is a constant.

Proof Assume that A is a positive quantity such that $\frac{dA}{dt} = kA$, for some constant $k \neq 0$.

Then A is strictly increasing or A is strictly decreasing, depending on whether $k > 0$ or $k < 0$. Thus A is a one-to-one function of t, and so from Chapter 9 we can write

$$\frac{dt}{dA} = \frac{1}{kA}$$

Antidifferentiating with respect to A gives

$$\begin{aligned} t &= \frac{1}{k}\log_e A + c \\ \therefore\quad \log_e A &= k(t - c) \end{aligned}$$

Thus $A = e^{kt-kc} = e^{-kc}e^{kt}$. We let $A_0 = e^{-kc}$ and we have the result $A = A_0 e^{kt}$.

Section summary

- If $r \in \mathbb{Q} \setminus \{-1\}$, then

$$\int (ax+b)^r\,dx = \frac{1}{a(r+1)}(ax+b)^{r+1} + c$$

- For $ax + b > 0$,

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log_e(ax+b) + c$$

- For $ax + b < 0$,

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log_e(-ax-b) + c$$

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Or more conveniently:

- For $ax + b \neq 0$,

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log_e|ax+b| + c$$

Exercise 11C

Skillsheet

Example 7

1 Find:

a $\int (2x-1)^2\,dx$ **b** $\int (2-t)^3\,dt$ **c** $\int (5x-2)^3\,dx$

d $\int (4x-6)^{-2}\,dx$ **e** $\int (6-4x)^{-3}\,dx$ **f** $\int (4x+3)^{-3}\,dx$

g $\int (3x+6)^{\frac{1}{2}}\,dx$ **h** $\int (3x+6)^{-\frac{1}{2}}\,dx$ **i** $\int (2x-4)^{\frac{7}{2}}\,dx$

j $\int (3x+11)^{\frac{4}{3}}\,dx$ **k** $\int \sqrt{2-3x}\,dx$ **l** $\int (5-2x)^4\,dx$

Example 8

2 Find an antiderivative of each of the following:

a $\dfrac{1}{2x},\ x > 0$ **b** $\dfrac{1}{3x+2},\ x > -\dfrac{2}{3}$ **c** $\dfrac{4}{1+4x},\ x > -\dfrac{1}{4}$

d $\dfrac{5}{3x-2},\ x > \dfrac{2}{3}$ **e** $\dfrac{3}{1-4x},\ x < \dfrac{1}{4}$ **f** $\dfrac{3}{2-\frac{x}{2}},\ x < 4$

Example 9

3 Find (using the absolute value function in your answer):

a $\int \dfrac{5}{x}\,dx$ **b** $\int \dfrac{3}{x-4}\,dx$ **c** $\int \dfrac{10}{2x+1}\,dx$

d $\int \dfrac{6}{5-2x}\,dx$ **e** $\int 6(1-2x)^{-1}\,dx$ **f** $\int (4-3x)^{-1}\,dx$

4 Find an antiderivative of each of the following. (Where appropriate, use the absolute value function in your answer.)

a $\dfrac{3x+1}{x}$ **b** $\dfrac{x+1}{x}$ **c** $\dfrac{1}{(x+1)^2}$

d $\dfrac{(x+1)^2}{x}$ **e** $\dfrac{3}{(x-1)^3}$ **f** $\dfrac{1-2x}{x}$

Example 8, 9

5 Find y in terms of x for each of the following:

a $\dfrac{dy}{dx} = \dfrac{1}{2x}$ and $y = 2$ when $x = e^2$ **b** $\dfrac{dy}{dx} = \dfrac{2}{5-2x}$ and $y = 10$ when $x = 2$

6 A curve with equation $y = f(x)$ passes through the point $(5 + e, 10)$ and $f'(x) = \dfrac{10}{x-5}$. Find the equation of the curve.

7 Find an antiderivative of each of the following. (Where appropriate, use the absolute value function in your answer.)

a $\dfrac{x}{x+1}$ **b** $\dfrac{1-2x}{x+1}$ **c** $\dfrac{2x+1}{x+1}$

8 Given that $\frac{dy}{dx} = \frac{3}{x-2}$ and $y = 10$ when $x = 0$, find an expression for y in terms of x.

9 Given that $\frac{dy}{dx} = \frac{5}{2-4x}$ and $y = 10$ when $x = -2$, find an expression for y in terms of x.

10 Given that $\frac{dy}{dx} = \frac{5}{2-4x}$ and $y = 10$ when $x = 1$, find an expression for y in terms of x.

11D The antiderivative of e^{kx}

In Chapter 9 we found that, if $f(x) = e^{kx}$, then $f'(x) = ke^{kx}$.

Thus:

$$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + c, \quad k \neq 0$$

Example 10

Find the general antiderivative of each of the following:

a e^{4x} **b** $e^{5x} + 6x$ **c** $e^{3x} + 2$ **d** $e^{-x} + e^{x}$

Solution

a $\int e^{4x}\,dx = \frac{1}{4}e^{4x} + c$

b $\int e^{5x} + 6x\,dx = \frac{1}{5}e^{5x} + 3x^2 + c$

c $\int e^{3x} + 2\,dx = \frac{1}{3}e^{3x} + 2x + c$

d $\int e^{-x} + e^{x}\,dx = -e^{-x} + e^{x} + c$

Example 11

If the gradient of the tangent at a point (x, y) on a curve is given by $5e^{2x}$ and the curve passes through the point $(0, 7.5)$, find the equation of the curve.

Solution

Let the curve have equation $y = f(x)$. Then $f'(x) = 5e^{2x}$.

$$\int 5e^{2x}\,dx = \frac{5}{2}e^{2x} + c$$

$$\therefore \quad f(x) = \frac{5}{2}e^{2x} + c$$

But $f(0) = 7.5$ and therefore

$$\begin{aligned} 7.5 &= \frac{5}{2}e^{0} + c \\ &= 2.5 + c \end{aligned}$$

$$\therefore \quad c = 5$$

Hence $f(x) = \frac{5}{2}e^{2x} + 5$.

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Section summary

$$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + c, \quad k \neq 0$$

Exercise 11D

Skillsheet

Example 10

1 Find the general antiderivative of each of the following:

a e^{6x} **b** $e^{2x} + 3x$ **c** $e^{-3x} + 2x$ **d** $e^{-2x} + e^{2x}$

2 Find:

a $\int e^{2x} - e^{\frac{x}{2}}\,dx$ **b** $\int \frac{e^{2x} + 1}{e^{x}}\,dx$ **c** $\int 2e^{3x} - e^{-x}\,dx$

d $\int 5e^{\frac{x}{3}} - 2e^{\frac{x}{5}}\,dx$ **e** $\int 3e^{\frac{2x}{3}} - 3e^{\frac{7x}{5}}\,dx$ **f** $\int 5e^{\frac{4x}{3}} - 3e^{\frac{2x}{3}}\,dx$

Example 11

3 Find y in terms of x for each of the following:

a $\frac{dy}{dx} = e^{2x} - x$ and $y = 5$ when $x = 0$

b $\frac{dy}{dx} = \frac{3 - e^{2x}}{e^{x}}$ and $y = 4$ when $x = 0$

4 Given that $\frac{dy}{dx} = ae^{-x} + 1$ and that when $x = 0$, $\frac{dy}{dx} = 3$ and $y = 5$, find the value of y when $x = 2$.

5 A curve for which $\frac{dy}{dx} = e^{kx}$, where k is a constant, is such that the tangent at $(1, e^2)$ passes through the origin. Find the gradient of this tangent and hence determine:

a the value of k **b** the equation of the curve.

6 A curve for which $\frac{dy}{dx} = -e^{kx}$, where k is a constant, is such that the tangent at $(1, -e^3)$ passes through the origin. Find the gradient of this tangent and hence determine:

a the value of k **b** the equation of the curve.

11E The fundamental theorem of calculus and the definite integral

The integrals that you have learned to evaluate in the previous sections are known as **indefinite integrals** because they are only defined to within an arbitrary constant: for example, we have $\int 3x^2\,dx = x^3 + c$. In general terms, we can write $\int f(x)\,dx = F(x) + c$; that is, the integral of $f(x)$ is $F(x)$ plus a constant, where $F(x)$ is an antiderivative of $f(x)$.

We now resume our consideration of the **definite integral** and investigate its connection with the indefinite integral.

Signed area

We first look at regions below the x-axis as well as those above the x-axis.

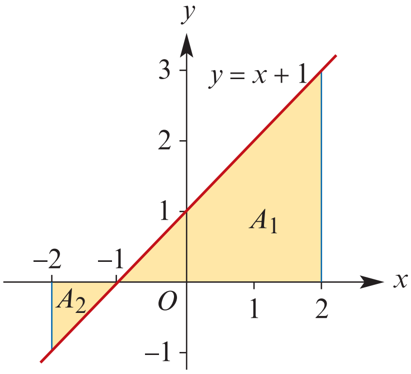
Consider the graph of $y = x + 1$ shown to the right.

$A_1 = \frac{1}{2} \times 3 \times 3 = 4\frac{1}{2}$ (area of a triangle)

$A_2 = \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$

The total area is $A_1 + A_2 = 5$.

The **signed area** is $A_1 - A_2 = 4$.

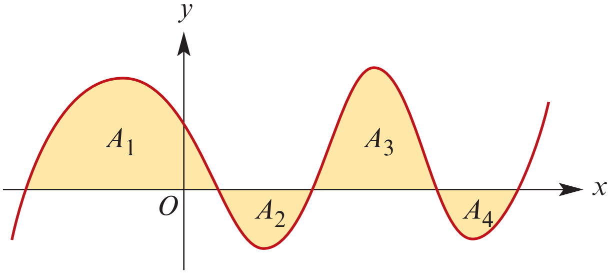


Regions above the x-axis have **positive signed area**.

Regions below the x-axis have **negative signed area**.

The total area of the shaded region is $A_1 + A_2 + A_3 + A_4$.

The signed area of the shaded region is $A_1 - A_2 + A_3 - A_4$.



For any continuous function f on an interval $[a, b]$, the **definite integral** $\int_a^b f(x)\,dx$ gives the **signed area** enclosed by the graph of $y = f(x)$ between $x = a$ and $x = b$.

In this more general setting, the definite integral can still be determined by a limiting process as discussed in the first section of this chapter.

The fundamental theorem of calculus

The fundamental theorem of calculus provides a connection between the area definition of the definite integral and the antiderivatives discussed previously. An outline of the proof is given in the final section of this chapter.

Fundamental theorem of calculus

If f is a continuous function on an interval $[a, b]$, then

$$\int_a^b f(x)\,dx = G(b) - G(a)$$

where G is any antiderivative of f.

To facilitate setting out, we sometimes write

$$G(b) - G(a) = [G(x)]_a^b$$

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Example 12

Evaluate the definite integral $\int_1^2 x\,dx$.

Solution

We have $\int x\,dx = \frac{1}{2}x^2 + c$ and so

$$\begin{aligned}\int_1^2 x\,dx &= \frac{1}{2}\times 2^2 + c - \left(\frac{1}{2}\times 1^2 + c\right)\\ &= 2 - \frac{1}{2} = \frac{3}{2}\end{aligned}$$

Note: The arbitrary constant cancels out. Because of this, we ignore it when evaluating definite integrals. We also use the more compact notation $G(b) - G(a) = [G(x)]_a^b$ to help with setting out:

$$\int_1^2 x\,dx = \left[\frac{x^2}{2}\right]_1^2 = \frac{2^2}{2} - \frac{1^2}{2} = \frac{3}{2}$$

Example 13

Evaluate each of the following the definite integrals:

a $\int_2^3 x^2\,dx$ **b** $\int_3^2 x^2\,dx$ **c** $\int_0^1 x^{\frac{1}{2}} + x^{\frac{3}{2}}\,dx$

Solution

a $$\begin{aligned}&\int_2^3 x^2\,dx\\ &= \left[\frac{x^3}{3}\right]_2^3\\ &= \frac{27}{3} - \frac{8}{3}\\ &= \frac{19}{3}\end{aligned}$$

b $$\begin{aligned}&\int_3^2 x^2\,dx\\ &= \left[\frac{x^3}{3}\right]_3^2\\ &= \frac{8}{3} - \frac{27}{3}\\ &= -\frac{19}{3}\end{aligned}$$

c $$\begin{aligned}&\int_0^1 x^{\frac{1}{2}} + x^{\frac{3}{2}}\,dx\\ &= \left[\frac{2}{3}x^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}}\right]_0^1\\ &= \frac{2}{3} + \frac{2}{5}\\ &= \frac{16}{15}\end{aligned}$$

Example 14

Evaluate each of the following definite integrals:

a $\int_0^1 2e^{-2x}\,dx$ **b** $\int_0^4 e^{2x} + 1\,dx$ **c** $\int_1^4 2x^{\frac{1}{2}} + e^{\frac{x}{2}}\,dx$

Solution

a $$\begin{aligned}\int_0^1 2e^{-2x}\,dx &= \left[\frac{2}{-2}e^{-2x}\right]_0^1\\ &= -1(e^{-2\times 1} - e^{-2\times 0})\\ &= -1(e^{-2} - 1)\\ &= 1 - e^{-2}\end{aligned}$$

b $$\begin{aligned}\int_0^4 e^{2x} + 1\,dx &= \left[\frac{1}{2}e^{2x} + x\right]_0^4\\ &= \frac{1}{2}e^8 + 4 - \left(\frac{1}{2}e^0 + 0\right)\\ &= \frac{1}{2}(e^8 + 7)\end{aligned}$$

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c $\int_1^4 2x^{\frac{1}{2}} + e^{\frac{x}{2}}\, dx = \left[\frac{4}{3}x^{\frac{3}{2}} + 2e^{\frac{x}{2}}\right]_1^4$

$= \frac{4}{3} \times 8 + 2e^2 - \left(\frac{4}{3} + 2e^{\frac{1}{2}}\right)$

$= \frac{28}{3} + 2e^2 - 2e^{\frac{1}{2}}$

$= 2\left(\frac{14}{3} + e^2 - e^{\frac{1}{2}}\right)$

Example 15

Evaluate each of the following definite integrals:

a $\int_6^8 \frac{1}{x-5}\, dx$

b $\int_4^5 \frac{1}{2x-5}\, dx$

Solution

a $\int_6^8 \frac{1}{x-5}\, dx = \left[\log_e(x-5)\right]_6^8$

$= \log_e 3 - \log_e 1$

$= \log_e 3$

b $\int_4^5 \frac{1}{2x-5}\, dx = \frac{1}{2}\left[\log_e(2x-5)\right]_4^5$

$= \frac{1}{2}(\log_e 5 - \log_e 3)$

$= \frac{1}{2}\log_e\left(\frac{5}{3}\right)$

Important properties of the definite integral are listed in the summary below.

Section summary

- For any continuous function f on an interval $[a, b]$, the **definite integral** $\int_a^b f(x)\, dx$ gives the **signed area** enclosed by the graph of $y = f(x)$ between $x = a$ and $x = b$.
- Fundamental theorem of calculus

  If f is a continuous function on the interval $[a, b]$, then

  $$\int_a^b f(x)\, dx = [G(x)]_a^b = G(b) - G(a)$$

  where G is any antiderivative of f.
- Properties of the definite integral
  - $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$
  - $\int_a^a f(x)\, dx = 0$
  - $\int_a^b k\, f(x)\, dx = k \int_a^b f(x)\, dx$
  - $\int_a^b f(x) \pm g(x)\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx$
  - $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$

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Exercise 11E

Skillsheet

Example 12, 13

1 Evaluate each of the following:

a $\int_1^2 x^2\,dx$ **b** $\int_{-1}^3 x^3\,dx$ **c** $\int_0^1 x^3 - x\,dx$

d $\int_{-1}^2 (x+1)^2\,dx$ **e** $\int_1^2 \frac{1}{x^2}\,dx$ **f** $\int_1^4 x^{\frac{1}{2}} + 2x^2\,dx$

g $\int_0^2 x^3 + 2x^2 + x + 2\,dx$ **h** $\int_1^4 2x^{\frac{3}{2}} + 5x^3\,dx$

2 Evaluate each of the following:

a $\int_0^1 (2x+1)^3\,dx$ **b** $\int_0^2 (4x+1)^{-\frac{1}{2}}\,dx$ **c** $\int_1^2 (1-2x)^2\,dx$

d $\int_0^1 (3-2x)^{-2}\,dx$ **e** $\int_0^2 (3+2x)^{-3}\,dx$ **f** $\int_{-1}^1 (4x+1)^3\,dx$

g $\int_0^1 \sqrt{2-x}\,dx$ **h** $\int_3^4 \frac{1}{\sqrt{2x-4}}\,dx$ **i** $\int_0^1 \frac{1}{(3+2x)^2}\,dx$

Example 14

3 Evaluate each of the following:

a $\int_0^1 e^{2x}\,dx$ **b** $\int_0^1 e^{-2x} + 1\,dx$ **c** $\int_0^1 2e^{\frac{x}{3}} + 2\,dx$ **d** $\int_{-2}^2 \frac{e^x + e^{-x}}{2}\,dx$

4 Given that $\int_0^4 h(x)\,dx = 5$, evaluate:

a $\int_0^4 2h(x)\,dx$ **b** $\int_0^4 h(x) + 3\,dx$ **c** $\int_4^0 h(x)\,dx$

d $\int_0^4 h(x) + 1\,dx$ **e** $\int_0^4 h(x) - x\,dx$

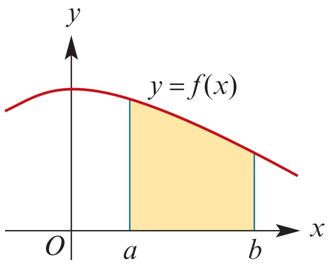
Example 15

5 **a** Find $\int_0^4 \frac{1}{x-6}\,dx$. **b** Find $\int_2^4 \frac{1}{2x-3}\,dx$. **c** Find $\int_5^6 \frac{3}{2x+7}\,dx$.

11F Finding the area under a curve

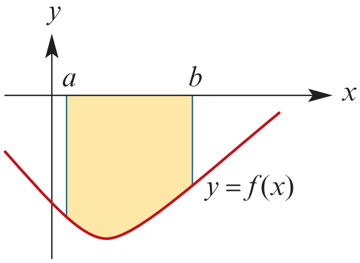
Recall that the definite integral $\int_a^b f(x)\,dx$ gives the net signed area 'under' the curve.

▶ Finding the area of a region



- If $f(x) \geq 0$ for all $x \in [a, b]$, then the area A of the region contained between the curve, the x-axis and the lines $x = a$ and $x = b$ is given by

$$A = \int_a^b f(x)\,dx$$



- If $f(x) \leq 0$ for all $x \in [a, b]$, then the area A of the region contained between the curve, the x-axis and the lines $x = a$ and $x = b$ is given by

$$A = -\int_a^b f(x)\,dx$$
$$= \int_b^a f(x)\,dx$$

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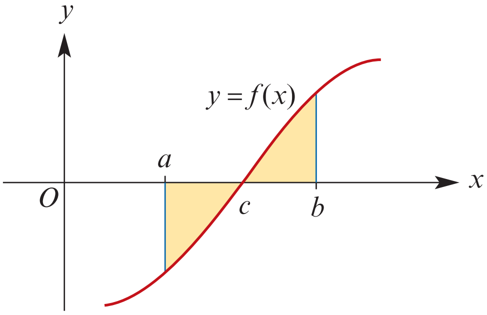
- If $c \in (a, b)$ with $f(c) = 0$ and $f(x) \geq 0$ for $x \in (c, b]$ and $f(x) \leq 0$ for $x \in [a, c)$, then the area A of the shaded region is given by
$$A = \int_c^b f(x)\,dx + \left(-\int_a^c f(x)\,dx\right)$$

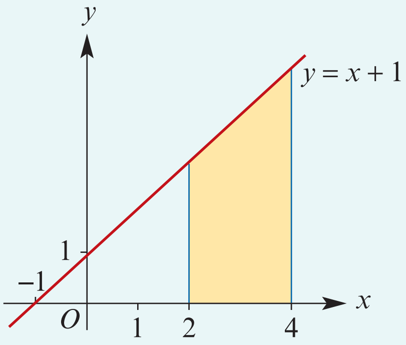


Note: In determining the area 'under' a curve $y = f(x)$, the sign of $f(x)$ in the given interval is the critical factor.

Example 16

a Find the area of the region between the x-axis, the line $y = x + 1$ and the lines $x = 2$ and $x = 4$. Check the answer by working out the area of the trapezium.

b Find the area under the line $y = x + 1$ between $x = -4$ and $x = -2$.



Solution

a $$\text{Area} = \int_2^4 x + 1\,dx = \left[\frac{x^2}{2} + x\right]_2^4$$
$$= \left(\frac{4^2}{2} + 4\right) - \left(\frac{2^2}{2} + 2\right)$$
$$= 12 - 4 = 8$$

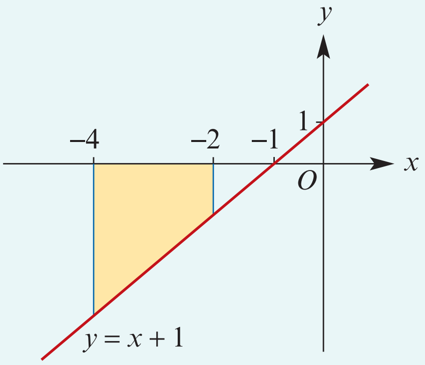
The area of the shaded region is 8 square units.

Check: Area of trapezium = average height × base = $\frac{3+5}{2} \times 2 = 8$

b $$\text{Area} = -\int_{-4}^{-2} x + 1\,dx = -\left[\frac{x^2}{2} + x\right]_{-4}^{-2}$$
$$= -(0 - 4) = 4$$

The area of the shaded region is 4 square units.

Note: The negative sign is introduced as the integral gives the *signed area* from −4 to −2, which is negative.



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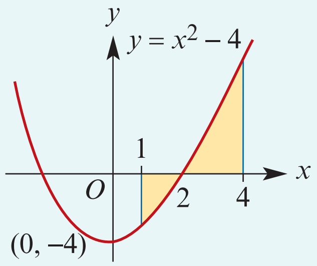
Example 17

Find the exact area of the shaded region.



Solution

$$\text{Area} = \int_2^4 (x^2 - 4)\,dx + -\int_1^2 (x^2 - 4)\,dx$$

$$= \left[\frac{x^3}{3} - 4x\right]_2^4 - \left[\frac{x^3}{3} - 4x\right]_1^2$$

$$= \left(\frac{64}{3} - 16\right) - \left(\frac{8}{3} - 8\right) - \left(\left(\frac{8}{3} - 8\right) - \left(\frac{1}{3} - 4\right)\right)$$

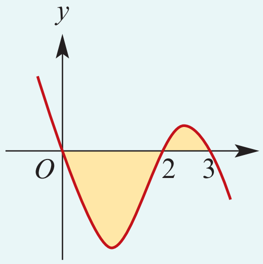
$$= \frac{56}{3} - 8 - \left(\frac{7}{3} - 4\right) = \frac{37}{3}$$

The area is $\frac{37}{3}$ square units.

Example 18

Find the exact area of the regions enclosed by the graph of $y = x(2 - x)(x - 3)$ and the x-axis.

Solution



$$y = x(-x^2 + 5x - 6)$$

$$= -x^3 + 5x^2 - 6x$$

$$\text{Area} = \int_2^3 (-x^3 + 5x^2 - 6x)\,dx + -\int_0^2 (-x^3 + 5x^2 - 6x)\,dx$$

$$= \left[\frac{-x^4}{4} + \frac{5x^3}{3} - \frac{6x^2}{2}\right]_2^3 - \left[\frac{-x^4}{4} + \frac{5x^3}{3} - \frac{6x^2}{2}\right]_0^2$$

$$= \left(\frac{-81}{4} + 45 - 27\right) - \left(-4 + \frac{40}{3} - 12\right) - \left(-4 + \frac{40}{3} - 12\right)$$

$$= \frac{-81}{4} + 18 + 32 - \frac{80}{3}$$

$$= 50 - \frac{243 + 320}{12} = \frac{37}{12}$$

The area is $\frac{37}{12}$ square units.

Note: There is no need to find the coordinates of stationary points.

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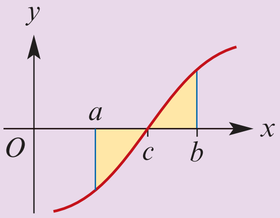
Section summary

Finding areas:

- If $f(x) \geq 0$ for all $x \in [a, b]$, then the area of the region contained between the curve, the x-axis and the lines $x = a$ and $x = b$ is given by $\int_a^b f(x)\,dx$.
- If $f(x) \leq 0$ for all $x \in [a, b]$, then the area of the region contained between the curve, the x-axis and the lines $x = a$ and $x = b$ is given by $-\int_a^b f(x)\,dx$.
- If $c \in (a, b)$ with $f(c) = 0$ and $f(x) \geq 0$ for $x \in (c, b]$ and $f(x) \leq 0$ for $x \in [a, c)$, then the area of the shaded region is given by $\int_c^b f(x)\,dx + \left(-\int_a^c f(x)\,dx\right)$.



Exercise 11F

Example 16, 17

1 Sketch the graph and find the exact area of the region(s) bounded by the x-axis and the graph of each of the following:

a $y = 3x^2 + 2$ between $x = 0$ and $x = 1$

b $y = x^3 - 8$ between $x = 2$ and $x = 4$

c $y = 4 - x$ between:

i $x = 0$ and $x = 4$ **ii** $x = 0$ and $x = 6$

Example 18

2 Find the exact area bounded by the x-axis and the graph of each of the following:

a $y = x^2 - 2x$ **b** $y = (4 - x)(3 - x)$ **c** $y = (x + 2)(7 - x)$

d $y = x^2 - 5x + 6$ **e** $y = 3 - x^2$ **f** $y = x^3 - 6x^2$

3 For each of the following, sketch a graph to illustrate the region for which the definite integral gives the area:

a $\int_1^4 2x + 1\,dx$ **b** $\int_0^3 3 - x\,dx$ **c** $\int_0^4 x^2\,dx$

d $\int_{-1}^1 4 - 2x^2\,dx$ **e** $\int_2^4 \sqrt{x}\,dx$ **f** $\int_0^1 (1 - x)(1 + x)^2\,dx$

4 Find the exact area of the region bounded by the curve $y = 3x + 2x^{-2}$, the lines $x = 2$ and $x = 5$ and the x-axis.

5 Sketch the graph of $f(x) = 1 + x^3$ and find the exact area of the region bounded by the curve and the axes.

6 Sketch the graph of $f(x) = 4e^{2x} + 3$ and find the exact area of the region enclosed by the curve, the axes and the line $x = 1$.

7 Sketch the graph of $y = x(2 - x)(x - 1)$ and find the exact area of the region enclosed by the curve and the x-axis.

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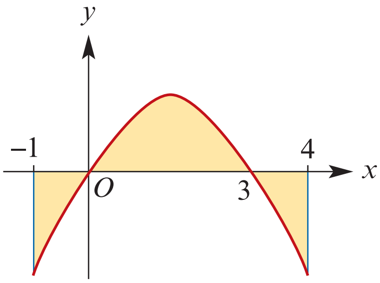
8 **a** Evaluate $\int_{-1}^{4} x(3-x)\,dx$.

b Find the exact area of the shaded region in the figure.

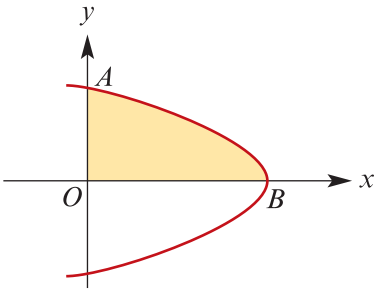


9 **a** In the figure, the graph of $y^2 = 9(1-x)$ is shown. Find the coordinates of A and B.

b Find the exact area of the shaded region by evaluating

$$\int_0^b 1 - \frac{y^2}{9}\,dy$$

for a suitable choice of b.



10 Sketch the graph of $y = \dfrac{1}{2-3x}$ and find the exact area of the region enclosed by the curve, the x-axis and the lines with equations $x = -3$ and $x = -2$.

11 Sketch the graph of $y = 2 + \dfrac{1}{x+4}$ and find the exact area of the region enclosed by the curve, the axes and the line $x = -2$.

12 Let $a > 0$ with $a \neq 1$.

a Show that $a^x = e^{x(\log_e a)}$.

b Hence find the derivative and an antiderivative of a^x.

c Hence, or otherwise, show that the area under the curve $y = a^x$ between the lines $x = 0$ and $x = b$ is $\dfrac{1}{\log_e a}(a^b - 1)$.

11G Integration of circular functions

Recall the following results from Chapter 9:

- If $f(x) = \sin(kx + a)$, then $f'(x) = k\cos(kx + a)$.
- If $g(x) = \cos(kx + a)$, then $g'(x) = -k\sin(kx + a)$.

Thus:

$$\int \sin(kx + a)\,dx = -\frac{1}{k}\cos(kx + a) + c$$

$$\int \cos(kx + a)\,dx = \frac{1}{k}\sin(kx + a) + c$$

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Example 19

Find an antiderivative of each of the following:

a $\sin\left(3x + \frac{\pi}{4}\right)$

b $\frac{1}{4}\sin(4x)$

Solution

a $-\frac{1}{3}\cos\left(3x + \frac{\pi}{4}\right) + c$

b $-\frac{1}{16}\cos(4x) + c$

Example 20

Find the exact value of each of the following definite integrals:

a $\int_0^{\frac{\pi}{4}} \sin(2x)\, dx$

b $\int_0^{\frac{\pi}{2}} 2\cos x + 1\, dx$

Solution

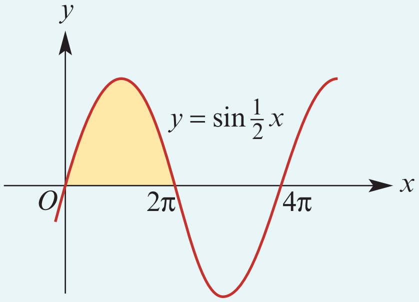
a
$$\begin{aligned}\int_0^{\frac{\pi}{4}} \sin(2x)\, dx &= \left[\frac{-1}{2}\cos(2x)\right]_0^{\frac{\pi}{4}}\\ &= \frac{-1}{2}\cos\left(\frac{\pi}{2}\right) - \left(\frac{-1}{2}\cos 0\right)\\ &= 0 + \frac{1}{2} = \frac{1}{2}\end{aligned}$$

b
$$\begin{aligned}\int_0^{\frac{\pi}{2}} 2\cos x + 1\, dx &= \left[2\sin x + x\right]_0^{\frac{\pi}{2}}\\ &= 2\sin\left(\frac{\pi}{2}\right) + \frac{\pi}{2} - (2\sin 0 + 0)\\ &= 2 + \frac{\pi}{2}\end{aligned}$$

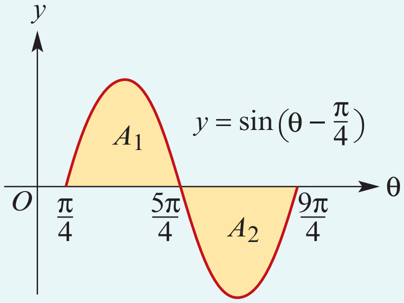
Example 21

Find the exact area of the shaded region for each graph:

a



b



Solution

a
$$\begin{aligned}\text{Area} &= \int_0^{2\pi} \sin\left(\frac{1}{2}x\right) dx\\ &= \left[-2\cos\left(\frac{1}{2}x\right)\right]_0^{2\pi}\\ &= -2\cos\pi - (-2\cos 0)\\ &= 4\end{aligned}$$

∴ Area of shaded region is 4 square units.

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b Regions A_1 and A_2 must be considered separately:

$$\begin{aligned}\text{Area } A_1 &= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \sin\left(\theta - \frac{\pi}{4}\right) d\theta \\ &= \left[-\cos\left(\theta - \frac{\pi}{4}\right)\right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \\ &= -(\cos\pi - \cos 0) \\ &= 2\end{aligned}$$

$$\begin{aligned}\text{Area } A_2 &= -\int_{\frac{5\pi}{4}}^{\frac{9\pi}{4}} \sin\left(\theta - \frac{\pi}{4}\right) d\theta \\ &= -\left[-\cos\left(\theta - \frac{\pi}{4}\right)\right]_{\frac{5\pi}{4}}^{\frac{9\pi}{4}} \\ &= \cos(2\pi) - \cos\pi \\ &= 2\end{aligned}$$

$\therefore$ Total area of shaded region is 4 square units.

Section summary

$$\int \sin(kx + a)\, dx = -\frac{1}{k}\cos(kx + a) + c$$

$$\int \cos(kx + a)\, dx = \frac{1}{k}\sin(kx + a) + c$$

Exercise 11G

Example 19 **1** Find an antiderivative of each of the following:

a $\cos(3x)$ **b** $\sin\left(\frac{1}{2}x\right)$ **c** $3\cos(3x)$

d $2\sin\left(\frac{1}{2}x\right)$ **e** $\sin\left(2x - \frac{\pi}{3}\right)$ **f** $\cos(3x) + \sin(2x)$

g $\cos(4x) - \sin(4x)$ **h** $-\frac{1}{2}\sin(2x) + \cos(3x)$ **i** $-\frac{1}{2}\cos\left(2x + \frac{\pi}{3}\right)$

j $\sin(\pi x)$

Example 20 **2** Find the exact value of each of the following definite integrals:

a $\int_0^{\frac{\pi}{4}} \sin x\, dx$ **b** $\int_0^{\frac{\pi}{4}} \cos(2x)\, dx$

c $\int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \cos\theta\, d\theta$ **d** $\int_0^{\frac{\pi}{2}} \sin\theta + \cos\theta\, d\theta$

e $\int_0^{\frac{\pi}{2}} \sin(2\theta)\, d\theta$ **f** $\int_0^{\frac{\pi}{3}} \cos(3\theta) + \sin(3\theta)\, d\theta$

g $\int_0^{\frac{\pi}{3}} \cos(3\theta) + \sin\left(\theta - \frac{\pi}{3}\right) d\theta$ **h** $\int_0^{\pi} \sin\left(\frac{x}{4}\right) + \cos\left(\frac{x}{4}\right) dx$

i $\int_0^{\frac{\pi}{4}} \sin\left(2x - \frac{\pi}{3}\right) dx$ **j** $\int_0^{\pi} \cos(2x) - \sin\left(\frac{x}{2}\right) dx$

Example 21 **3** Calculate the exact area of the region bounded by the curve $y = \sin\left(\frac{1}{2}x\right)$, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{2}$.

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4 For each of the following, draw a graph to illustrate the area given by the definite integral and evaluate the integral:

a $\int_0^{\frac{\pi}{4}} \cos x \, dx$ **b** $\int_0^{\frac{\pi}{3}} \sin(2x) \, dx$ **c** $\int_{\frac{-\pi}{6}}^{\frac{\pi}{6}} \cos(2x) \, dx$

d $\int_0^{\frac{\pi}{2}} \cos\theta + \sin\theta \, d\theta$ **e** $\int_0^{\frac{\pi}{2}} \sin(2\theta) + 1 \, d\theta$ **f** $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} 1 - \cos(2\theta) \, d\theta$

5 Find the exact value of each of the following definite integrals:

a $\int_0^{\frac{\pi}{2}} \sin\left(2x + \frac{\pi}{4}\right) dx$ **b** $\int_0^{\frac{\pi}{3}} \cos\left(3x + \frac{\pi}{6}\right) dx$

c $\int_0^{\frac{\pi}{3}} \cos\left(3x + \frac{\pi}{3}\right) dx$ **d** $\int_0^{\frac{\pi}{4}} \cos(3\pi - x) \, dx$

6 Sketch the curve $y = 2 + \sin(3x)$ for the interval $0 \le x \le \frac{2\pi}{3}$ and calculate the exact area enclosed by the curve, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{3}$.

11H Miscellaneous exercises

In this section we look at some further integrals to provide additional practice and introduce new approaches.

Example 22

Let $f(x) = \log_e(x^2 + 1)$.

a Show that $f'(x) = \dfrac{2x}{x^2 + 1}$.

b Hence evaluate $\int_0^2 \dfrac{x}{x^2 + 1} \, dx$.

Solution

a Let $y = \log_e(x^2 + 1)$ and $u = x^2 + 1$.
Then $y = \log_e u$. By the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= \frac{1}{u} \cdot 2x$$
$$\therefore \quad f'(x) = \frac{2x}{x^2 + 1}$$

b
$$\int_0^2 \frac{x}{x^2 + 1} \, dx = \frac{1}{2} \int_0^2 \left(\frac{2x}{x^2 + 1}\right) dx$$
$$= \frac{1}{2}\Big[\log_e(x^2 + 1)\Big]_0^2$$
$$= \frac{1}{2}(\log_e 5 - \log_e 1)$$
$$= \frac{1}{2} \log_e 5$$

Example 23

Let $f(x) = \dfrac{\cos x}{\sin x}$.

a Show that $f'(x) = \dfrac{-1}{\sin^2 x}$.

b Hence evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \dfrac{1}{\sin^2 x} \, dx$.

Solution

a Using the quotient rule:

$$\begin{aligned} f'(x) &= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} \\ &= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} \\ &= \frac{-1}{\sin^2 x} \end{aligned}$$

b

$$\begin{aligned} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\sin^2 x}\, dx &= -\left[\frac{\cos x}{\sin x}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\ &= -\frac{\cos(\frac{\pi}{2})}{\sin(\frac{\pi}{2})} + \frac{\cos(\frac{\pi}{4})}{\sin(\frac{\pi}{4})} \\ &= 1 \end{aligned}$$

Example 24

a If $f(x) = x\log_e(kx)$, find $f'(x)$ and hence find $\int \log_e(kx)\, dx$, where k is a positive real constant.

b If $f(x) = x^2\log_e(kx)$, find $f'(x)$ and hence find $\int x\log_e(kx)\, dx$, where k is a positive real constant.

Solution

a

$$\begin{aligned} f'(x) &= \log_e(kx) + x \times \frac{1}{x} \\ &= \log_e(kx) + 1 \end{aligned}$$

Antidifferentiate both sides of the equation with respect to x:

$$\begin{aligned} \int f'(x)\, dx &= \int \log_e(kx)\, dx + \int 1\, dx \\ x\log_e(kx) + c_1 &= \int \log_e(kx)\, dx + x + c_2 \end{aligned}$$

Thus

$$\begin{aligned} \int \log_e(kx)\, dx &= x\log_e(kx) - x + c_1 - c_2 \\ &= x\log_e(kx) - x + c \end{aligned}$$

b

$$\begin{aligned} f'(x) &= 2x\log_e(kx) + x^2 \times \frac{1}{x} \\ &= 2x\log_e(kx) + x \end{aligned}$$

Antidifferentiate both sides of the equation with respect to x:

$$\begin{aligned} \int f'(x)\, dx &= \int 2x\log_e(kx)\, dx + \int x\, dx \\ x^2\log_e(kx) + c_1 &= \int 2x\log_e(kx)\, dx + \frac{x^2}{2} + c_2 \end{aligned}$$

Thus $\int x\log_e(kx)\, dx = \frac{1}{2}x^2\log_e(kx) - \frac{x^2}{4} + c$

It is not possible to find rules for antiderivatives of all continuous functions: for example, for e^{-x^2}. However, for these functions we can find approximations of definite integrals.

For some functions, a CAS calculator can be used to find exact values of definite integrals where finding an antiderivative by hand is beyond the scope of the course. The following example illustrates this case.

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Example 25

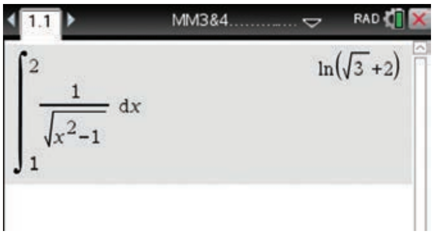
a Find $\int_1^2 \frac{1}{\sqrt{x^2-1}}\,dx$.

b Find $\int_0^{\frac{\pi}{2}} e^x \sin x \, dx$.

Using the TI-Nspire

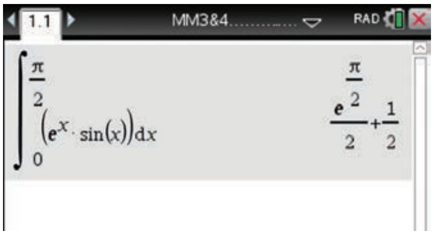
Use the **Integral** template from the **Calculus** menu and complete as shown.

a



$\int_1^2 \frac{1}{\sqrt{x^2-1}}\,dx \qquad \ln(\sqrt{3}+2)$

b

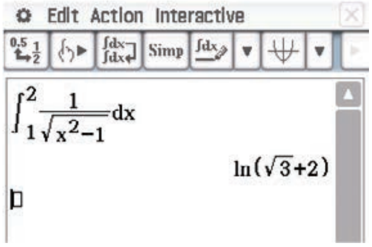


$\int_0^{\frac{\pi}{2}} (e^x \cdot \sin(x))\,dx \qquad \frac{e^{\frac{\pi}{2}}}{2}+\frac{1}{2}$

Using the Casio ClassPad

- Enter and highlight the expression $\frac{1}{\sqrt{x^2-1}}$ or the expression $e^x \sin(x)$.
- Go to **Interactive** > **Calculation** > $\int$ and select **Definite**.
- Enter the lower limit and upper limit and tap OK.

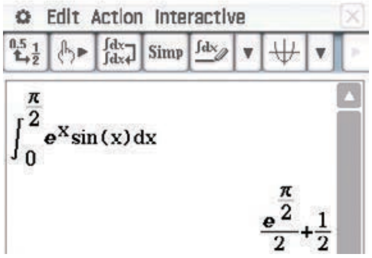
a



$\int_1^2 \frac{1}{\sqrt{x^2-1}}\,dx$

$\ln(\sqrt{3}+2)$

b



$\int_0^{\frac{\pi}{2}} e^x \sin(x)\,dx$

$\frac{e^{\frac{\pi}{2}}}{2}+\frac{1}{2}$

Exercise 11H

1 Find the exact value of each of the following:

a $\int_1^4 \sqrt{x}\,dx$

b $\int_{-1}^1 (1+x)^2\,dx$

c $\int_0^8 \sqrt[3]{x}\,dx$

d $\int_0^{\frac{\pi}{3}} \cos(2x) - \sin\left(\frac{1}{2}x\right)dx$

e $\int_1^2 e^{2x} + \frac{4}{x}\,dx$

f $\int_0^{\frac{\pi}{2}} \sin(2x) + \cos(3x)\,dx$

g $\int_0^{\pi} \sin\left(\frac{x}{4}\right) + \cos\left(\frac{x}{4}\right)dx$

h $\int_0^{\frac{\pi}{2}} 5x + \sin(2x)\,dx$

i $\int_1^4 \left(2+\frac{1}{x}\right)^2 dx$

j $\int_0^1 x^2(1-x)\,dx$

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2 Find the exact area of the region bounded by the graph of $f(x) = \sin x$, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{3}$.

Example 22, 23

3 **a** Differentiate $\frac{\sin x}{\cos x}$ and hence find an antiderivative of $\frac{1}{\cos^2 x}$.

b Differentiate $\frac{\cos(2x)}{\sin(2x)}$ and hence find an antiderivative of $\frac{1}{\sin^2(2x)}$.

c Differentiate $\log_e(3x^2 + 7)$ and hence evaluate $\int_0^2 \frac{x}{3x^2 + 7}\,dx$.

d Differentiate $x \sin x$ and hence evaluate $\int_0^{\frac{\pi}{4}} x \cos x\,dx$.

Example 24

4 **a** If $f(x) = x \log_e(2x)$, find $f'(x)$ and hence find $\int \log_e(2x)\,dx$.

b If $f(x) = x^2 \log_e(2x)$, find $f'(x)$ and hence find $\int x \log_e(2x)\,dx$.

c Find the derivatives of $x + \sqrt{1 + x^2}$ and $\log_e(x + \sqrt{1 + x^2})$.

By simplifying your last result if necessary, evaluate $\int_0^1 \frac{1}{\sqrt{1 + x^2}}\,dx$.

5 Find $\frac{d}{dx}(e^{\sqrt{x}})$ and hence evaluate $\int_1^2 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$.

6 Find $\frac{d}{dx}\left(\sin^3(2x)\right)$ and hence evaluate $\int_0^{\frac{\pi}{4}} \sin^2(2x)\cos(2x)\,dx$.

Example 25

7 Find the value of each of the following definite integrals, correct to two decimal places:

a $\int_0^{20} 10\cos\left(\frac{\pi x}{40}\right) e^{\frac{x}{80}}\,dx$

b $\int_2^5 \frac{e^x}{(x-1)^2}\,dx$

c $\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\cos x}{(x-1)^2}\,dx$

d $\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{100\cos x}{x^2}\,dx$

e $\int_0^{\pi} e^{\left(\frac{x}{10}\right)^2} \sin x\,dx$

f $\int_0^{\frac{\pi}{4}} \cos^3(x)\,e^{-x}\,dx$

8 **a** Show that $\frac{2x+3}{x-1} = 2 + \frac{5}{x-1}$.

b Hence evaluate $\int_2^4 \frac{2x+3}{x-1}\,dx$.

9 **a** Show that $\frac{5x-4}{x-2} = 5 + \frac{6}{x-2}$.

b Hence evaluate $\int_3^4 \frac{5x-4}{x-2}\,dx$.

10 **a** If $y = (1 - \frac{1}{2}x)^8$, find $\frac{dy}{dx}$. Hence, or otherwise, find $\int (1 - \frac{1}{2}x)^7\,dx$.

b If $y = \log_e(\cos x)$ for $\cos x > 0$, find $\frac{dy}{dx}$. Hence evaluate $\int_0^{\frac{\pi}{3}} \tan x\,dx$.

11 Find a function f such that $f'(x) = \sin\left(\frac{1}{2}x\right)$ and $f\left(\frac{4\pi}{3}\right) = 2$.

12 For each of the following, find $f(x)$:

a $f'(x) = \cos(2x)$ and $f(\pi) = 1$

b $f'(x) = \frac{3}{x}$ and $f(1) = 6$

c $f'(x) = e^{\frac{x}{2}}$ and $f(0) = 1$

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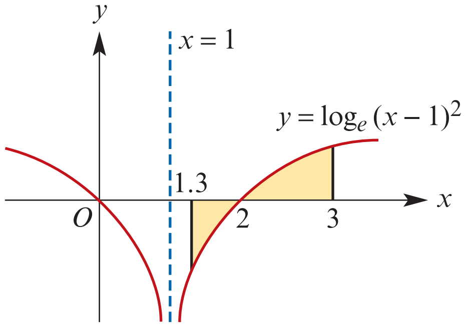
13 Find $\frac{d}{dx}\big(x\sin(3x)\big)$ and hence evaluate $\int_0^{\frac{\pi}{6}} x\cos(3x)\,dx$.

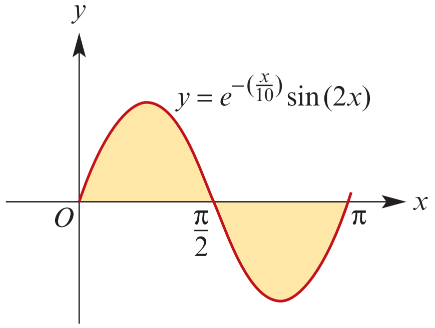
14 The curve with equation $y = a + b\sin\left(\frac{\pi x}{2}\right)$ passes through the points $(0, 1)$ and $(3, 3)$. Find a and b. Find the area of the region enclosed by this curve, the x-axis and the lines $x = 0$ and $x = 1$.

15 For each of the following, find the area of the shaded region correct to three decimal places:

a



b



16 Evaluate $\int_0^{\pi} e^{-\left(\frac{x}{10}\right)}\sin(2x)\,dx$, correct to four decimal places.

17 The gradient of a curve with equation $y = f(x)$ is given by $f'(x) = x + \sin(2x)$ and $f(0) = 1$. Find $f(x)$.

18 Let $f(x) = g'(x)$ and $h(x) = k'(x)$, where $g(x) = (x^2 + 1)^3$ and $k(x) = \sin(x^2)$. Find:

a $\int f(x)\,dx$

b $\int h(x)\,dx$

c $\int f(x) + h(x)\,dx$

d $\int -f(x)\,dx$

e $\int f(x) - 4\,dx$

f $\int 3h(x)\,dx$

19 Sketch the graph of $y = \frac{2}{x-1} + 4$ and evaluate $\int_2^3 \frac{2}{x-1} + 4\,dx$.
Indicate on your graph the region for which you have determined the area.

20 Sketch the graph of $y = \sqrt{2x-4} + 1$ and evaluate $\int_2^3 \sqrt{2x-4} + 1\,dx$.
Indicate on your graph the region for which you have determined the area.

21 Evaluate each of the following:

a $\int_3^4 \sqrt{x-2}\,dx$

b $\int_0^2 \sqrt{2-x}\,dx$

c $\int_0^1 \frac{1}{3x+1}\,dx$

d $\int_1^2 \frac{1}{2x-1} + 3\,dx$

e $\int_{2.5}^3 \sqrt{2x-5} - 6\,dx$

f $\int_3^4 \frac{1}{\sqrt{x-2}}\,dx$

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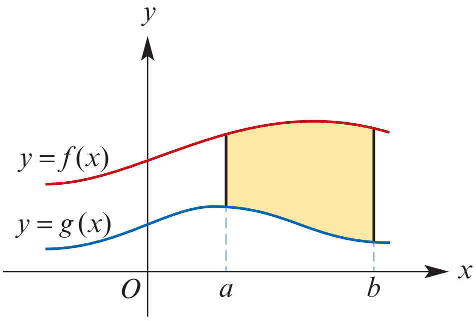
11I The area of a region between two curves

Let f and g be continuous functions on the interval $[a, b]$ such that

$$f(x) \geq g(x) \quad \text{for all } x \in [a, b]$$

Then the area of the region bounded by the two curves and the lines $x = a$ and $x = b$ can be found by evaluating

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b f(x) - g(x)\,dx$$



Example 26

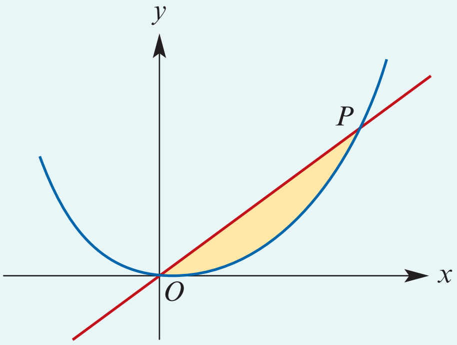
Find the area of the region bounded by the parabola $y = x^2$ and the line $y = 2x$.

Solution

We first find the coordinates of the point P:

$$x^2 = 2x$$
$$x(x - 2) = 0$$
$$\therefore \quad x = 0 \text{ or } x = 2$$

Therefore the coordinates of P are $(2, 4)$.



$$\begin{aligned}\text{Required area} &= \int_0^2 2x - x^2\,dx \\ &= \left[x^2 - \frac{x^3}{3}\right]_0^2 \\ &= 4 - \frac{8}{3} = \frac{4}{3}\end{aligned}$$

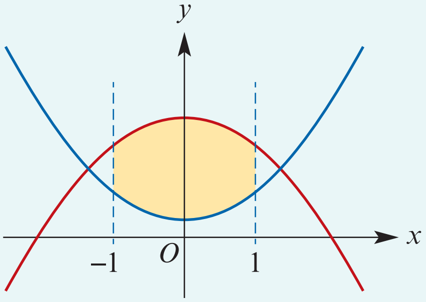
The area is $\frac{4}{3}$ square units.

Example 27

Calculate the area of the region enclosed by the curves with equations $y = x^2 + 1$ and $y = 4 - x^2$ and the lines $x = -1$ and $x = 1$.

Solution

$$\begin{aligned}\text{Required area} &= \int_{-1}^1 4 - x^2 - (x^2 + 1)\,dx \\ &= \int_{-1}^1 3 - 2x^2\,dx \\ &= \left[3x - \frac{2x^3}{3}\right]_{-1}^1 \\ &= 3 - \frac{2}{3} - \left(-3 + \frac{2}{3}\right) \\ &= \frac{14}{3}\end{aligned}$$



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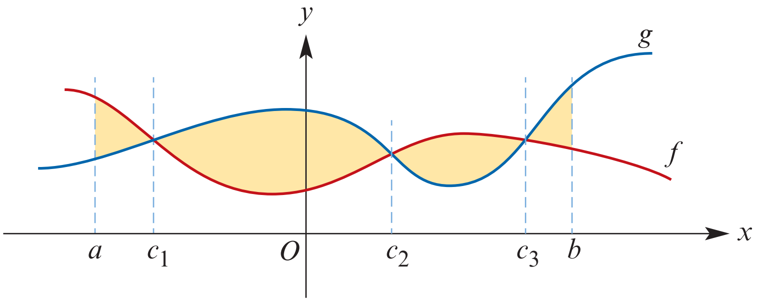
In Examples 26 and 27, the graph of one function is always 'above' the graph of the other for the intervals considered. What happens if the graphs cross?



To find the area of the shaded region, we must consider the intervals $[a, c_1]$, $[c_1, c_2]$, $[c_2, c_3]$ and $[c_3, b]$ separately. Thus, the shaded area is given by

$$\int_a^{c_1} f(x) - g(x)\,dx + \int_{c_1}^{c_2} g(x) - f(x)\,dx + \int_{c_2}^{c_3} f(x) - g(x)\,dx + \int_{c_3}^{b} g(x) - f(x)\,dx$$

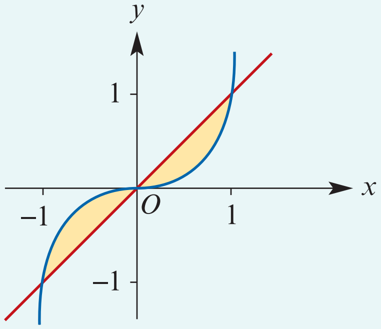
Example 28

Find the area of the region enclosed by the graphs of $f(x) = x^3$ and $g(x) = x$.

Solution

The graphs intersect where $f(x) = g(x)$:

$$\begin{aligned} x^3 &= x \\ x^3 - x &= 0 \\ x(x^2 - 1) &= 0 \\ \therefore\quad x &= 0 \text{ or } x = \pm 1 \end{aligned}$$



We see that:

- $f(x) \geq g(x)$ for $-1 \leq x \leq 0$
- $f(x) \leq g(x)$ for $0 \leq x \leq 1$

Thus the area is given by

$$\begin{aligned} \int_{-1}^{0} f(x) - g(x)\,dx + \int_{0}^{1} g(x) - f(x)\,dx &= \int_{-1}^{0} x^3 - x\,dx + \int_{0}^{1} x - x^3\,dx \\ &= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{0}^{1} \\ &= -\left(-\frac{1}{4}\right) + \frac{1}{4} \\ &= \frac{1}{2} \end{aligned}$$

The area is $\frac{1}{2}$ square unit.

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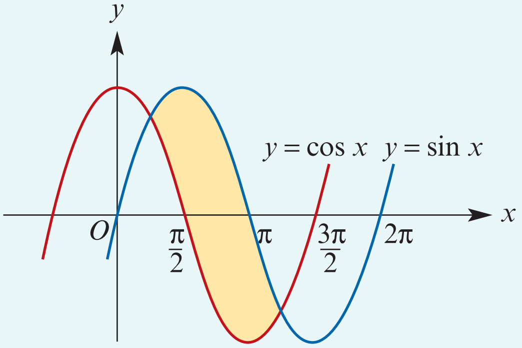
Example 29

Find the area of the shaded region.



Solution

First find the x-coordinates of the two points of intersection.

If $\sin x = \cos x$, then $\tan x = 1$ and so $x = \frac{\pi}{4}$ or $x = \frac{5\pi}{4}$.

$$\begin{aligned}\text{Area} &= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \sin x - \cos x \, dx \\ &= \Big[-\cos x - \sin x\Big]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \\ &= -\cos\left(\frac{5\pi}{4}\right) - \sin\left(\frac{5\pi}{4}\right) - \left[-\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\right] \\ &= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \\ &= \frac{4}{\sqrt{2}} = 2\sqrt{2}\end{aligned}$$

The area is $2\sqrt{2}$ square units.

Example 30

For the function $f\colon \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$, $f(x) = \log_e(x+1)$:

a Find f^{-1} and sketch the graphs of f and f^{-1} on the one set of axes.

b Find the exact value of the area $\int_0^{\log_e 2} f^{-1}(x)\, dx$.

c Find the exact value of $\int_0^1 f(x)\, dx$.

Solution

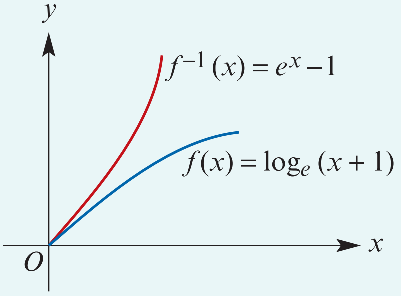
a Let $x = \log_e(y+1)$. Then

$$e^x = y + 1$$

$$\therefore \quad y = e^x - 1$$

Hence the inverse function is

$$f^{-1}\colon \mathbb{R}^+ \cup \{0\} \to \mathbb{R}, \ f^{-1}(x) = e^x - 1$$



b $\int_0^{\log_e 2} f^{-1}(x)\,dx = \int_0^{\log_e 2} e^x - 1\,dx$

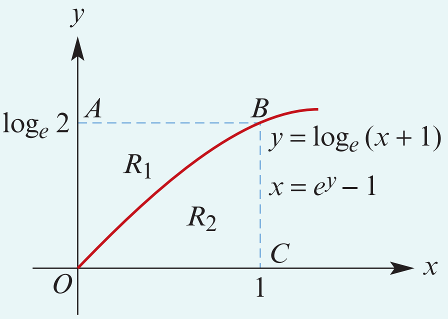
$$\begin{aligned} &= [e^x - x]_0^{\log_e 2} \\ &= e^{\log_e 2} - \log_e 2 - (e^0 - 0) \\ &= 2 - \log_e 2 - 1 \\ &= 1 - \log_e 2 \end{aligned}$$

c Area of rectangle $OABC = \log_e 2$

$$\begin{aligned} \text{Area of region } R_1 &= \int_0^{\log_e 2} e^y - 1\,dy \\ &= [e^y - y]_0^{\log_e 2} \\ &= 1 - \log_e 2 \qquad \text{(see } \mathbf{b}\text{)} \end{aligned}$$

$$\begin{aligned} \text{Area of region } R_2 &= \text{area of rectangle } OABC \\ &\quad - \text{area of region } R_1 \\ &= \log_e 2 - (1 - \log_e 2) \\ &= 2\log_e 2 - 1 \end{aligned}$$

$$\therefore \quad \int_0^1 f(x)\,dx = 2\log_e 2 - 1$$

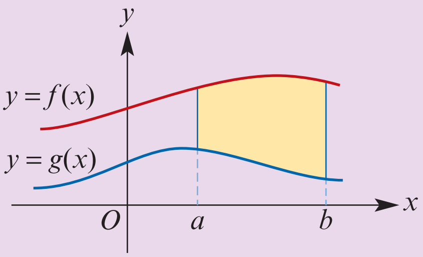


Section summary

To find the area of the shaded region bounded by the two curves and the lines $x = a$ and $x = b$, use

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b f(x) - g(x)\,dx$$

where f and g are continuous functions on $[a, b]$ such that $f(x) \geq g(x)$ for all $x \in [a, b]$.



Exercise 11I

Example 26 **1** Find the exact area of the region bounded by the graphs of $y = 12 - x - x^2$ and $y = x + 4$.

Example 27 **2** Find the exact area of the region bounded by the graphs of $f(x) = 5 - x^2$ and $g(x) = (x - 1)^2$.

Example 28 **3** Find the exact area of the region bounded by the graphs with equations:

a $y = x + 3$ and $y = 12 + x - x^2$

b $y = 3x + 5$ and $y = x^2 + 1$

c $y = 3 - x^2$ and $y = 2x^2$

d $y = x^2$ and $y = 3x$

e $y^2 = x$ and $x - y = 2$

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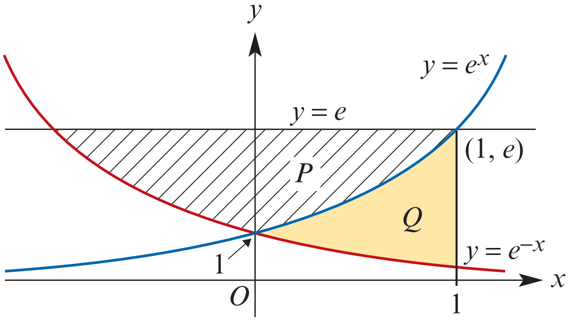
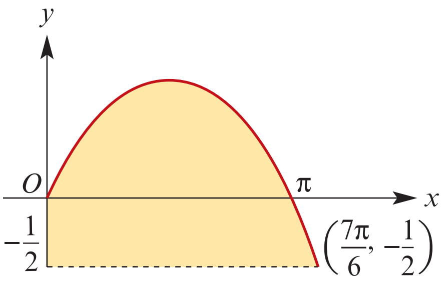
4 **a** Find the area of region P.

b Find the area of region Q.

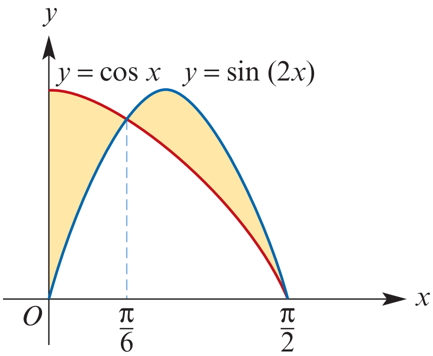


5 The figure shows part of the curve $y = \sin x$. Calculate the area of the shaded region, correct to three decimal places.



Example 29 **6** Using the same axes, sketch the curves $y = \sin x$ and $y = \sin(2x)$ for $0 \le x \le \pi$. Calculate the smaller of the two areas enclosed by the curves.

7 Find the area of the shaded region.



8 Find the coordinates of P, the point of intersection of the curves $y = e^x$ and $y = 2 + 3e^{-x}$. If these curves cut the y-axis at points A and B respectively, calculate the area bounded by AB and the arcs AP and BP. Give your answer correct to three decimal places.

Example 30 **9** For the function $f\colon \mathbb{R}^+ \to \mathbb{R}$, $f(x) = \log_e(2x)$:

a Find f^{-1} and sketch the graphs of f and f^{-1} on the one set of axes.

b Find the exact value of the area $\int_0^{\log_e 4} f^{-1}(x)\,dx$.

c Find the exact value of $\int_{\frac{1}{2}}^{2} f(x)\,dx$.

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11J Applications of integration

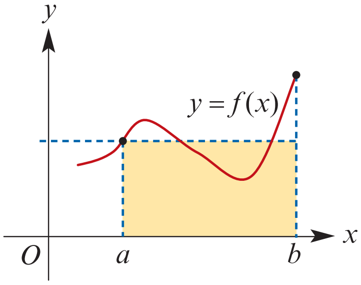
In this section we look at three applications of integration.

▶ Average value of a function

The average value of a function f for an interval $[a, b]$ is defined as:

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

In terms of the graph of $y = f(x)$, the average value is the height of a rectangle having the same area as the area under the graph for the interval $[a, b]$.

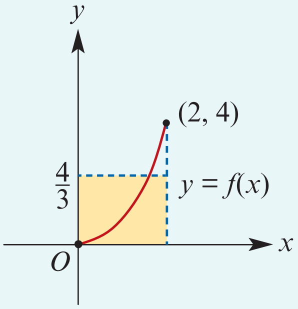


Example 31

Find the average value of $f(x) = x^2$ for the interval $[0, 2]$. Illustrate with a horizontal line determined by this value.

Solution

$$\begin{aligned}\text{Average} &= \frac{1}{2-0}\int_0^2 x^2\,dx\\ &= \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2\\ &= \frac{1}{2}\times\frac{8}{3}\\ &= \frac{4}{3}\end{aligned}$$



Note: Area of rectangle $= \int_0^2 f(x)\,dx$

▶ Motion in a straight line

We may be given a rule for acceleration and, by the use of antidifferentiation and some additional information, we can deduce rules for both velocity and position.

Example 32

A body starts from O and moves in a straight line. After t seconds ($t \geq 0$) its velocity, v m/s, is given by $v = 2t - 4$. Find:

a its position x in terms of t

b its position after 3 seconds

c its average velocity in the first 3 seconds

d the distance travelled in the first 3 seconds

e its average speed in the first 3 seconds.

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Solution

a Antidifferentiate v to find the expression for position, x m, at time t seconds:

$$x = t^2 - 4t + c$$

When $t = 0$, $x = 0$, and so $c = 0$.

$$\therefore \quad x = t^2 - 4t$$

b When $t = 3$, $x = -3$. The body is 3 m to the left of O.

c Average velocity $= \dfrac{-3 - 0}{3} = -1$ m/s

d First find when the body is at rest: $v = 0$ implies $2t - 4 = 0$, i.e. $t = 2$.

When $t = 2$, $x = -4$. Therefore the body goes from $x = 0$ to $x = -4$ in the first 2 seconds, and then back to $x = -3$ in the next second.

Thus it has travelled 5 m in the first 3 seconds.

e Average speed $= \dfrac{5}{3}$ m/s

It is useful to observe that, for a time interval $[t_1, t_2]$,

$$\text{Average velocity} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} v(t)\, dt$$

where $v(t)$ is the velocity at time t.

Example 33

A particle starts from rest 3 metres from a fixed point and moves in a straight line with an acceleration of $a = 6t + 8$. Find its position and velocity at time t seconds.

Solution

We are given the acceleration:

$$a = \frac{dv}{dt} = 6t + 8$$

Find the velocity by antidifferentiating:

$$v = 3t^2 + 8t + c$$

At $t = 0$, $v = 0$, and so $c = 0$.

$$\therefore \quad v = 3t^2 + 8t$$

Find the position by antidifferentiating again:

$$x = t^3 + 4t^2 + d$$

At $t = 0$, $x = 3$, and so $d = 3$.

$$\therefore \quad x = t^3 + 4t^2 + 3$$

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Example 34

A stone is projected vertically upwards from the top of a building 20 m high with an initial velocity of 15 m/s. Find:

a the time taken for the stone to reach its maximum height

b the maximum height reached by the stone

c the time taken for the stone to reach the ground

d the velocity of the stone as it hits the ground.

In this case we only consider the stone's motion in a vertical direction, so we can treat it as motion in a straight line. Also we will assume that the acceleration due to gravity is approximately -10 m/s^2. (Note that downwards is considered the negative direction.)

Solution

Antidifferentiating $a = -10$ gives $v = -10t + c$.

At $t = 0$, $v = 15$ and therefore $v = -10t + 15$.

Antidifferentiating v gives $x = -5t^2 + 15t + d$.

At $t = 0$, $x = 20$ and so $x = -5t^2 + 15t + 20$.

a The stone will reach its maximum height when $v = 0$:

$$\begin{aligned} -10t + 15 &= 0 \\ \therefore \quad t &= 1.5 \end{aligned}$$

The stone takes 1.5 seconds to reach its maximum height.

b At $t = 1.5$,
$$\begin{aligned} x &= -5(1.5)^2 + 15(1.5) + 20 \\ &= 31.25 \end{aligned}$$

The maximum height reached by the stone is 31.25 metres.

c The stone reaches the ground when $x = 0$:

$$\begin{aligned} -5t^2 + 15t + 20 &= 0 \\ -5(t^2 - 3t - 4) &= 0 \\ -5(t - 4)(t + 1) &= 0 \\ \therefore \quad t &= 4 \end{aligned}$$

(The solution $t = -1$ is rejected as $t \geq 0$.)
The stone takes 4 seconds to reach the ground.

d At $t = 4$,
$$\begin{aligned} v &= -10(4) + 15 \\ &= -25 \end{aligned}$$

The velocity on impact is -25 m/s.

Rates of change

Given the rate of change of a quantity we can obtain information about how the quantity varies. For example, we have seen that if the velocity of an object travelling in a straight line is given at time t, then the position of the object at time t can be determined using information about the initial position of the object.

Example 35

The rate of change of temperature with respect to time of a liquid which has been boiled and then allowed to cool is given by $\frac{dT}{dt} = -0.5(T - 30)$, where T is the temperature (°C) at time t (minutes).

a Sketch the graph of $\frac{dT}{dt}$ against T for $T > 30$.

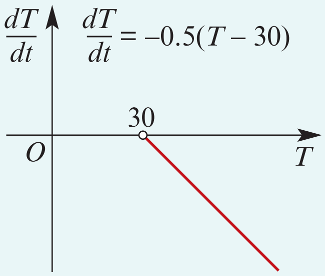
b Sketch the graph of $\frac{dt}{dT}$ against T for $T > 30$.

c **i** Find the area of the region enclosed by the graph of **b**, the x-axis and the lines $T = 35$ and $T = 120$. Give your answer correct to two decimal places.

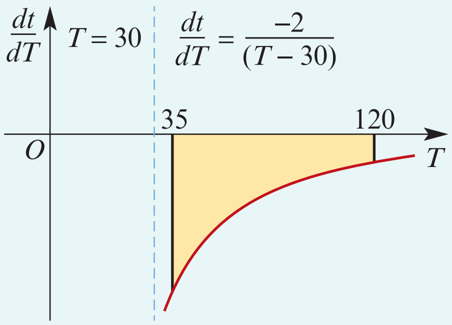
ii What does this area represent?

Solution

a



b



c **i** Area $= -\int_{35}^{120} \frac{-2}{(T-30)}\, dT = 5.78$

ii The area represents the time taken for the liquid to cool from 120°C to 35°C.

Exercise 11J

Example 31

1 Find the average value of each of the following functions for the stated interval:

a $f(x) = x(2 - x), \; x \in [0, 2]$

b $f(x) = \sin x, \; x \in [0, \pi]$

c $f(x) = \sin x, \; x \in \left[0, \frac{\pi}{2}\right]$

d $f(x) = \sin(nx), \; x \in \left[0, \frac{2\pi}{n}\right]$

e $f(x) = e^x + e^{-x}, \; x \in [-2, 2]$

2 An object is cooling and its temperature, T°C, after t minutes is given by $T = 50e^{\frac{-t}{2}}$. What is its average temperature over the first 10 minutes of cooling?

3 Find the average speed over the given interval for each of the following speed functions. For each of them, sketch a graph and mark in the average as a horizontal line. Time is in seconds and speed in metres per second.

a $v = 20t,\ t \in [0, 5]$ **b** $v = 24\sin\left(\frac{1}{4}\pi t\right),\ t \in [0, 4]$ **c** $v = 5(1 - e^{-t}),\ t \in [0, 5]$

4 An object falls from rest. Its velocity, v m/s, at time t seconds is given by $v = 9.8t$. Find the average velocity of the object over the first 3 seconds of its motion.

5 Find the mean value of $x(a - x)$ from $x = 0$ to $x = a$.

6 A quantity of gas expands according to the law $pv^{0.9} = 300$, where v m^3 is the volume of the gas and p N/m^2 is the pressure.

a What is the average pressure as the volume changes from $\frac{1}{2}$ m^3 to 1 m^3?

b If the change in volume in terms of t is given by $v = 3t + 1$, what is the average pressure over the time interval from $t = 0$ to $t = 1$?

Example 32

7 An object starts from point O and moves in a straight line. After t seconds ($t \geq 0$) its velocity, v m/s, is given by $v = 2t - 3$. Find:

a its position x in terms of t

b its position after 3 seconds

c its average velocity in the first 3 seconds

d the distance travelled in the first 3 seconds

e its average speed in the first 3 seconds.

8 The velocity of a particle, v m/s, at time t seconds ($t \geq 0$) is given by $v = 2t^2 - 8t + 6$. It is initially 4 m to the right of a point O. Find:

a its position and acceleration at time t

b its position when the velocity is zero

c its acceleration when the velocity is zero.

9 An object moves in a straight line with an acceleration of 8 m/s^2. If after 1 second it passes through point O and after 3 seconds it is 30 metres from O, find its initial position relative to O.

Example 33

10 A particle moves in a straight line so that its acceleration, a m/s^2, after t seconds ($t \geq 0$) is given by $a = 2t - 3$. If the initial position of the object is 2 m to the right of a point O and its initial velocity is 3 m/s, find the particle's position and velocity after 10 seconds.

Example 34

11 An object is projected vertically upwards with a velocity of 25 m/s. (Its acceleration due to gravity is -10 m/s^2.) Find:

a the object's velocity at time t

b its height above the point of projection at time t

c the time it takes to reach its maximum height

d the maximum height reached

e the time taken to return to the point of projection.

Example 35 **12** Heat escapes from a storage tank such that the rate of heat loss, in kilojoules per day, is given by

$$\frac{dH}{dt} = 1 + \frac{3}{4}\sin\left(\frac{\pi t}{60}\right), \qquad 0 \le t \le 200$$

where $H(t)$ is the total accumulated heat loss at time t days after noon on 1 April.

a Sketch the graph of $\frac{dH}{dt}$ against t for $0 \le t \le 200$.

b Find the values of t for which the rate of heat loss, i.e. $\frac{dH}{dt}$, is greater than 1.375.

c Find the values of t for which the rate of heat loss reaches its maximum.

d Find the heat lost between:

- **i** $t = 0$ and $t = 120$
- **ii** $t = 0$ and $t = 200$

13 The rate of flow of water from a reservoir is given by $\frac{dV}{dt} = 1000 - 30t^2 + 2t^3$ for $0 \le t \le 15$, where V is measured in millions of litres and t is the number of hours after the sluice gates are opened.

a Find the rate of flow (in million litres per hour) when $t = 0$ and $t = 2$.

b **i** Find the times when the rate of flow is a maximum.

- **ii** Find the maximum flow.

c Sketch the graph of $\frac{dV}{dt}$ against t for $0 \le t \le 15$.

d **i** Find the area beneath the graph between $t = 0$ and $t = 10$.

- **ii** What does this area represent?

14 The population of penguins on an island off the coast of Tasmania is increasing steadily. The rate of growth is given by the function $R\colon [0, \infty) \to \mathbb{R}$, $R(t) = 10\log_e(t + 1)$. The rate is measured in number of penguins per year. The date 1 January 1875 coincides with $t = 0$.

a Find the rate of growth of penguins when $t = 5$, $t = 10$, $t = 100$.

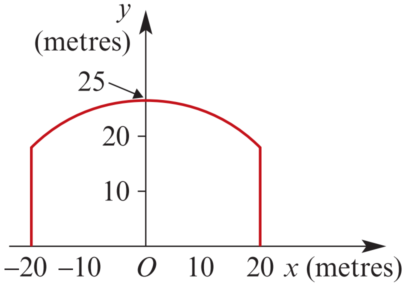
b Sketch the graph of $y = R(t)$.

c Find the inverse function R^{-1}.

d **i** Find the area under the graph of $y = R(t)$ between $t = 0$ and $t = 100$. (Use the inverse function to help find this area.)

- **ii** What does this area represent?

15 The roof of an exhibition hall has the shape of the function $f\colon [-20, 20] \to \mathbb{R}$ where $f(x) = 25 - 0.02x^2$. The hall is 80 metres long. A cross-section of the hall is shown in the figure. An air-conditioning company wishes to find the volume of the hall so that a suitable system may be installed. Find this volume.



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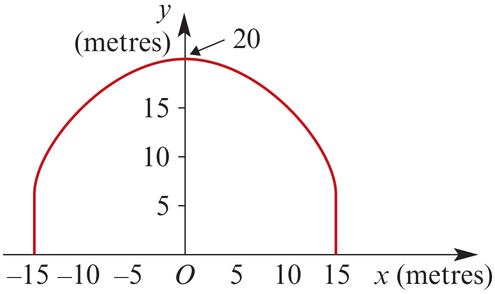
16 An aircraft hangar has the cross-section illustrated. The roof has the shape of the function $f\colon [-15, 15] \to \mathbb{R}$ where $f(x) = 20 - 0.06x^2$.

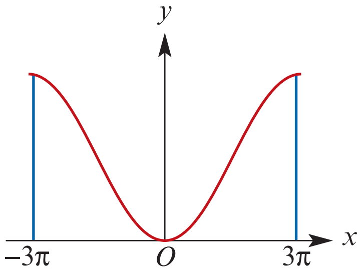
a Find the area of the cross-section.

b Find the volume of the hangar if it is 100 metres long.



17 A long trough with a parabolic cross-section is $1\frac{1}{2}$ metres wide at the top and 2 metres deep. Find the depth of water when the trough is half full.

18 A sculpture has cross-section as shown. The equation of the curve is $y = 3 - 3\cos\left(\frac{x}{3}\right)$ for $x \in [-3\pi, 3\pi]$. All measurements are in metres.



a Find the maximum value of the function and hence the height of the sculpture.

b The sculpture has a flat metal finish on one face, which in the diagram is represented by the region between the curve and the x-axis. Find the area of this region.

c There is a strut that meets the right side of the curve at right angles and passes through the point $(9, 0)$.

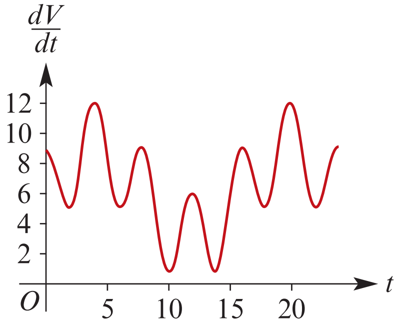
  i Find the equation of the normal to the curve where $x = a$.

  ii Find, correct to three decimal places, the value of a if the normal passes through $(9, 0)$.

19 The graph shows the number of litres per minute of water flowing through a pipe against the number of minutes since the machine started. The pipe is attached to the machine, which requires the water for cooling.

The curve has equation

$$\frac{dV}{dt} = 3\left[\cos\left(\frac{\pi t}{2}\right) + \sin\left(\frac{\pi t}{8}\right) + 2\right]$$



a What is the rate of flow of water when:

  i $t = 0$ **ii** $t = 2$ **iii** $t = 4$?

b Find, correct to three decimal places, the maximum and minimum flow through the pipe.

c Find the volume of water which flows through the pipe in the first 8 minutes.

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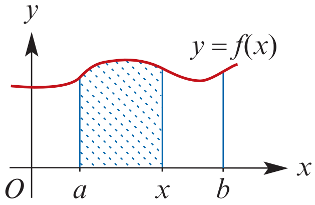
11K The fundamental theorem of calculus

▶ The derivative of the area function

Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function such that $f(x) \geq 0$ for all $x \in [a,b]$.

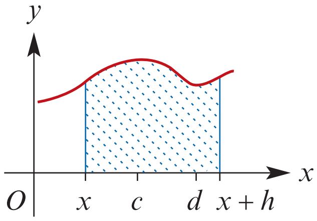


We define the function A geometrically by saying that $A(x)$ is the measure of the area under the curve $y = f(x)$ between a and x. We thus have $A(a) = 0$. We will see that $A'(x) = f(x)$, and thus A is an antiderivative of f.

First consider the quotient $\dfrac{A(x+h) - A(x)}{h}$ for $h > 0$.

By our definition of $A(x)$, it follows that $A(x+h) - A(x)$ is the area between x and $x + h$.

Let c be the point in the interval $[x, x+h]$ such that $f(c) \geq f(z)$ for all $z \in [x, x+h]$, and let d be the point in the same interval such that $f(d) \leq f(z)$ for all $z \in [x, x+h]$.



Thus $f(d) \leq f(z) \leq f(c)$ for all $z \in [x, x+h]$.

Therefore $hf(d) \leq A(x+h) - A(x) \leq hf(c)$.

That is, the shaded region has an area less than the area of the rectangle with base h and height $f(c)$ and an area greater than the area of the rectangle with base h and height $f(d)$.

Dividing by h gives

$$f(d) \leq \frac{A(x+h) - A(x)}{h} \leq f(c)$$

As $h \to 0$, both $f(c)$ and $f(d)$ approach $f(x)$.

Thus we have shown that $A'(x) = f(x)$, and therefore A is an antiderivative of f.

Now let G be any antiderivative of f. Since both A and G are antiderivatives of f, they must differ by a constant. That is,

$$A(x) = G(x) + k$$

where k is a constant. First let $x = a$. We then have

$$0 = A(a) = G(a) + k$$

and so $k = -G(a)$.

Thus $A(x) = G(x) - G(a)$, and letting $x = b$ yields

$$A(b) = G(b) - G(a)$$

The area under the curve $y = f(x)$ between $x = a$ and $x = b$ is equal to $G(b) - G(a)$, where G is any antiderivative of f.

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A similar argument could be used if $f(x) \le 0$ for all $x \in [a, b]$, but in this case we must take $A(x)$ to be the negative of the area under the curve. In general:

Fundamental theorem of calculus

If f is a continuous function on an interval $[a, b]$, then

$$\int_a^b f(x)\,dx = G(b) - G(a)$$

where G is any antiderivative of f.

The area as the limit of a sum

Finally, we consider the limit of a sum in a special case. This discussion gives an indication of how the limiting process can be undertaken in general.

Notation

We first introduce a notation to help us express sums. We do this through examples:

$$\sum_{i=1}^{3} i^2 = 1^2 + 2^2 + 3^2$$

$$\sum_{i=1}^{5} x_i = x_1 + x_2 + x_3 + x_4 + x_5$$

$$\sum_{i=1}^{n} x_i f(x_i) = x_1 f(x_1) + x_2 f(x_2) + x_3 f(x_3) + \cdots + x_n f(x_n)$$

The symbol $\sum$ is the uppercase Greek letter ‘sigma’, which is used in mathematics to denote *sum*.

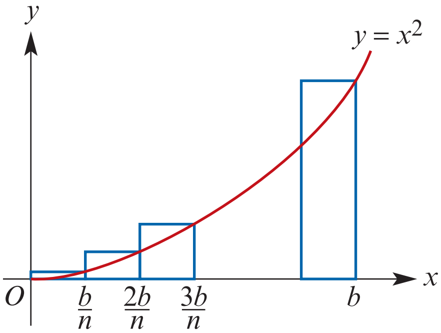
The area under a parabola

Consider the graph of $y = x^2$. We will find the area under the curve from $x = 0$ to $x = b$ using a technique due to Archimedes.

Divide the interval $[0, b]$ into n equal subintervals:

$$\left[0, \frac{b}{n}\right], \left[\frac{b}{n}, \frac{2b}{n}\right], \left[\frac{2b}{n}, \frac{3b}{n}\right], \ldots, \left[\frac{(n-1)b}{n}, b\right]$$

Each subinterval is the base of a rectangle with height determined by the right endpoint of the subinterval.



$$\begin{aligned}\text{Area of rectangles} &= \frac{b}{n}\left[\left(\frac{b}{n}\right)^2 + \left(\frac{2b}{n}\right)^2 + \left(\frac{3b}{n}\right)^2 + \cdots + \left(\frac{nb}{n}\right)^2\right] \\ &= \frac{b}{n}\left(\frac{b^2}{n^2} + \frac{4b^2}{n^2} + \frac{9b^2}{n^2} + \cdots + \frac{n^2b^2}{n^2}\right) \\ &= \frac{b^3}{n^3}(1 + 4 + 9 + \cdots + n^2)\end{aligned}$$

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There is a rule for working out the sum of the first n square numbers:

$$\sum_{i=1}^{n} i^2 = \frac{n}{6}(n+1)(2n+1)$$

$$\begin{aligned}\text{Area of rectangles} &= \frac{b^3}{n^3}\sum_{i=1}^{n} i^2\\ &= \frac{b^3}{n^3} \times \frac{n}{6}(n+1)(2n+1)\\ &= \frac{b^3}{6n^2}(2n^2+3n+1)\\ &= \frac{b^3}{6}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)\end{aligned}$$

As n becomes very large, the terms $\frac{3}{n}$ and $\frac{1}{n^2}$ become very small. We write:

$$\lim_{n\to\infty} \frac{b^3}{6}\left(2+\frac{3}{n}+\frac{1}{n^2}\right) = \frac{b^3}{3}$$

We read this as: the limit of the sum as n approaches infinity is $\frac{b^3}{3}$.

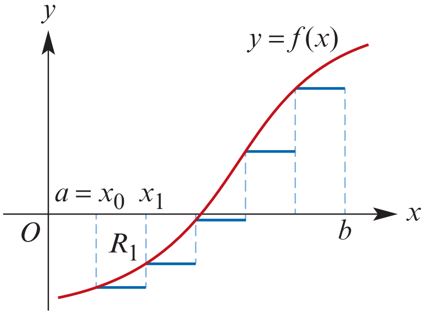
Using n left-endpoint rectangles, and considering the limit as $n \to \infty$, also gives the area $\frac{b^3}{3}$.

The signed area enclosed by a curve

This technique may be applied in general to a continuous function f on an interval $[a, b]$. For convenience, we will consider an increasing function.

Divide the interval $[a, b]$ into n equal subintervals. Each subinterval is the base of a rectangle with its 'height' determined by the left endpoint of the subinterval.

The contribution of rectangle R_1 is $(x_1 - x_0)f(x_0)$. Since $f(x_0) < 0$, the result is negative and so we have found the *signed area* of R_1.



The sum of the signed areas of the rectangles is

$$\frac{b-a}{n}\sum_{i=0}^{n-1} f(x_i)$$

If the limit as $n \to \infty$ exists, then we can make the following definition:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left(\frac{b-a}{n}\sum_{i=0}^{n-1} f(x_i)\right)$$

We could also have used the right-endpoint estimate: the left- and right-endpoint estimates will converge to the same limit as n approaches infinity. Definite integrals may be defined as the limit of suitable sums, and the fundamental theorem of calculus holds true under this definition.

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Review

Chapter summary

Spreadsheet AS Nrich

Antidifferentiation

- To find the general antiderivative:
  If $F'(x) = f(x)$, then $\int f(x)\,dx = F(x) + c$, where c is an arbitrary real number.
- Basic antiderivatives:

$$\int x^r\,dx = \frac{x^{r+1}}{r+1} + c \qquad \text{where } r \in \mathbb{Q} \setminus \{-1\}$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log_e(ax+b) + c \qquad \text{for } ax+b > 0$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log_e(-ax-b) + c \qquad \text{for } ax+b < 0$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log_e|ax+b| + c \qquad \text{for } ax+b \neq 0$$

$$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + c \qquad \text{where } k \neq 0$$

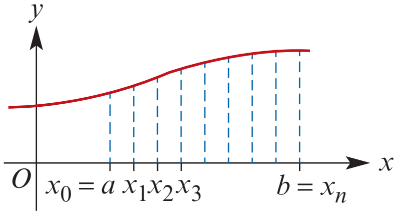
$$\int \sin(kx)\,dx = -\frac{1}{k}\cos(kx) + c \qquad \text{where } k \neq 0$$

$$\int \cos(kx)\,dx = \frac{1}{k}\sin(kx) + c \qquad \text{where } k \neq 0$$

- Properties of antidifferentiation:
  - $\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$
  - $\int k\,f(x)\,dx = k\int f(x)\,dx$, where k is a real number

Integration

- Numerical methods for approximating the area under a graph: Divide the interval $[a, b]$ on the x-axis into n equal subintervals $[a, x_1], [x_1, x_2], [x_2, x_3], \dots, [x_{n-1}, b]$.
  - Left-endpoint estimate
    $$L_n = \frac{b-a}{n}\Big[f(x_0) + f(x_1) + \dots + f(x_{n-1})\Big]$$
  - Right-endpoint estimate
    $$R_n = \frac{b-a}{n}\Big[f(x_1) + f(x_2) + \dots + f(x_n)\Big]$$



- Definite integral The signed area enclosed by the graph of $y = f(x)$ between $x = a$ and $x = b$ is denoted by $\int_a^b f(x)\,dx$.
- Fundamental theorem of calculus
  If f is a continuous function on an interval $[a, b]$, then
  $$\int_a^b f(x)\,dx = \big[G(x)\big]_a^b = G(b) - G(a)$$
  where G is any antiderivative of f.

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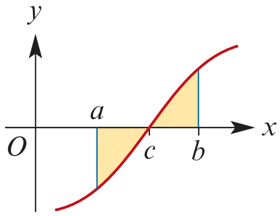
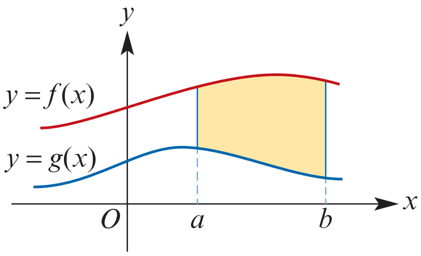
- Finding areas:
  - If $f(x) \geq 0$ for all $x \in [a, b]$, then the area of the region contained between the curve, the x-axis and the lines $x = a$ and $x = b$ is given by $\int_a^b f(x)\,dx$.
  - If $f(x) \leq 0$ for all $x \in [a, b]$, then the area of the region contained between the curve, the x-axis and the lines $x = a$ and $x = b$ is given by $-\int_a^b f(x)\,dx$.
  - If $c \in (a, b)$ with $f(c) = 0$ and $f(x) \geq 0$ for $x \in (c, b]$ and $f(x) \leq 0$ for $x \in [a, c)$, then the area of the shaded region is given by $\int_c^b f(x)\,dx + \left(-\int_a^c f(x)\,dx\right)$.



  - To find the area of the shaded region bounded by the two curves and the lines $x = a$ and $x = b$, use

    $$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b f(x) - g(x)\,dx$$

    where f and g are continuous functions on $[a, b]$ such that $f(x) \geq g(x)$ for all $x \in [a, b]$.



- Properties of the definite integral:
  - $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
  - $\int_a^a f(x)\,dx = 0$
  - $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$
  - $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
  - $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
- The **average value** of a continuous function f for an interval $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$.

Technology-free questions

1 Evaluate each of the following definite integrals:

a $\int_2^3 x^3\,dx$

b $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin\theta\,d\theta$

c $\int_a^{4a} (a^{\frac{1}{2}} - x^{\frac{1}{2}})\,dx$, where a is a positive constant

d $\int_1^4 \frac{3}{\sqrt{x}} - 5\sqrt{x} - x^{-\frac{3}{2}}\,dx$

e $\int_0^{\frac{\pi}{4}} \cos(2\theta)\,d\theta$

f $\int_1^e \frac{1}{x}\,dx$

g $\int_0^{\frac{\pi}{2}} \sin 2\left(\theta + \frac{\pi}{4}\right)\,d\theta$

h $\int_0^{\pi} \sin(4\theta)\,d\theta$

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2 Find $\int_{-1}^{2} x + 2f(x)\,dx$ if $\int_{-1}^{2} f(x)\,dx = 5$.

3 Find $\int_{1}^{5} f(x)\,dx$ if $\int_{0}^{1} f(x)\,dx = -2$ and $\int_{0}^{5} f(x)\,dx = 1$.

4 Find $\int_{3}^{-2} f(x)\,dx$ if $\int_{-2}^{1} f(x)\,dx = 2$ and $\int_{1}^{3} f(x)\,dx = -6$.

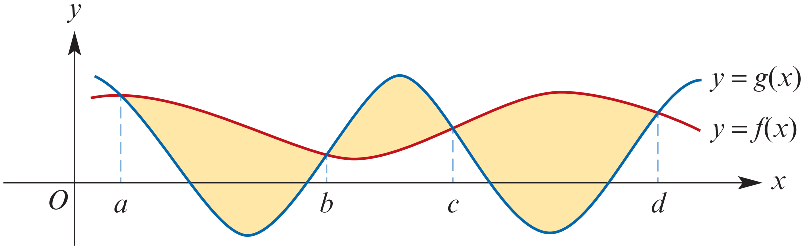
5 Evaluate $\int_{0}^{2} (x+1)^7\,dx$.

6 Evaluate $\int_{0}^{1} (3x+1)^3\,dx$.

7 Find $\int_{0}^{3} f(3x)\,dx$ if $\int_{0}^{9} f(x)\,dx = 5$.

8 Find $\int_{0}^{1} f(3x+1)\,dx$ if $\int_{1}^{4} f(x)\,dx = 5$.

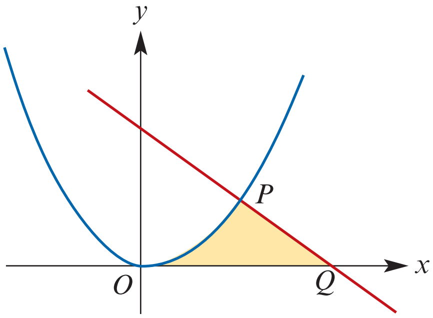
9 Set up a sum of definite integrals that represents the total shaded area between the curves $y = f(x)$ and $y = g(x)$.



10 The figure shows the curve $y = x^2$ and the straight line $2x + y = 15$. Find:

a the coordinates of P and Q

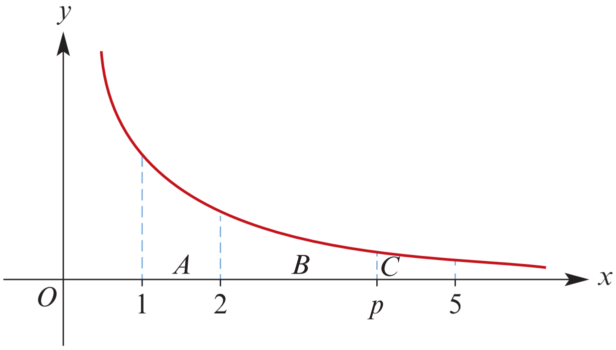
b the area of the shaded region.



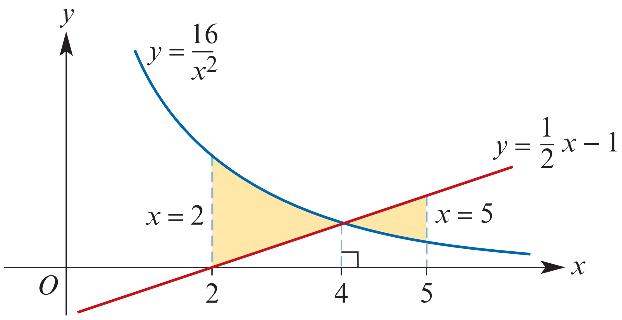
11 The figure shows part of the curve $y = \dfrac{10}{x^2}$. Find:

a the area of region A

b the value of p for which the regions B and C are of equal area.



12 Find the area of the shaded region.

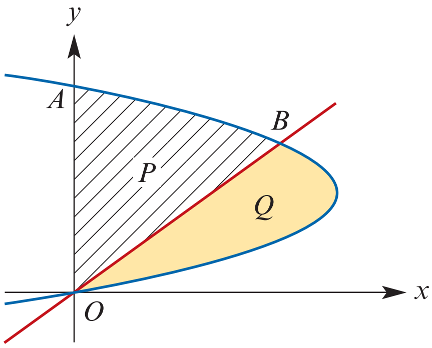


13 The figure shows part of the curve $x = 6y - y^2$ and part of the line $y = x$.

a Find the coordinates of A and B.

b Find the area of region P.

c Find the area of region Q.



14 **a** Sketch the graph of $y = e^x + 1$ and clearly indicate, by shading the region, the area given by the definite integral $\int_0^2 e^x + 1\, dx$.

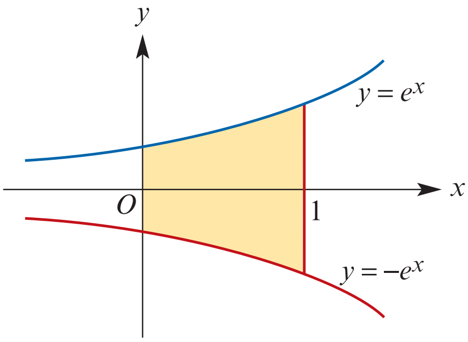
b Evaluate $\int_0^2 e^x + 1\, dx$.

15 **a** Sketch the graphs of $y = e^{-x}$ and $y = e^x$ on the one set of axes and clearly indicate, by shading the region, the area given by $\int_0^2 e^{-x}\, dx + \int_{-2}^0 e^x\, dx$.

b Evaluate $\int_0^2 e^{-x}\, dx + \int_{-2}^0 e^x\, dx$.

16 **a** Evaluate $\int_0^1 e^x\, dx$.

b By symmetry, find the area of the region shaded in the figure.



17 Sketch the graph of $f(x) = 2e^{2x} + 3$ and find the area of the region enclosed between the curve, the axes and the line $x = 1$.

18 Sketch the graph of $y = x(x-2)(x+1)$ and find the area of the region contained between the graph and the x-axis. (Do not attempt to find the coordinates of the turning points.)

19 Evaluate each of the following definite integrals:

a $\int_0^2 e^{-x} + x\, dx$ **b** $\int_{-2}^{-1} x + \frac{1}{x-1}\, dx$

c $\int_0^{\frac{\pi}{2}} \sin x + x\, dx$ **d** $\int_{-4}^{-5} e^x + \frac{1}{2-2x}\, dx$

Multiple-choice questions

1 $\int_0^2 3f(x) + 2\, dx =$

A $3\int_0^2 f(x)\, dx + 3x$ **B** $3\int_0^2 f(x)\, dx + x$ **C** $3\int_0^2 f(x)\, dx + 4$

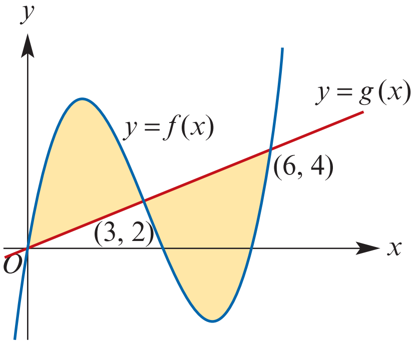
D $3f'(x) + 4$ **E** $\int_0^2 f(x)\, dx + 4$

2 For a and b positive real constants, the antiderivative of $\sqrt{(ax-b)^3}$ for $x > \frac{b}{a}$ is

A $\frac{3}{2}(ax-b)^{\frac{1}{2}} + c$ **B** $\frac{1}{5\sqrt{ax-b}} + c$ **C** $\frac{2}{5a}(ax-b)^{\frac{5}{2}} + c$

D $\frac{1}{5a}\sqrt{(ax-b)^5} + c$ **E** $\frac{2}{5\sqrt{(ax-b)^5}} + c$

3 An expression using integral notation for the area of the shaded region shown is



A $\int_0^6 f(x) - g(x)\, dx$

B $\int_0^3 f(x) - g(x)\, dx + \int_3^6 g(x) - f(x)\, dx$

C $\int_0^4 f(x) - g(x)\, dx$

D $\int_0^2 f(x) - g(x)\, dx + \int_2^4 f(x) - g(x)\, dx$

E $\int_0^2 f(x) - g(x)\, dx + \int_2^4 g(x) - f(x)\, dx$

4 $\int_a^b c\, dx$, where a, b and c are distinct real number constants, is equal to

A ca **B** $cb - ca$ **C** $ca - b$ **D** $cb - a$ **E** $c(a + b)$

5 An expression for y if $\frac{dy}{dx} = \frac{ax}{2} + 1$ and $y = 1$ when $x = 0$ is

A $y = \frac{ax^2}{4} + x + 1$ **B** $y = a$ **C** $y = ax^2 + x - 1$

D $y = ax^2 + x + a$ **E** $y = ax^2 + ax + a$

6 The function f such that $f'(x) = -6\sin(3x)$ and $f\left(\frac{2\pi}{3}\right) = 3$ is given by $f(x) =$

A $-18\cos(3x) + 21$ **B** $-2\cos(2x) + 5$ **C** $-2\sin(3x) + 1$

D $2\cos(3x) + 1$ **E** $2\sin(4x) + 3$

7 The area of the region enclosed by the curve $y = e^{5x} - 2\sin(4x)$, the x-axis and the lines $x = -1$ and $x = 1$, correct to two decimal places, is

A 0.17 **B** 29.55 **C** 29.68 **D** 29.85 **E** 30.02

8 If $\dfrac{dy}{dx} = ae^{-x} + 2$ and when $x = 0$, $\dfrac{dy}{dx} = 5$ and $y = 1$, then when $x = 2$, $y =$

A $-\dfrac{3}{e^2} + 2$ **B** $-\dfrac{3}{e^2} + 4$ **C** $-\dfrac{3}{e^2} + 8$ **D** $3e^2 + 4$ **E** $3e^2 + 8$

9 The rate of flow of water from a tap follows the rule $R(t) = 5e^{-0.1t}$, where $R(t)$ litres per minute is the rate of flow after t minutes. The number of litres, to the nearest litre, which flowed out in the first 3 minutes is

A 0 **B** 5 **C** 13 **D** 50 **E** 153

10 The graph represents the function $y = \sin x$ where $0 \le x \le 2\pi$. The total area of the shaded regions is

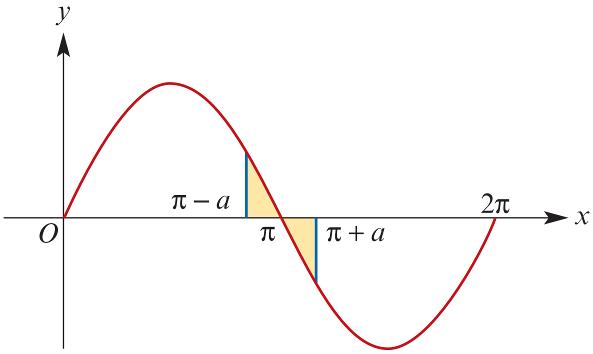
A $\frac{1}{2}\cos a$

B $2\cos a$

C $\frac{1}{2}(1 - \cos a)$

D $2(1 - \cos a)$

E $2\sin^2 a$



Extended-response questions

1 The diagram shows part of the curve with equation

$$y = x - \frac{1}{x^2}$$

The point C has coordinates $(2, 0)$. Find:

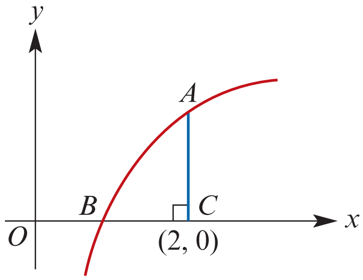
a the equation of the tangent to the curve at point A

b the coordinates of the point T where this tangent meets the x-axis

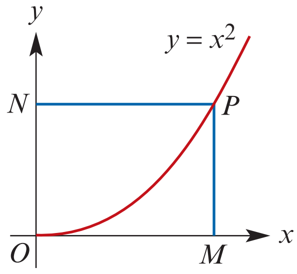
c the coordinates of the point B where the curve meets the x-axis

d the area of the region enclosed by the curve and the lines AT and BT

e the ratio of the area found in part **d** to the area of the triangle ATC.



2 **a** In the figure, the point P is on the curve $y = x^2$. Prove that the curve divides the rectangle $OMPN$ into two regions whose areas are in the ratio 2:1.



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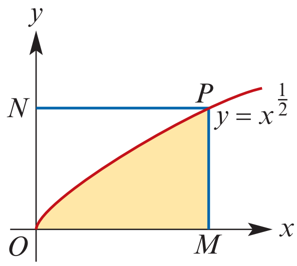
b In the figure, the point P is on the curve $y = x^{\frac{1}{2}}$. Prove that the area of the shaded region is two-thirds the area of the rectangle $OMPN$.



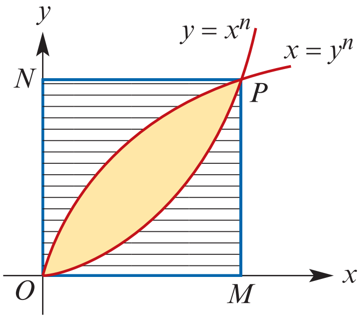
c Consider a point P on the curve $y = x^n$, with PM and PN the perpendiculars from P to the x-axis and the y-axis respectively. Prove that the area of the region enclosed between PM, the x-axis and the curve is equal to $\frac{1}{n+1}$ of the area of the rectangle $OMPN$.

3 **a** Find the area enclosed between the parabolas $y = x^2$ and $y^2 = x$.

b Show that the curves with equations $y = x^n$ and $y^n = x$ intersect at $(1, 1)$, where $n = 1, 2, 3, \ldots$.

c Show that the area of the region contained between the curves $y = x^n$ and $y^n = x$ is $\frac{n-1}{n+1}$.

d Find the area of the region indicated by horizontal shading in the diagram.



e Use your result from **c** to find the area of the region between the curves for $n = 10$, $n = 100$ and $n = 1000$.

f Describe the result for n very large.

4 It is thought that the temperature, θ, of a piece of charcoal in a barbecue will increase at a rate $\frac{d\theta}{dt}$ given by $\frac{d\theta}{dt} = e^{2.6t}$, where θ is in degrees and t is in minutes.

a If the charcoal starts at a temperature of 30°C, find the expected temperature of the charcoal after 3 minutes.

b Sketch the graph of θ against t.

c At what time does the temperature of the charcoal reach 500°C?

d Find the average rate of increase of temperature from $t = 1$ to $t = 2$.

5 It is believed that the velocity of a certain subatomic particle t seconds after a collision will be given by the expression

$$\frac{dx}{dt} = ve^{-t}, \quad v = 5 \times 10^4 \text{ m/s}$$

where x is the distance travelled in metres.

a What is the initial velocity of the particle?

b What happens to the velocity as $t \to \infty$ (i.e. as t becomes very large)?

c How far will the particle travel between $t = 0$ and $t = 20$?

d Find an expression for x in terms of t.

e Sketch the graph of x against t.

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6 **a** Differentiate $e^{-3x}\sin(2x)$ and $e^{-3x}\cos(2x)$ with respect to x.

b Hence show that

$$e^{-3x}\sin(2x) + c_1 = -3\int e^{-3x}\sin(2x)\,dx + 2\int e^{-3x}\cos(2x)\,dx$$

and $$e^{-3x}\cos(2x) + c_2 = -3\int e^{-3x}\cos(2x)\,dx - 2\int e^{-3x}\sin(2x)\,dx$$

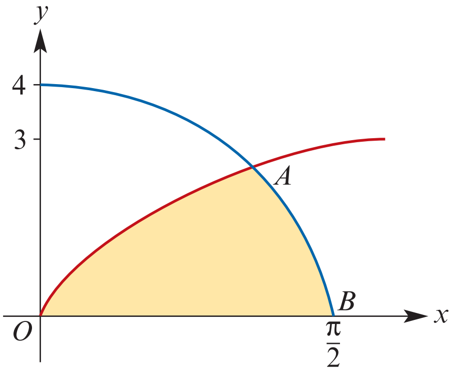
c Use the two equations from **b** to determine $\int e^{-3x}\sin(2x)\,dx$.

7 The curves $y = 3\sin x$ and $y = 4\cos x$, where $0 \le x \le \frac{\pi}{2}$, intersect at a point A.

a If $x = a$ at the point of intersection of the two curves:

- **i** Find $\tan a$.
- **ii** Hence find $\sin a$ and $\cos a$.

b Hence find the area of the shaded region in the diagram.



8 **a** If $y = x\log_e x$, find $\frac{dy}{dx}$. Hence find the value of $\int_1^e \log_e x\,dx$.

b If $y = x(\log_e x)^n$, where n is a positive integer, find $\frac{dy}{dx}$.

c Let $I_n = \int_1^e (\log_e x)^n\,dx$. For $n > 1$, show that $I_n + nI_{n-1} = e$.

d Hence find the value of $\int_1^e (\log_e x)^3\,dx$.

9 The curves $y^2 = ax$ and $x^2 = by$, where a and b are both positive, intersect at the origin and at the point (r, s). Find r and s in terms of a and b. Prove that the two curves divide the rectangle with corners $(0, 0)$, $(0, s)$, (r, s), $(r, 0)$ into three regions of equal area.

10 **a** Sketch the graph of $f(x) = 2\sin x - 1$ for $\frac{-\pi}{2} \le x \le \frac{\pi}{2}$.

b Evaluate $\int_0^{\frac{\pi}{6}} f(x)\,dx$ and indicate the area given by this integral on the graph of **a**.

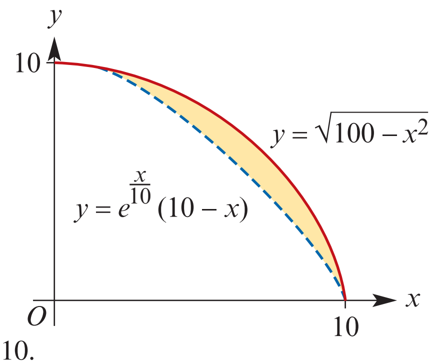
c Find the inverse function f^{-1}.

d Evaluate $\int_0^1 f^{-1}(x)\,dx$ and indicate the area given by this integral on the graph of **a**.

11 A teacher attempts to draw a quarter circle of radius 10 on the white board. However, the first attempt results in a curve with equation $y = e^{\frac{x}{10}}(10 - x)$.
The quarter circle has equation $y = \sqrt{100 - x^2}$.

a Find $\frac{dy}{dx}$ for both functions.

b Find the gradient of each of the functions when $x = 0$.



c Find the gradient of $y = e^{\frac{x}{10}}(10 - x)$ when $x = 10$.

d Find the area of the shaded region correct to two decimal places using a calculator.

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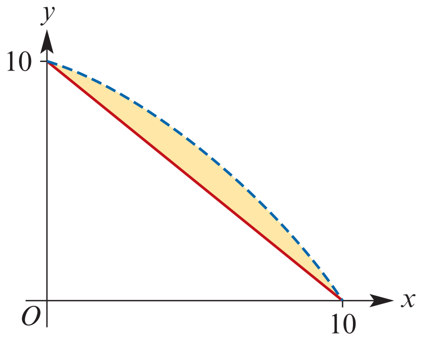
e Find the percentage error for the calculation of the area of the quarter circle.

f The teacher draws in a chord from (0, 10) to (10, 0). Find the area of the shaded region using a calculator.



g **i** Use the result that the derivative of $e^{\frac{x}{10}}(10 - x)$ is $-e^{\frac{x}{10}} + \frac{1}{10}e^{\frac{x}{10}}(10 - x)$ to find $\int_0^{10} e^{\frac{x}{10}}(10 - x)\,dx$ by analytic techniques.

ii Find the exact area of the original shaded region and compare it to the answer of **d**.

12 A water-cooling device has a system of water circulation for the first 30 minutes of its operation. The circulation follows the following sequence:

- For the first 3 minutes water is flowing in.
- For the second 3 minutes water is flowing out.
- For the third 3 minutes water is flowing in.

This pattern is continued for the first 30 minutes. The rate of flow of water is given by the function

$$R(t) = 10e^{\frac{-t}{10}} \sin\left(\frac{\pi t}{3}\right)$$

where $R(t)$ litres per minute is the rate of flow at time t minutes. Initially there are 4 litres of water in the device.

a **i** Find $R(0)$. **ii** Find $R(3)$.

b Find $R'(t)$.

c **i** Solve the equation $R'(t) = 0$ for $t \in [0, 12]$.

ii Find the coordinates of the stationary points of $y = R(t)$ for $t \in [0, 12]$.

d Solve the equation $R(t) = 0$ for $t \in [0, 12]$.

e Sketch the graph of $y = R(t)$ for $t \in [0, 12]$.

f **i** How many litres of water flowed into the device for $t \in [0, 3]$?

ii How many litres of water flowed out of the device for $t \in [3, 6]$?

iii How many litres of water are in the device when $t = 6$? (Remember there are initially 4 litres of water.)

g How many litres of water are there in the device when $t = 30$?

13 **a** Use the identities $\cos(2x) = 2\cos^2 x - 1$ and $\cos(2x) = 1 - 2\sin^2 x$ to show that

$$\frac{1 - \cos(2x)}{1 + \cos(2x)} = \sec^2 x - 1$$

b Hence evaluate $\int_0^{\frac{\pi}{4}} \frac{1 - \cos(2x)}{1 + \cos(2x)}\,dx$.

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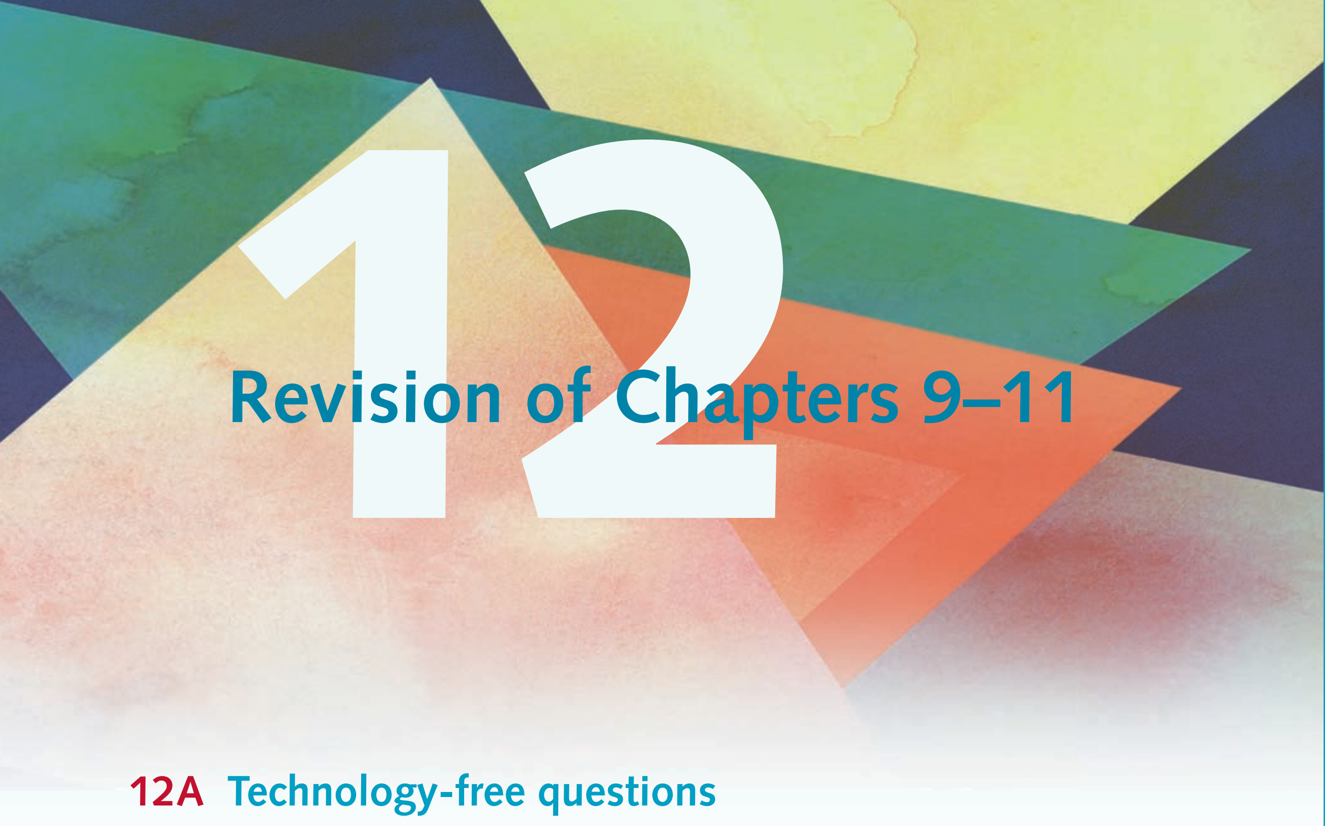

12

Revision of Chapters 9–11

12A Technology-free questions

1 Let $y = \dfrac{x^2 - 1}{x^4 - 1}$.

a Find $\dfrac{dy}{dx}$. **b** Find $\left\{ x : \dfrac{dy}{dx} = 0 \right\}$.

2 Let $y = (3x^2 - 4x)^4$. Find $\dfrac{dy}{dx}$.

3 Let $f : \mathbb{R}^+ \to \mathbb{R}$, $f(x) = x^2 \log_e(2x)$. Find $f'(x)$.

4 **a** Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{2x+1}$. The tangent to the graph of f at the point where $x = b$ passes through the point $(0, 0)$. Find b.

b Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{2x+1} + k$ where k is a real number. The tangent to the graph of f at the point where $x = b$ passes through the point $(0, 0)$. Find k in terms of b.

5 The line $y = mx - 8$ is tangent to the curve $y = x^{\frac{1}{3}} + c$ at the point $(8, a)$. Find the values of a, c and m.

6 Find the average value of the function with rule $f(x) = \dfrac{1}{3x + 1}$ over the interval $[0, 2]$.

7 Find an antiderivative of:

a $\dfrac{3}{5x - 2}$, $x > \dfrac{2}{5}$ **b** $\dfrac{3}{(5x - 2)^2}$, $x \neq \dfrac{2}{5}$

8 If $f(3) = -2$ and $f'(3) = 5$, find $g'(3)$ where:

a $g(x) = 3x^2 - 5f(x)$ **b** $g(x) = \dfrac{3x + 1}{f(x)}$ **c** $g(x) = [f(x)]^2$

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9 If $f(4) = 6$ and $f'(4) = 2$, find $g'(4)$ where:

a $g(x) = \sqrt{x}\, f(x)$ **b** $g(x) = \dfrac{f(x)}{x}$

10 Given that $f'(x) = \sqrt{3x+4}$ and $g(x) = x^2 - 1$, find $F'(x)$ if $F(x) = f(g(x))$.

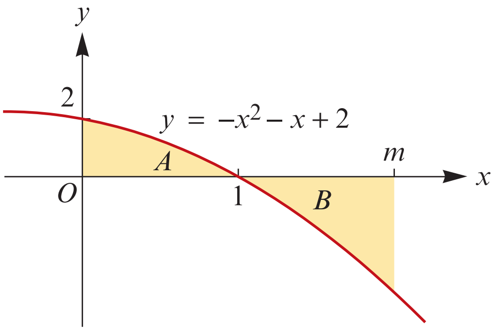
11 If $f(x) = 2x^2 - 3x + 5$, find:

a $f'(x)$ **b** $f'(0)$ **c** $\{x : f'(x) = 1\}$

12 Find the derivative of $\log_e(3f(x))$ with respect to x.

13 The tangent to the graph of $y = \sqrt{a - x}$ at $x = 1$ has a gradient of -6. Find the value of a.

14 The graph of $y = -x^2 - x + 2$ is shown. Find the value of m such that regions A and B have the same area.



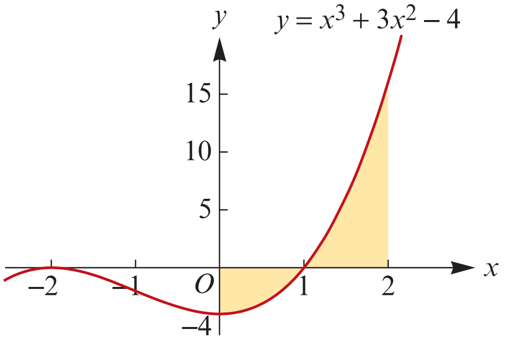
15 Let $f(x) = x^3 + 3x^2 - 4$. The graph of $y = f(x)$ is as shown. Find:

a the coordinates of the stationary points

b $\int_{-2}^{2} f(x)\, dx$

c $\int_{0}^{2} f(x)\, dx$

d the area of the shaded region.



16 If $f(x) = \dfrac{1}{3x-1}$, find $f'(2)$.

17 If $y = 1 - x^2$, prove that $x\dfrac{dy}{dx} + 2 = 2y$ for all values of x.

18 If $A = 4\pi r^2$, calculate $\dfrac{dA}{dr}$ when $r = 3$.

19 At what point on the graph of $y = 1.8x^2$ is the gradient 1?

20 If $y = 3x^2 - 4x + 7$, find the value of x such that $\dfrac{dy}{dx} = 0$.

21 If $y = \dfrac{x^2+2}{x^2-2}$, find $\dfrac{dy}{dx}$.

22 If $z = 3y + 4$ and $y = 2x - 1$, find $\dfrac{dz}{dx}$.

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23 If $y = (5 - 7x)^9$, calculate $\dfrac{dy}{dx}$.

24 If $y = 3x^{\frac{1}{3}}$, find $\dfrac{dy}{dx}$ when $x = 27$.

25 If $y = \sqrt{5 + x^2}$, find $\dfrac{dy}{dx}$ when $x = 2$.

26 Find $\dfrac{dy}{dx}$ when $x = 1$, given that $y = (x^2 + 3)(2 - 4x - 5x^2)$.

27 If $y = \dfrac{x}{1 + x^2}$, find $\dfrac{dy}{dx}$ when $x = 1$.

28 If $y = \dfrac{2 + x}{x^2 + x + 1}$, find $\dfrac{dy}{dx}$ when $x = 0$.

29 Let $f(x) = \dfrac{1}{2x + 1}$.

a Use the definition of derivative to find $f'(x)$.

b Find the gradient of the tangent to the graph of f at the point $(0, 1)$.

30 Let $f(x) = x^3 + 3x^2 - 1$. Find:

a $\{x : f'(x) = 0\}$ **b** $\{x : f'(x) > 0\}$ **c** $\{x : f'(x) < 0\}$

31 Let $y = \dfrac{x}{1 - x}$.

a Find $\dfrac{dy}{dx}$. **b** Write $\dfrac{dy}{dx}$ in terms of y.

32 If $y = (x^2 + 1)^{-\frac{3}{2}}$, find $\dfrac{dy}{dx}$.

33 If $y = x^4$, prove that $x\dfrac{dy}{dx} = 4y$.

34 Show that $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x^5$ is a strictly increasing function for $\mathbb{R}$ by showing that $f'(x) > 0$, for all non-zero x, and showing that, if $b > 0$, then $f(b) > f(0)$, and if $0 > b$, then $f(0) > f(b)$.

35 Evaluate each of the following integrals:

a $\displaystyle\int_0^{\frac{\pi}{2}} 2\sin\left(\frac{x}{2}\right) dx$ **b** $\displaystyle\int_0^{\frac{3}{2}} e^{\frac{x}{2}}\, dx$ **c** $\displaystyle\int_{\frac{1}{2}}^{1} \frac{1}{2x}\, dx$

d $\displaystyle\int_{-1}^{-\frac{1}{2}} \frac{1}{2x}\, dx$ **e** $\displaystyle\int_3^4 \frac{1}{2(x-2)^2}\, dx$ **f** $\displaystyle\int_2^4 \frac{1}{(3x-2)^2}\, dx$

36 Show that $f : \mathbb{R} \to \mathbb{R}$, $f(x) = -2x^3 + 1$ is a strictly decreasing function for $\mathbb{R}$.

37 Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{-mx+2} + 4x$ where m is a positive rational number.

a Find the x-coordinate of the stationary point of the graph of $y = f(x)$ in terms of m.

b Find the values of m for which the x-coordinate of this stationary point is negative.

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12B Multiple-choice questions

1 If $y = \dfrac{x^4 + x}{x^2}$, then $\dfrac{dy}{dx}$ equals

A $\dfrac{4x^3 + 2x}{2x}$ **B** $x^2 + 1$ **C** $2x$ **D** $\dfrac{2x^3 + 1}{x^2}$ **E** $2x - \dfrac{1}{x^2}$

2 If $f(x) = x^5 + x^3 + x$, the value of $f'(1)$ is

A 0 **B** 1 **C** 2 **D** 9 **E** -3

3 If $y = (4 - 9x^4)^{\frac{1}{2}}$, then $\dfrac{dy}{dx}$ equals

A $-\dfrac{9}{2}(4 - 9x^4)^{-\frac{1}{2}}$ **B** $\dfrac{1}{2}(4 - 9x^4)^{-\frac{1}{2}}$ **C** $2(4 - 9x^4)^{-\frac{1}{2}}$

D $-3x(4 - 9x^4)^{-\frac{1}{2}}$ **E** $-18x^3(4 - 9x^4)^{-\frac{1}{2}}$

4 The gradient of the curve with equation $y = \sin(2x) + 1$ at $(0, 1)$ is

A 1 **B** -1 **C** 0 **D** 2 **E** -2

5 $\dfrac{d}{dx}(e^{x^2+1})$ is

A $2x$ **B** $2xe^{x^2+1}$ **C** $2xe^{2x}$ **D** $(x^2 + 1)e^{x^2}$ **E** $(x^2 + 1)e^{x^2+1}$

6 The derivative of $\dfrac{1}{1 + x}$ is

A $\dfrac{1}{(1 + x)^2}$ **B** $\dfrac{1}{1 - x}$ **C** $\dfrac{-1}{(1 + x)^2}$ **D** 1 **E** $\dfrac{1}{2}$

7 Points P and Q lie on the curve $y = x^3$. The x-coordinates of P and Q are 2 and $2 + h$ respectively. The gradient of the secant PQ is

A $\dfrac{h^3 - 8}{h - 2}$ **B** $12 + 6h$ **C** 12 **D** $\dfrac{(2 + h)^3 - h^3}{h}$ **E** $12 + 6h + h^2$

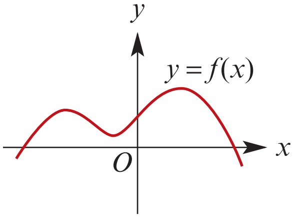
8 If $f(x) = \dfrac{3}{x}$, then $\dfrac{f(x + h) - f(x)}{h}$ is equal to

A $\dfrac{-3}{x(x + h)}$ **B** $\dfrac{3}{x^2}$ **C** $\dfrac{-3}{x^2}$ **D** $\dfrac{-3}{h(x + h)}$ **E** $f'(x)$

9 The gradient of $y = ce^{2x}$ is equal to 11 when $x = 0$. The value of c is

A 0 **B** 1 **C** 5 **D** 5.5 **E** $5e^{-2}$

10 For the graph of $y = f(x)$ shown, $f'(x) = 0$ at



A 3 points **B** 2 points

C 5 points **D** 0 points

E none of these

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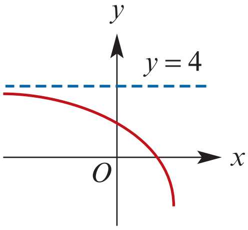
11 Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 4 - e^{-2x}$. The graph of $f'(x)$ is best represented by

A



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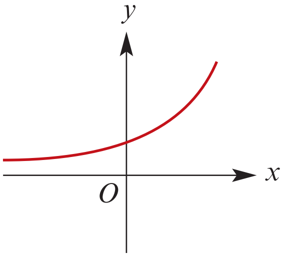

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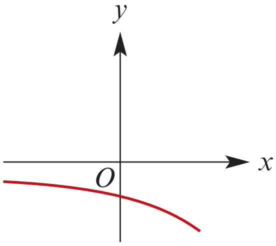

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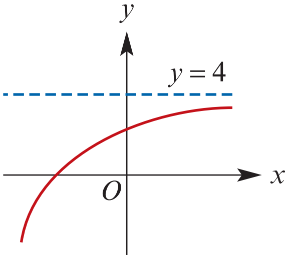

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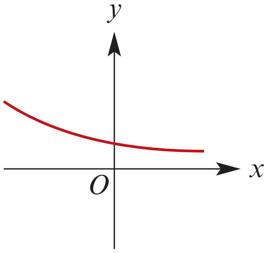
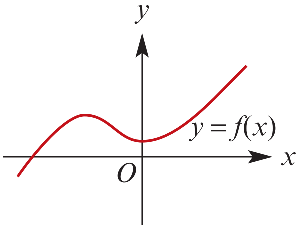


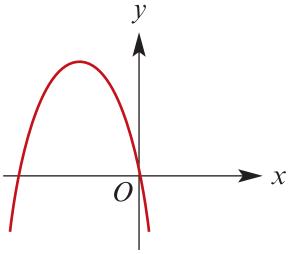
12 The graph of $y = bx^2 - cx$ crosses the x-axis at the point $(4, 0)$. The gradient at this point is 1. The value of c is

A 8 **B** 1 **C** 4 **D** −8 **E** 2

13 The graph of $y = f(x)$ is shown on the right. The graph that best represents the graph of $y = f'(x)$ is



A



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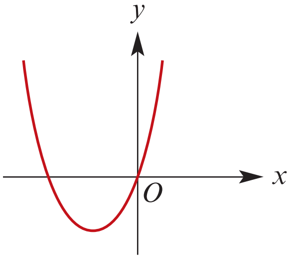

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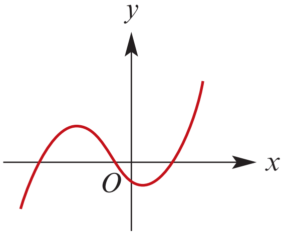

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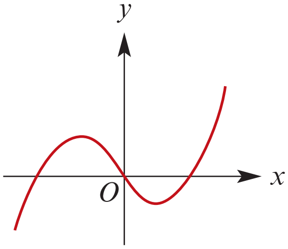

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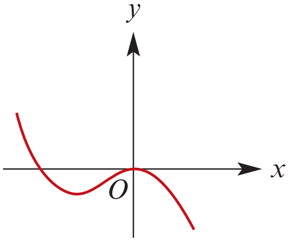

14 If $y = (3x^4 - 2)^4$, then $\dfrac{dy}{dx}$ equals

A $x^4(3x^4 - 2)^3$ **B** $4(3x^4 - 2)^3$ **C** $12x^{12}$

D $(12x^2 - 2)^4$ **E** $48x^3(3x^4 - 2)^3$

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15 Let $f(x) = 3x^2 + 2$. If $g'(x) = f'(x)$ and $g(2) = 29$, then $g(x) =$

A $3x^3 + 5$ **B** $3x^2 - 3$ **C** $\dfrac{x^3}{3} + 2x$ **D** $3x^2 + 17$ **E** $6x + 17$

16 If $f(x) = e^{kx} + e^{-kx}$, then $f'(x) > 0$ for

A $x \in \mathbb{R}$ **B** $x \geq 0$ **C** $x < 0$ **D** $x \leq 0$ **E** $x > 0$

17 If $y = 3x^2 + 2x - \dfrac{4}{x^2}$, then $\dfrac{dy}{dx}$ is equal to

A $6x + 2 - \dfrac{8}{x^3}$ **B** $6x + 2 + \dfrac{8}{x^3}$ **C** $6x + 2 + \dfrac{8}{x}$

D $x^3 + x^2 + \dfrac{4}{x}$ **E** $x^3 + x^2 - \dfrac{4}{x}$

18 Rainwater is being collected in a water tank. The volume, V m^3, of water in the tank after time t minutes is given by $V = 2t^2 + 3t + 1$. The average rate of change of volume of water between times $t = 2$ and $t = 4$, in m^3/min, is

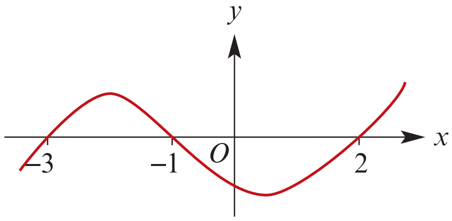
A 11 **B** 13 **C** 15 **D** 17 **E** 19

19 $P(x, f(x))$ and $Q(x + h, f(x + h))$ are two points on the graph of the function $f(x) = x^2 - 2x + 1$. The gradient of the line joining P and Q is given by

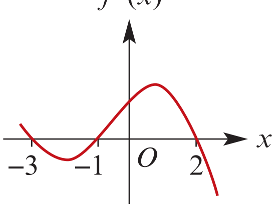
A $2x - 2$ **B** $2xh - 4x - 2h + 2$ **C** $2xh - 2h - h^2$

D $2xh - 2h + h^2$ **E** $2x - 2 + h$

20 The graph of $y = f(x)$ is shown.
A possible graph of the gradient function f' with rule given by $f'(x)$ is



A



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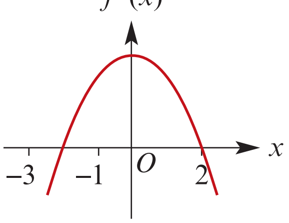

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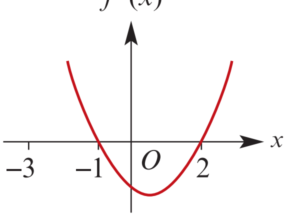

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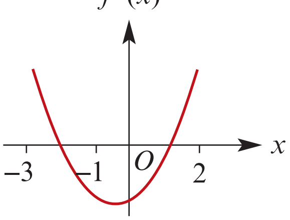
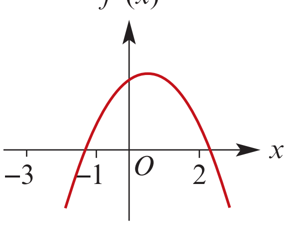
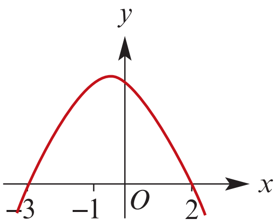


E



21 The graph of the derivative function f' given by $y = f'(x)$ is shown. The function f is increasing for



A $\{x : x \geq 0\}$
B $\{x : -3 \leq x \leq 2\}$
C $\{x : x \geq 2\}$
D $\{x : x \leq -3\} \cup \{x : x \geq 2\}$
E $\{x : x \leq 0\}$

22 Which one of the following gives the gradient of the tangent to a curve with the equation $y = f(x)$ at the point $x = 2$?

A $\dfrac{f(x+h) - f(x)}{h}$ **B** $f(2+h) - f(2)$ **C** $\dfrac{f(2+h) - f(2)}{h}$
D $\lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$ **E** $\lim\limits_{h\to 0} \dfrac{f(2+h) - f(2)}{h}$

23 The derivative of $\dfrac{e^{2x} + e^{-2x}}{e^x}$ is

A $e^x + e^{-3x}$ **B** $e^x - 3e^{-3x}$ **C** $xe^x - 3xe^{-3x}$ **D** $\dfrac{2e^{2x} - e^{-2x}}{e^x}$ **E** $\dfrac{e^{3x} - 3e^{-x}}{e^{x^2}}$

24 The maximum value of $-x^2 + 4x + 3$ is

A 2 **B** 3 **C** $2 + 2\sqrt{7}$ **D** 7 **E** 15

25 The equation of the tangent to the curve with equation $y = 1 + e^{2x}$ at the point $(0, 2)$ is

A $y = 2e^{2x}$ **B** $y = 2x + 2$ **C** $y = \dfrac{-1}{2}x + 2$ **D** $y = 2$ **E** $y = 2e^{2x} + 2$

26 The graph of the curve with equation $y = x^2 - x^3$ has stationary points where x is equal to

A 0 and $\dfrac{2}{3}$ **B** 0 and 1 **C** −1 and 0 **D** 0 and $\dfrac{3}{2}$ **E** 2 and −3

27 The derivative of $\dfrac{4x^2 + 6}{x}$ is

A $8x + 6$ **B** $8x^3 - 6$ **C** $\dfrac{4x^2 + 6}{x^2}$ **D** $\dfrac{4x^2 - 6}{x^2}$ **E** $\dfrac{8x^3 - 6}{x^2}$

28 If $f(x) = 4x^3 - 3x + 7 - \dfrac{2}{x}$, then $f'(1)$ is equal to

A −2 **B** 18 **C** 7 **D** 11 **E** 14

29 If $f'(x) = x^2 + \dfrac{1}{x}$ and $f(1) = \dfrac{1}{3}$, then $f(x)$ is equal to

A $\dfrac{x^3}{3} + \log_e x$ **B** $\dfrac{x^3}{3} + \log_e x + \dfrac{2}{3}$ **C** $\dfrac{x^3}{3} - \log_e x - \dfrac{1}{3}$
D $\dfrac{-x^3}{3} + \log_e x + \dfrac{2}{3}$ **E** $\dfrac{x^3}{3} - \log_e x + \dfrac{1}{3}$

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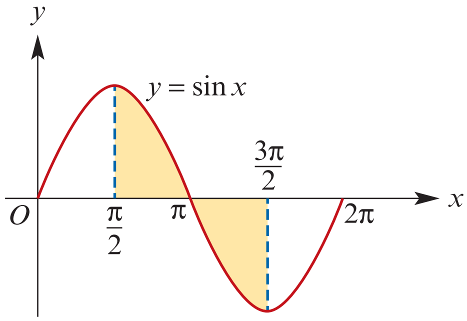
30 If $y = F(x)$ and $\dfrac{dy}{dx} = f(x)$, then $\int_2^3 f(x)\,dx$ is equal to

A $f(3) - f(2)$ **B** $F'(3) - F'(2)$ **C** $F(3) - F(2)$ **D** $f(x) + c$ **E** $F(3) - f(2)$

31 The area of the shaded region of the graph is given by



A $\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin x\,dx$

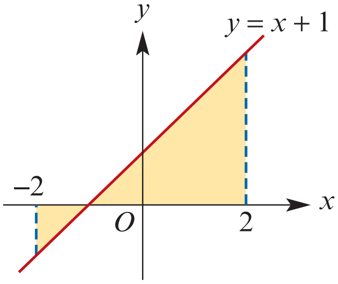
B $\int_{\pi}^{\frac{3\pi}{2}} \sin x\,dx + \int_{\frac{\pi}{2}}^{\pi} \sin x\,dx$

C $\int_{\frac{3\pi}{2}}^{\pi} \sin x\,dx + \int_{\frac{\pi}{2}}^{\pi} \sin x\,dx$

D $\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin x\,dx + \int_{\pi}^{\frac{\pi}{2}} \sin x\,dx$

E $\pi \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin^2 x\,dx$

32 The area of the shaded region of the graph is given by



A $\int_0^2 (x+1)\,dx - \int_2^0 (x+1)\,dx$

B $\int_{-2}^2 (x+1)\,dx$

C $\int_0^2 (x+1)\,dx + \int_{-2}^0 (x+1)\,dx$

D $\int_{-1}^2 (x+1)\,dx - \int_{-2}^{-1} (x+1)\,dx$

E $\int_{-1}^2 (x+1)\,dx + \int_{-2}^{-1} (x+1)\,dx$

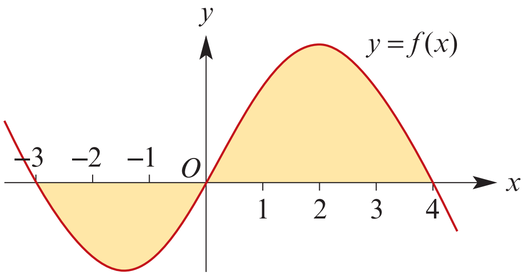
33 If $\dfrac{dy}{dx} = \dfrac{1}{x^2}$ and $y = 2$ when $x = 1$, then

A $y = \dfrac{-1}{x}$ **B** $y = \dfrac{-1}{x} + 3$ **C** $y = \dfrac{-2}{x^3}$ **D** $y = \dfrac{2}{x^3}$ **E** $y = \dfrac{1}{x} + 1$

34 If $\int_0^{36} \dfrac{1}{2x+9}\,dx = \log_e k$, then k is

A 3 **B** $\dfrac{9}{2}$ **C** $6\sqrt{2}$ **D** 9 **E** 81

35 The area of the shaded region of the graph is given by



A $\int_{-3}^4 f(x)\,dx$

B $\int_{-3}^0 f(x)\,dx + \int_0^4 f(x)\,dx$

C $\int_{-3}^1 f(x)\,dx + \int_1^4 f(x)\,dx$

D $\int_4^0 f(x)\,dx + \int_{-3}^0 f(x)\,dx$

E none of these

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36 $\int x^2 - \frac{1}{x^2} + \sin x \, dx$ is

A $\frac{x^3}{3} + \frac{1}{x} + \cos x + c$ **B** $\frac{x^3}{3} - \frac{2}{x} - \cos x + c$ **C** $2x - \frac{3}{x^2} + \cos x + c$

D $\frac{x^3}{3} + \frac{1}{x} - \cos x + c$ **E** $2x + \frac{2}{x^2} + \cos x + c$

37 The area bounded by the curve $y = \frac{1}{3 - x}$, the x-axis, the y-axis and the line $x = 2$ is

A $\log_e 3$ **B** $\log_e\left(\frac{1}{3}\right)$ **C** $-\log_e(3 - x)$ **D** $\log_e 2$ **E** $\log_e\left(\frac{1}{2}\right)$

38 If $\int_a^b \sin(2x) \, dx = 0$, then

A $b = \frac{3\pi}{4},\ a = \frac{\pi}{4}$ **B** $b = \frac{\pi}{2},\ a = 0$ **C** $b = \pi,\ a = \frac{\pi}{2}$

D $b = \frac{\pi}{6},\ a = \frac{\pi}{3}$ **E** $b = \pi,\ a = \frac{\pi}{4}$

39 The area of the shaded region of the graph is given by

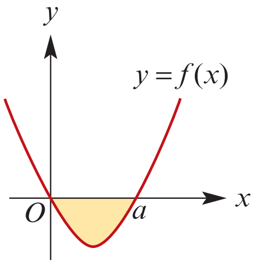
A $\int_0^a -f(x) \, dx$

B $\int_0^a f(x) \, dx$

C $\int_0^a x - f(x) \, dx$

D $\int_0^a f(x) - x \, dx$

E none of these



40 An antiderivative of $x^2 - \frac{1}{x}$ is

A $2x - \frac{2}{x^2}$ **B** $\frac{x^3}{3} - \log_e x$ **C** $x^3 + \frac{1}{x^2}$ **D** $x^3 - \log_e x$ **E** none of these

41 The function $f(x) = x^3 + 3x^2 - 9x + 7$ is increasing only when

A $x > 0$ **B** $-3 < x < 1$ **C** $x < -1$ or $x > 3$

D $x < -3$ or $x > 1$ **E** $-1 < x < 3$

42 For $f(x) = \frac{\sin x}{x}$, $f'(x) =$

A $\cos x$ **B** $\frac{x \cos x - \sin x}{x^2}$ **C** $\frac{x \cos x - \sin x}{x}$

D $\frac{\sin x - x \cos x}{x^2}$ **E** $\frac{x \cos x - \sin x}{\sin^2 x}$

43 If $y = \log_e(\cos(2x))$ for $0 < x < \frac{\pi}{4}$, then $\frac{dy}{dx}$ is equal to

A $\frac{2}{\cos(2x)}$ **B** $-\frac{2\sin(2x)}{\cos(2x)}$ **C** $\frac{1}{x}\cos(2x) - 2\sin(2x)\log_e x$

D $\frac{\sin(2x)}{2\cos(2x)}$ **E** $\frac{1}{\cos(2x)}$

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44 If $f'(x) = \sin(2x)$ and $f(0) = 3$, then

A $f(x) = -\frac{1}{2}\cos(2x) + 3$ **B** $f(x) = \frac{1}{2}\cos(2x) + 3$ **C** $f(x) = -\frac{1}{2}\cos(2x) + 3\frac{1}{2}$
D $f(x) = -\frac{1}{2}\cos(2x) + 2\frac{1}{2}$ **E** $f(x) = \frac{1}{2}\cos(2x) + 2\frac{1}{2}$

45 The equation of the tangent to the curve with equation $y = 4e^{3x} - x$ at the point $(0, 4)$ is

A $y = 12x + 4$ **B** $y = -4x + 4$ **C** $y = 4$ **D** $y = 11x + 4$ **E** $y = 4x + 4$

46 The function $f(x) = x^3 - x^2 - x + 2$ has a local minimum at the point

A $(-1, 0)$ **B** $(1, 1)$ **C** $(2, 0)$ **D** $(-1, 1)$ **E** $(1, 0)$

47 $\frac{d}{dx}\left(\frac{x-1}{\sqrt{x}}\right)$ equals

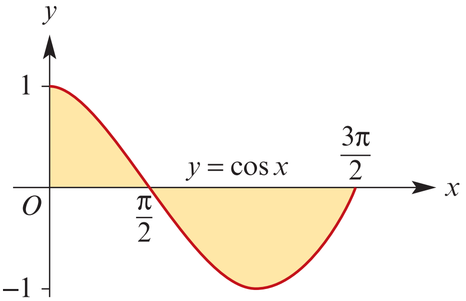
A $2\sqrt{x}$ **B** $\frac{x+1}{x\sqrt{x}}$ **C** $\frac{3x-1}{2\sqrt{x}}$ **D** $\frac{x+1}{2x\sqrt{x}}$ **E** $\frac{3x-1}{2x\sqrt{x}}$

48 $\frac{d}{dx}(e^{\cos x}) =$

A $e^{\cos x}$ **B** $e^{\cos x} \cdot \sin x$ **C** $-e^{\cos x} \cdot \sin x$ **D** $e^{\sin x}$ **E** $e^{\sin x} \cdot \cos x$

49 The total area, in square units, of the shaded regions is

A 3 **B** −1 **C** 1 **D** 2 **E** −2



50 Given that $x + y = 1$, the maximum value of $P = x^2 + xy - y^2$ occurs for x equal to

A 2 **B** −1 **C** $\frac{3}{2}$ **D** 1 **E** $\frac{2}{3}$

51 The gradient of the normal to the curve of $y = e^{-\cos x}$ at the point where $x = \frac{\pi}{3}$ is

A $\frac{\sqrt{3}}{2e^{\frac{1}{2}}}$ **B** $\frac{-2e^{\frac{1}{2}}}{\sqrt{3}}$ **C** $\frac{1}{2e^{\frac{\sqrt{3}}{2}}}$ **D** $\frac{2e^{\frac{1}{2}}}{\sqrt{3}}$ **E** $-e^{-\frac{1}{2}}$

52 $\int_0^{\frac{\pi}{2}} (\cos x + \sin x)\, dx$ equals

A −2 **B** −1 **C** 1 **D** $\frac{\pi}{2}$ **E** 2

53 Let f be differentiable for all values of x in $[0, 2]$. The graph with equation $y = f(x)$ has a maximum point at $(1, 3)$. The equation of the tangent at $(1, 3)$ is

A $x + 3y = 0$ **B** $x = 1$ **C** $y = 3$ **D** $x - 3y = 0$ **E** $3x + y = 0$

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54 The equation of the tangent to the curve with equation $y = 4 - x^2$ at the point $(1, 3)$ is

A $y = 3x$
B $y = -x + 4$
C $y = -2x + 5$
D $y = x + 2$
E $y = 2x + 1$

55 The maximum value of $P = -x^2 + 6x + 4$ is

A 3
B $-6 + 2\sqrt{5}$
C 4
D 13
E 24

56 The graph of the function whose rule is $f(x) = x^3 - x^2 - 1$ has stationary points when x equals

A $\frac{2}{3}$ only
B 0 and $\frac{2}{3}$
C 0 and $-\frac{2}{3}$
D $-\frac{1}{3}$ and 1
E $\frac{1}{3}$ and -1

57 If $\dfrac{f(2 + h) - f(2)}{h} = h^2 + 6h + 12$, then $f'(2)$ equals

A 4
B 6
C 10
D 12
E 28

58 If $f(x) = \log_e(3x)$, then $f'(1)$ is

A $\frac{1}{3}$
B $\log_e 3$
C 1
D $3 \log_e 3$
E 3

59 If $y = x^2 e^x$, then the minimum value of y is

A -2
B 0
C $4e^{-2}$
D $-4e^{-2}$
E e

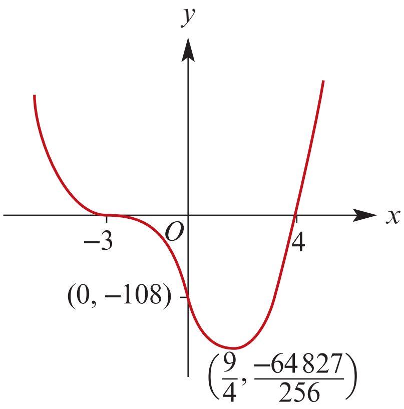
60 If $f(x) = a \sin(3x)$ where a is constant and $f'(\pi) = 2$, then a is equal to

A -3
B $-\frac{3}{2}$
C $\frac{3}{2}$
D $\frac{2}{3}$
E $-\frac{2}{3}$

61 An antiderivative of $\dfrac{1}{(2x - 5)^{\frac{5}{2}}}$ is equal to

A $\dfrac{-3}{(2x - 5)^{\frac{5}{2}}}$
B $\dfrac{-1}{3(2x - 5)^{\frac{3}{2}}}$
C $\dfrac{5}{(2x - 5)^{\frac{5}{2}}}$
D $\dfrac{7}{2(2x - 5)^{\frac{7}{2}}}$
E $\dfrac{1}{3(2x - 5)^{\frac{3}{2}}}$

62 The graph shown is of a function with rule $y = (x + 3)^3(x - 4)$. Which of the following is *not* true?



A $\dfrac{dy}{dx} = 0$ when $x = \dfrac{9}{4}$ and $x = -3$ and at no other point.
B There is only one turning point on the graph.
C The x-axis is a tangent to the graph where $x = -3$.
D There is only one stationary point on the graph.
E $y \geq \dfrac{-64\,827}{256}$ for all values of x.

12C Extended-response questions

1 The amount of salt (s grams) in 100 litres of salt solution at time t minutes is given by $s = 50 + 30e^{-\frac{1}{5}t}$.

a Find the amount of salt in the mixture after 10 minutes.

b Sketch the graph of s against t for $t \geq 0$.

c Find the rate of change of the amount of salt at time t (in terms of t).

d Find the rate of change of the amount of salt at time t (in terms of s).

e Find the concentration (grams per litre) of salt at time $t = 0$.

f Find the value of t for which the salt solution first reaches a concentration of 0.51 grams per litre.

2 A medium is kept at a constant temperature of 20°C. An object is placed in this medium. The temperature, T°C, of the object at time t minutes is given by

$$T = 40e^{-0.36t} + 20, \quad t \geq 0$$

a Find the initial temperature of the object.

b Sketch the graph of T against t for $t \geq 0$.

c Find the rate of change of temperature with respect to time (in terms of t).

d Find the rate of change of temperature with respect to time (in terms of T).

3 A certain food is susceptible to contamination from bacterial spores of two types, F and G. In order to kill the spores, the food is heated to a temperature of 120°C. The number of live spores after t minutes can be approximated by $f(t) = 1000e^{-0.5t}$ for F-type spores and by $g(t) = 1200e^{-0.7t}$ for G-type spores.

a Find the time required to kill 50% of the F-type spores.

b Find the total number of live spores of both types when $t = 0$, and find the percentage of these that are still alive when $t = 5$.

c Find the rate at which the total number of live spores is decreasing when $t = 5$.

d Find the value of t for which the number of live F-type spores and the number of live G-type spores are equal.

e On the same set of axes, sketch the graphs of $y = f(t)$ and $y = g(t)$ for $t \geq 0$.

4 An object falls from rest in a medium and its velocity, V m/s, after t seconds is given by $V = 100(1 - e^{-0.2t})$.

a Sketch the graph of V against t for $t \geq 0$.

b Express the acceleration at any instant:

i in terms of t **ii** in terms of V.

c Find the value of t for which the velocity of the object is 80 m/s.

5 A manufacturer determines that the total cost, \$$C$ per year, of producing a product is given by $C = 0.05x^2 + 5x + 500$, where x is the number of units produced per year. At what level of output will the average cost per unit be a minimum? (Use a continuous function to model this discrete situation.)

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6 An object that is at a higher temperature than its surroundings cools according to Newton's law of cooling: $T = T_0\, e^{-kt}$, where T_0 is the original excess of temperature and T is the excess of temperature after time t minutes.

a Prove that $\dfrac{dT}{dt}$ is proportional to T.

b If the original temperature of the object is 100°C, the temperature of its surroundings is 30°C and the object cools to 70°C in 20 minutes, find the value of k correct to three decimal places.

c At what rate is the temperature decreasing after 30 minutes?

7 Suppose that the spread of a cold virus through a population is such that the proportion, $p(t)$, of the population which has had the virus up to time t days after its introduction into the population is given by

$$p(t) = 0.2 - 0.2e^{\frac{-t}{20}} + 0.1e^{\frac{-t}{10}}, \quad \text{for } t \geq 0$$

a **i** Find, correct to four decimal places, the proportion of the population which has had the virus up to 10 days after its introduction.

ii Find the proportion of the population that eventually catches the virus.

b The number of new cases on day t is proportional to $p'(t)$. Find how long after the introduction of the virus the number of new cases per day is at a maximum.

8 A real-estate firm owns the Shantytown Apartments, consisting of 70 garden-type apartments. The firm can find a tenant for all the apartments at \$500 each per month. However, for every \$20 per month increase, there will be two vacancies with no possibility of filling them. What price per apartment will maximise monthly revenue? (Use a continuous function to model this discrete situation.)

9 The amount of liquid, V m^3, in a large pool at time t days is given by $V = \dfrac{5 \times 10^4}{(t+1)^2}$ for $t \geq 0$.

a Find the initial volume of the pool.

b Find the rate of change of volume with respect to time when $t = 1$.

c Find the average rate of change for the interval $t = 1$ to $t = 4$.

d When is the amount of water in the pool less than 1 cubic metre?

e Sketch the graph of V against t for $t \geq 0$.

10 Each week a factory produced N thousand bottle tops and the cost of production is reckoned to be \$1000C, where $C = (N^3 + 16)^{\frac{1}{4}}$.

a Sketch the graph of C against N. (Use a continuous model.)

b Calculate $\dfrac{dC}{dN}$.

c What does $\dfrac{dC}{dN}$ represent?

11 A company produces items at a cost price of \$2 per item. Market research indicates that the likely number of items sold per month will be $\dfrac{800}{p^2}$, where p dollars is the selling price of each item. Find the value of p for which the company would expect to maximise its total monthly profit, and the corresponding number of items sold.

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12 A curve with equation $y = (ax + b)^{-2}$ has y-axis intercept $(0, \frac{1}{4})$ and at this point the gradient is $-\frac{3}{4}$. Find the value(s) of a and b and sketch the graph.

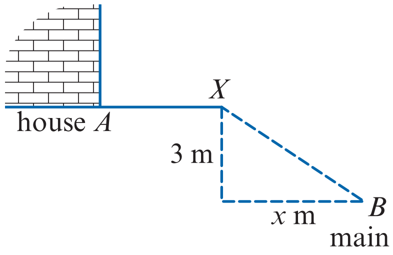
13 The cost of running a ship at a constant speed of V km/h is $160 + \dfrac{1}{100}V^3$ dollars per hour.

- **a** Find the cost of a journey of 1000 km at a speed of 10 km/h.
- **b** Find the cost, \$$C$, of a journey of 1000 km at a speed of V km/h.
- **c** Sketch the graph of C against V.
- **d** Find the most economical speed for the journey, and the minimum cost.
- **e** If the ship has a maximum speed of 16 km/h, find the minimum cost.

14 **a** A camper is on an island shore at point A, which is 12 km from the nearest point B on the straight shore of the mainland. He wishes to reach a town C, which is 30 km along the shore from B, in the least possible time. If he can row his boat at 5 km/h and walk at 8 km/h, how far along the shore from B towards C should he land?

- **b** Repeat **a** if C is only 24 km from B.

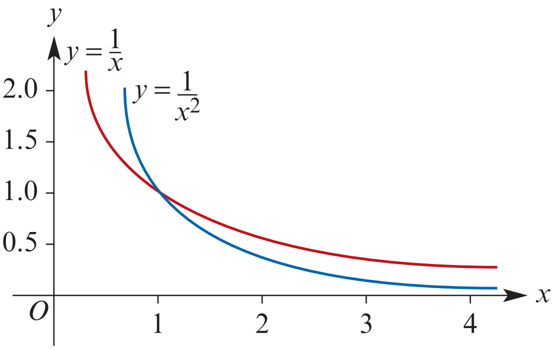
15 To connect a house to a gas supply, a pipe must be installed connecting the point A on the house to the point B on the main, where B is 3 m below ground level and at a horizontal distance of 4 m from the building. If it costs \$25 per metre to lay pipe underground and \$10 per metre on the surface, find the length of pipe which should be on the surface to minimise costs.



16 Define the functions

$$g\colon \mathbb{R}^+ \to \mathbb{R},\ g(x) = \frac{1}{x}$$

and $$h\colon \mathbb{R}^+ \to \mathbb{R},\ h(x) = \frac{1}{x^2}$$



- **a** Find $\{ x : g(x) > h(x) \}$.
- **b** Find $\{ x : g'(x) > h'(x) \}$, i.e. find the set of x for which the gradient of g is greater than the gradient of h.
- **c** On one set of axes, sketch the graphs of

  $$f\colon \mathbb{R}^+ \to \mathbb{R},\ f(x) = \frac{1}{x^3} \quad \text{and} \quad h\colon \mathbb{R}^+ \to \mathbb{R},\ h(x) = \frac{1}{x^2}$$

  Find $\{ x : h(x) > f(x) \}$ and $\{ x : h'(x) > f'(x) \}$.
- **d** For $f_1\colon \mathbb{R}^+ \to \mathbb{R}$, $f_1(x) = \dfrac{1}{x^n}$ and $f_2\colon \mathbb{R}^+ \to \mathbb{R}$, $f_2(x) = \dfrac{1}{x^{n+1}}$, find $\{ x : f_1(x) > f_2(x) \}$ and $\{ x : f_1'(x) > f_2'(x) \}$.

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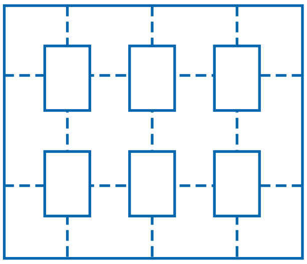
17 **a** Find the points $P\left(x, \frac{1}{x}\right)$ on the curve $y = \frac{1}{x}$ for which the distance OP is a minimum, where O is the origin $(0, 0)$.

b Find the points $P\left(x, \frac{1}{x^2}\right)$ on the curve $y = \frac{1}{x^2}$ for which the distance OP is a minimum.

c Find the points $P\left(x, \frac{1}{x^n}\right)$ on the curve $y = \frac{1}{x^n}$ for which the distance OP is a minimum, where n is a positive integer.

18 The figure represents an intended basic design for a workshop wall which is to have six equal windows spaced so that each dashed line has length 2 m. The total area of window space is to be 36 m^2.



a Express the total area, A m^2, of brickwork as a function of the window height, x m.

b Sketch the graph of A against x.

c Find the dimensions of each window which will give a minimum amount of brickwork.

d If building regulations require that both the height and the width of a window must not be less than 1 m, find the maximum amount of brickwork that could be used.

19 **a** Sketch the graph of the equation $y = x^2 - a^2$. Label the points A, B at which it cuts the x-axis. Write down the coordinates of A and B.

b Find the area of the region between the x-axis and the graph.

c Draw a rectangle $ABCD$ on your sketch, lying *below* the x-axis, with area equal to the area found in part **b**. What is the length of the side BC?

d If the vertex of the parabola is at point V, calculate the ratio $\dfrac{\text{length of } BC}{\text{length of } OV}$.

20 **a** Calculate $\int_{-3}^{1} (1 - t^2)\, dt$ and illustrate the region of the Cartesian plane for which this integral gives the signed area.

b Show that $\int_{a}^{1} (1 - t^2)\, dt = 0$ implies $a^3 - 3a + 2 = 0$.

c Find the values of a for which $\int_{a}^{1} (1 - t^2)\, dt = 0$.

21 The rate of flow of water into a tank is given by $\dfrac{dV}{dt} = 10e^{-(t+1)}(5 - t)$ for $0 \leq t \leq 5$, where V litres is the amount of water in the tank at time t minutes. Initially the tank is empty.

a **i** Find the initial rate of flow of water into the tank.

ii Find the value of t for which $\dfrac{dV}{dt} = 0$.

iii Find the time, to the nearest second, when the rate is 1 litre per minute.

iv Find the first time, to the nearest second, when $\dfrac{dV}{dt} < 0.1$.

b Find the amount of water in the tank when $t = 5$.

c Find the time, to the nearest second, when there are 10 litres of water in the tank.

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13

Discrete random variables and their probability distributions

Objectives

- To review the basic concepts of **probability**.
- To define **discrete random variables**.
- To define the **probability distribution** of a discrete random variable.
- To calculate and interpret **expected value** (**mean**) for a discrete random variable.
- To calculate and interpret **variance** and **standard deviation** for a discrete random variable.
- To illustrate the property that for many random variables **approximately 95% of the distribution is within two standard deviations of the mean**.

Uncertainty is involved in much of the reasoning we undertake every day of our lives. We are often required to make decisions based on the chance of a particular occurrence. Some events can be predicted from our present store of knowledge, such as the time of the next high tide. Others, such as whether a head or tail will show when a coin is tossed, are not predictable.

Ideas of uncertainty are pervasive in everyday life, and the use of chance and risk models makes an important impact on many human activities and concerns. Probability is the study of chance and uncertainty.

In this chapter we will extend our knowledge of probability by introducing the concept of the probability distribution (also known as the probability mass function) for a discrete random variable. Using this distribution we can determine the theoretical values of two important parameters which describe the random variable: the mean and the standard deviation. We will see that together the mean and the standard deviation tell us a lot about the distribution of the variable under consideration.

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13A Sample spaces and probability

In this section we will review the fundamental concepts of probability, the numerical value which we assign to give a measure of the likelihood of an outcome of an experiment. Probability takes a value between 0 and 1, where a probability of 0 means that the outcome is impossible, and a probability of 1 means that it is certain. Generally, the probability of an outcome will be somewhere in between, with a higher value meaning that the outcome is more likely.

▶ Sample spaces and events

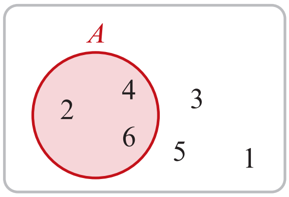
When a six-sided die is rolled, the possible outcomes are the numbers 1, 2, 3, 4, 5, 6. Rolling a six-sided die is an example of a **random experiment**, since while we can list all the possible outcomes, we do not know which one will be observed.

The possible outcomes are generally listed as the elements of a set, and the set of all possible outcomes is called the **sample space** and denoted by the Greek letter ε (epsilon). Thus, for this example:

$$\varepsilon = \{1, 2, 3, 4, 5, 6\}$$

An **event** is a subset of the sample space, usually denoted by a capital letter. If the event A is defined as 'an even number when a six-sided die is rolled', we write

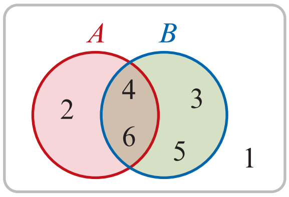
$$A = \{2, 4, 6\}$$



If A and B are two events, then the **union** of A and B, denoted by $A \cup B$, is equivalent to either event A or event B or both occurring.

Thus, if event A is 'an even number when a six-sided die is rolled' and event B is 'a number greater than 2 when a six-sided die is rolled', then $A = \{2, 4, 6\}$, $B = \{3, 4, 5, 6\}$ and

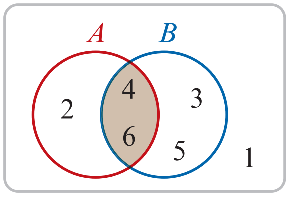
$$A \cup B = \{2, 3, 4, 5, 6\}$$



The **intersection** of A and B, denoted by $A \cap B$, is equivalent to both event A and event B occurring.

Thus, using the events A and B already described:

$$A \cap B = \{4, 6\}$$



In some experiments, it is helpful to list the elements of the sample space systematically by means of a tree diagram.

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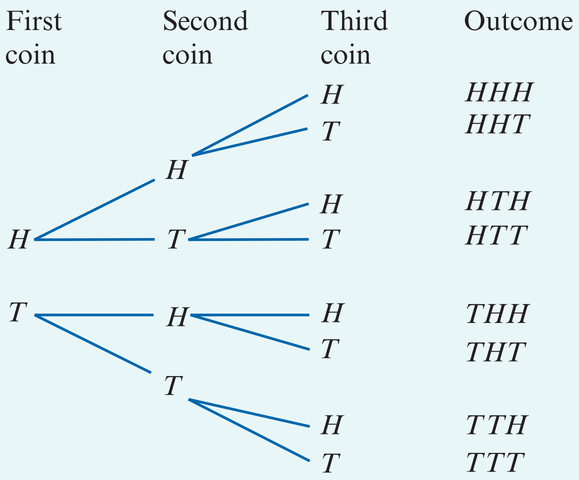
Example 1

Find the sample space when three coins are tossed and the results noted.

Solution

To list the elements of the sample space, construct a tree diagram:



Each path along the branches of the tree identifies an outcome, giving the sample space as

$$\varepsilon = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

Determining probabilities for equally likely outcomes

Probability is a numerical measure of the chance of a particular event occurring. There are many approaches to determining probability, but often we assume that all of the possible outcomes are equally likely.

We require that the probabilities of all the outcomes in the sample space sum to 1, and that the probability of each outcome is a non-negative number. This means that the probability of each outcome must lie in the interval $[0, 1]$. Since six outcomes are possible when rolling a die, we can assign the probability of each outcome to be $\frac{1}{6}$. That is,

$$\Pr(1) = \Pr(2) = \Pr(3) = \Pr(4) = \Pr(5) = \Pr(6) = \frac{1}{6}$$

When the sample space is finite, the **probability of an event** is equal to the sum of the probabilities of the outcomes in that event.

For example, let A be the event that an even number is rolled on the die. Then $A = \{2, 4, 6\}$ and $\Pr(A) = \Pr(2) + \Pr(4) + \Pr(6) = \frac{1}{2}$. Since the outcomes are equally likely, we can calculate this more easily as

$$\Pr(A) = \frac{\text{number of outcomes in } A}{\text{total number of outcomes}} = \frac{3}{6} = \frac{1}{2}$$

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Equally likely outcomes

In general, if the sample space ε for an experiment contains n outcomes, all of which are equally likely to occur, we assign a probability of $\frac{1}{n}$ to each of these outcomes.

Then the probability of any event A which contains m of these outcomes is the ratio of the number of elements in A to the number of elements in ε. That is,

$$\Pr(A) = \frac{n(A)}{n(\varepsilon)} = \frac{m}{n}$$

where the notation $n(S)$ is used to represent the number of elements in set S.

We will see that there are other methods of determining probabilities. But whichever method is used, the following rules of probability will hold:

- $\Pr(A) \geq 0$ for all events $A \subseteq \varepsilon$
- $\Pr(\varepsilon) = 1$
- The sum of the probabilities of all outcomes of an experiment is 1.
- $\Pr(\varnothing) = 0$, where $\varnothing$ represents the empty set
- $\Pr(A') = 1 - \Pr(A)$, where A' is the complement of A
- $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$, the **addition rule**

When two events A and B have no outcomes in common, i.e. when they cannot occur together, they are called **mutually exclusive** events. In this case, we have $\Pr(A \cap B) = 0$ and so the addition rule becomes:

$$\Pr(A \cup B) = \Pr(A) + \Pr(B), \quad \text{the addition rule when } A \text{ and } B \text{ are mutually exclusive}$$

We illustrate some of these rules in the following example.

Example 2

If one card is chosen at random from a well-shuffled deck of 52 cards, what is the probability that the card is:

a an ace **b** not a heart **c** an ace or a heart **d** either a king or an ace?

Solution

a Let A be the event 'the card drawn is an ace'. A standard deck of cards contains four aces, so

$$\Pr(A) = \frac{4}{52} = \frac{1}{13}$$

b Let H be the event 'the card drawn is a heart'. There are 13 cards in each suit, so

$$\Pr(H) = \frac{13}{52} = \frac{1}{4}$$

and therefore

$$\Pr(H') = 1 - \Pr(H) = 1 - \frac{1}{4} = \frac{3}{4}$$

c Using the addition rule:

$$\Pr(A \cup H) = \Pr(A) + \Pr(H) - \Pr(A \cap H)$$

Now $\Pr(A \cap H) = \frac{1}{52}$, since the event $A \cap H$ corresponds to drawing the ace of hearts. Therefore

$$\Pr(A \cup H) = \frac{1}{13} + \frac{1}{4} - \frac{1}{52} = \frac{4}{13}$$

d Let K be the event 'the card drawn is a king'. We observe that $K \cap A = \varnothing$. That is, the events K and A are mutually exclusive. Hence

$$\Pr(K \cup A) = \Pr(K) + \Pr(A) = \frac{1}{13} + \frac{1}{13} = \frac{2}{13}$$

Example 3

500 people were questioned and classified according to age and whether or not they regularly use social media. The results are shown in the table.

Do you regularly use social media?

	Age < 25	Age ≥ 25	Total
Yes	200	100	300
No	40	160	200
Total	240	260	500

One person is selected from these 500. Find the probability that:

a the person regularly uses social media

b the person is less than 25 years of age

c the person is less than 25 years of age and does not regularly use social media.

Solution	*Explanation*
a $\Pr(\text{Yes}) = \frac{300}{500} = \frac{3}{5}$	There are 300 out of 500 people who say yes.
b $\Pr(\text{Age} < 25) = \frac{240}{500} = \frac{12}{25}$	There are 240 out of 500 people who are less than 25 years of age.
c $\Pr(\text{No} \cap \text{Age} < 25) = \frac{40}{500} = \frac{2}{25}$	There are 40 out of 500 people who are less than 25 years of age and say no.

▶ Other methods of determining probabilities

When we are dealing with a random experiment which does not have equally likely outcomes, other methods of determining probability are required.

Subjective probabilities

Sometimes, the probability is assigned a value on the basis of judgement. For example, a farmer may look at the weather conditions and determine that there is a 70% chance of rain that day, and take appropriate actions. Such probabilities are called **subjective probabilities**.

Probabilities from data

A better way to estimate an unknown probability is by experimentation: by performing the random experiment many times and recording the results. This information can then be used to estimate the chances of the event happening again in the future. The proportion of trials that resulted in this event is called the **relative frequency** of the event. (For most purposes we can consider proportion and relative frequency as interchangeable.) That is,

$$\text{Relative frequency of event } A = \frac{\text{number of times event } A \text{ occurs}}{\text{number of trials}}$$

This information can then be used to estimate the probability of the event.

When the number of trials is sufficiently large, the observed relative frequency of an event A becomes close to the probability $\Pr(A)$. That is,

$$\Pr(A) \approx \frac{\text{number of times event } A \text{ occurs}}{\text{number of trials}} \qquad \text{for a large number of trials}$$

If the experiment was repeated, it would generally be found that the results were slightly different. One might conclude that relative frequency is not a very good way of estimating probability. In many situations, however, experiments are the only way to get at an unknown probability. One of the most valuable lessons to be learnt is that such estimates are not exact, and will in fact vary from sample to sample.

Understanding the variation between estimates is extremely important in the study of statistics, and this is the topic of Chapter 17. At this stage it is valuable to realise that the variation does exist, and that the best estimates of the probabilities will result from using as many trials as possible.

Example 4

Suppose that a die is tossed 1000 times and the following outcomes observed:

Outcome	1	2	3	4	5	6
Frequency	135	159	280	199	133	97

a Use this information to estimate the probability of observing a 6 when this die is rolled.

b What outcome would you predict to be most likely the next time the die is rolled?

Solution

a $\Pr(6) \approx \dfrac{97}{1000} = 0.097$

b The most likely outcome is 3, since it has the highest relative frequency.

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Probabilities from area

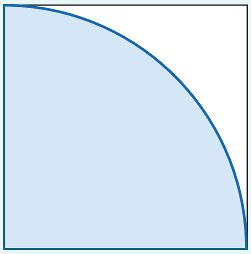
When we use the model of equally likely outcomes to determine probabilities, we count both the outcomes in the event and the outcomes in the sample space, and use the ratio to determine the probability of the event.

This idea can be extended to calculate probabilities when areas are involved, by assuming that the probabilities of all points in the region (which can be considered to be the sample space) are equally likely.

Example 5

A dartboard consists of a square of side length 2 metres containing a blue one-quarter of a circular disc centred at the bottom-left vertex of the square, as shown.

If a dart thrown at the square is equally likely to hit any part of the square, and it hits the square every time, find the probability of it hitting the blue region.



Solution

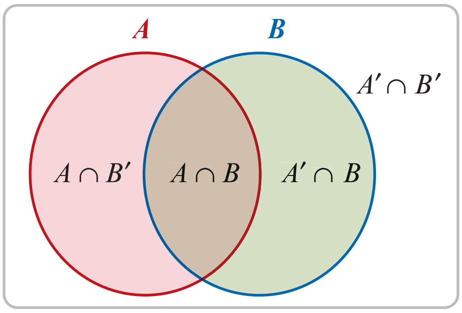
Area of blue region $= \frac{1}{4}\pi r^2 = \frac{1}{4}\pi \times 4 = \pi \text{ m}^2$

Area of dartboard $= 2 \times 2 = 4 \text{ m}^2$

$$\begin{aligned}\Pr(\text{hitting blue region}) &= \frac{\text{area of blue region}}{\text{area of dartboard}}\\ &= \frac{\pi}{4}\end{aligned}$$

Probability tables

A **probability table** is an alternative to a Venn diagram when illustrating a probability problem diagrammatically. Consider the Venn diagram which illustrates two intersecting sets A and B.



From the Venn diagram it can be seen that the sample space is divided by the sets into four disjoint regions: $A \cap B$, $A \cap B'$, $A' \cap B$ and $A' \cap B'$. These regions may be represented in a table as follows. Such a table is sometimes referred to as a **Karnaugh map**.

	B	B'
A	$A \cap B$	$A \cap B'$
A'	$A' \cap B$	$A' \cap B'$

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In a probability table, the entries give the probabilities of each of these events occurring.

	B	B'
A	$\Pr(A \cap B)$	$\Pr(A \cap B')$
A'	$\Pr(A' \cap B)$	$\Pr(A' \cap B')$

Summing the rows and columns, we can complete the table as shown.

	B	B'	
A	$\Pr(A \cap B)$	$\Pr(A \cap B')$	$\Pr(A)$
A'	$\Pr(A' \cap B)$	$\Pr(A' \cap B')$	$\Pr(A')$
	$\Pr(B)$	$\Pr(B')$	1

These tables can be useful when solving problems involving probability, as shown in the next example.

Example 6

Simone visits the dentist every 6 months for a checkup. The probability that she will need her teeth cleaned is 0.35, the probability that she will need a filling is 0.1 and the probability that she will need both is 0.05.

a What is the probability that she will not need her teeth cleaned on a visit, but will need a filling?

b What is the probability that she will not need either of these treatments?

Solution

The information in the question may be entered into a table as shown, where we use C to represent 'cleaning' and F to represent 'filling'.

	F	F'	
C	0.05		0.35
C'			
	0.1		1

All the empty cells in the table may now be filled in by subtraction:

	F	F'	
C	0.05	0.3	0.35
C'	0.05	0.6	0.65
	0.1	0.9	1

a The probability that she will not need her teeth cleaned but will need a filling is given by $\Pr(C' \cap F) = 0.05$.

b The probability that she will not need either of these treatments is $\Pr(C' \cap F') = 0.6$.

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Section summary

- The **sample space**, ε, for a random experiment is the set of all possible outcomes.
- An **event** is a subset of the sample space. The probability of an event A occurring is denoted by $\Pr(A)$.
- **Equally likely outcomes** If the sample space ε for an experiment contains n outcomes, all of which are equally likely to occur, we assign a probability of $\frac{1}{n}$ to each outcome. Then the probability of an event A is given by

$$\Pr(A) = \frac{\text{number of outcomes in } A}{\text{total number of outcomes}} = \frac{n(A)}{n(\varepsilon)}$$

- **Estimates of probability** When a probability is unknown, it can be estimated by the relative frequency obtained through repeated trials of the random experiment under consideration. In this case,

$$\Pr(A) \approx \frac{\text{number of times event } A \text{ occurs}}{\text{number of trials}} \qquad \text{for a large number of trials}$$

- Whichever method of determining probability is used, the rules of probability hold:
  - $\Pr(A) \geq 0$ for all events $A \subseteq \varepsilon$
  - $\Pr(\varnothing) = 0$ and $\Pr(\varepsilon) = 1$
  - The sum of the probabilities of all outcomes of an experiment is 1.
  - $\Pr(A') = 1 - \Pr(A)$, where A' is the complement of A
  - $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$, the **addition rule**
- If two events A and B are **mutually exclusive** (i.e. if A and B have no outcomes in common), then $\Pr(A \cap B) = 0$ and therefore $\Pr(A \cup B) = \Pr(A) + \Pr(B)$.

Exercise 13A

Example 1

1 An experiment consists of rolling a die and tossing a coin. Use a tree diagram to list the sample space for the experiment.

2 Two coins are tossed and a die is rolled. Use a tree diagram to show all the possible outcomes.

Example 2

3 If one card is chosen at random from a well-shuffled deck of 52 cards, what is the probability that the card is:

a a queen
b not a club
c a queen or a heart
d either a king or a queen?

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4 A blank six-sided die is marked with a 1 on two sides, a 2 on one side, and a 3 on the remaining three sides. Find the probability that when the die is rolled:

a a 3 shows **b** a 2 or a 3 shows.

5 Suppose that the probability that a student owns a smartphone is 0.7, the probability that they own a laptop is 0.6, and the probability that they own both is 0.5. What is the probability that a student owns either a smartphone or a laptop or both?

6 At a particular university, the probability that an Arts student studies a language is 0.3, literature is 0.6, and both is 0.25. What is the probability that an Arts student studies either a language or literature or both?

7 A computer manufacturer notes that 5% of their computers are returned owing to faulty disk drives, 2% are returned owing to faulty keyboards, and 0.3% are returned because both disk drives and keyboards are faulty. Find the probability that the next computer manufactured will be returned with:

a a faulty disk drive or a faulty keyboard

b a faulty disk drive and a working keyboard.

8 A new drug has been released and produces some minor side effects: 8% of users suffer only loss of sleep, 12% of users suffer only nausea, and 75% of users have no side effects at all. What percentage of users suffer from both loss of sleep and nausea?

9 In a particular town, the probability that an adult owns a car is 0.7, while the probability that an adult owns a car and is employed is 0.6. If a randomly selected adult is found to own a car, what is the probability that he or she is also employed?

Example 3 **10** An insurance company analysed the records of 500 drivers to determine the relationship between age and accidents in the last year.

	Accidents in the last year				
Age	0	1	2	3	Over 3
Under 20	19	35	25	17	10
20–29	30	45	33	39	17
30–39	40	33	15	6	2
40–49	18	15	10	3	1
Over 49	21	25	17	13	11

What is the probability that a driver chosen from this group at random:

a is under 20 years old and has had three accidents in the last year

b is from 40 to 49 years old and has had no accidents in the last year

c is from 20 to 29 years old

d has had more than three accidents in the last year?

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11 200 people were questioned and classified according to sex and whether or not they think private individuals should be allowed to carry guns. The results are shown in the table.

Do you think private individuals should be allowed to carry guns?

	Male	Female	Total
Yes	70	60	130
No	50	20	70
Total	120	80	200

One person is selected at random from these 200.

a What is the probability that the person thinks private individuals should be allowed to carry guns?

b What is the probability that the person is male and thinks private individuals should be allowed to carry guns?

Example 4 **12** Use the given data to estimate the probability of the specified event occurring:

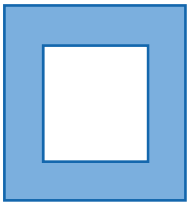
a Pr(head) if a coin is tossed 200 times and 114 heads observed

b Pr(ten) if a spinner is spun 380 times and lands on the 'ten' 40 times

c Pr(two heads) if two coins are tossed 200 times and two heads are observed on 54 occasions

d Pr(three sixes) if three dice are rolled 500 times and three sixes observed only twice

Example 5 **13** Suppose that a square dartboard consists of a white square of side length 30 cm inside a larger blue square of side length 50 cm, as shown. If a dart thrown at the board has equal chance of landing anywhere on the board, what is the probability it lands in the white area? (Ignore the possibility that it might land on the line or miss the board altogether.)

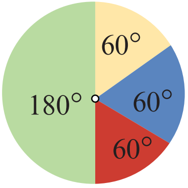


14 A spinner is as shown in the diagram. Find the probability that when spun the pointer will land on:

a the green section

b the yellow section

c any section except the yellow section.



Example 6 **15** In a particular country it has been established that the probability that a person drinks tea is 0.45, the probability that a person drinks coffee is 0.65, and the probability that a person drinks neither tea nor coffee is 0.22. Use the information to complete a probability table and hence determine the probability that a randomly selected person in that country:

a drinks tea but not coffee **b** drinks tea and coffee.

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16 A chocolate is chosen at random from a box of chocolates. It is known that in this box:

- the probability that the chocolate is dark but not soft-centred is 0.15
- the probability that the chocolate is not dark but is soft-centred is 0.42
- the probability that the chocolate is not dark is 0.60.

Find the probability that the randomly chosen chocolate is:

a dark **b** soft-centred **c** not dark and not soft-centred.

17 Records indicate that, in Australia, 65% of secondary students participate in sport, and 71% of secondary students are Australian by birth. They also show that 53% of students are Australian by birth and participate in sport. Use this information to find the probability that a student selected at random:

a does not participate in sport
b is Australian by birth and does not participate in sport
c is not Australian by birth and participates in sport
d is not Australian by birth and does not participate in sport.

13B Conditional probability and independence

The probability of an event A occurring when it is known that some event B has occurred is called conditional probability and is written $\Pr(A \mid B)$. This is usually read as 'the probability of A given B', and can be thought of as a means of adjusting probability in the light of new information.

Sometimes, the probability of an event is not affected by knowing that another event has occurred. For example, if two coins are tossed, then the probability of the second coin showing a head is independent of whether the first coin shows a head or a tail. Thus,

$$\begin{aligned}&\Pr(\text{head on second coin} \mid \text{head on first coin})\\&= \Pr(\text{head on second coin} \mid \text{tail on first coin})\\&= \Pr(\text{head on second coin})\end{aligned}$$

For other situations, however, a previous result may alter the probability. For example, the probability of rain today given that it rained yesterday will generally be different from the probability that it will rain today given that it didn't rain yesterday.

The **conditional probability** of an event A, given that event B has already occurred, is given by

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} \qquad \text{if } \Pr(B) \neq 0$$

This formula may be rearranged to give the **multiplication rule of probability**:

$$\Pr(A \cap B) = \Pr(A \mid B) \times \Pr(B)$$

The probabilities associated with a multi-stage experiment can be calculated by constructing an appropriate tree diagram and multiplying along the relevant branches (from the multiplication rule).

Example 7

In a certain town, the probability that it rains on any Monday is 0.21. If it rains on Monday, then the probability that it rains on Tuesday is 0.83. If it does not rain on Monday, then the probability of rain on Tuesday is 0.3. For a given week, find the probability that it rains:

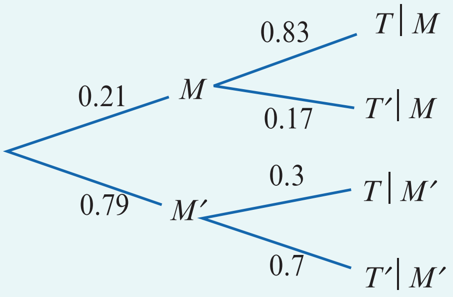
a on both Monday and Tuesday

b on Tuesday.

Solution

Let M represent the event 'rain on Monday' and T represent the event 'rain on Tuesday'.

The situation described in the question can be represented by a tree diagram. You can check that the probabilities are correct by seeing if they add to 1.



$$\Pr(T \cap M) = 0.21 \times 0.83 = 0.1743$$

$$\Pr(T' \cap M) = 0.21 \times 0.17 = 0.0357$$

$$\Pr(T \cap M') = 0.79 \times 0.3 = 0.237$$

$$\Pr(T' \cap M') = 0.79 \times 0.7 = 0.553$$

a The probability that it rains on both Monday and Tuesday is given by

$$\begin{aligned}\Pr(T \cap M) &= 0.21 \times 0.83 \\ &= 0.1743\end{aligned}$$

b The probability that it rains on Tuesday is given by

$$\begin{aligned}\Pr(T) &= \Pr(T \cap M) + \Pr(T \cap M') \\ &= 0.1743 + 0.237 \\ &= 0.4113\end{aligned}$$

The solution to part **b** of Example 7 is an application of a rule known as the law of total probability. This can be expressed in general terms as follows:

The **law of total probability** states that, in the case of two events A and B,

$$\Pr(A) = \Pr(A \mid B)\Pr(B) + \Pr(A \mid B')\Pr(B')$$

Example 8

Adrienne, Regan and Michael are doing the dishes. Since Adrienne is the oldest, she washes the dishes 40% of the time. Regan and Michael each wash 30% of the time. When Adrienne washes the probability of at least one dish being broken is 0.01, when Regan washes the probability is 0.02, and when Michael washes the probability is 0.03. Their parents don't know who is washing the dishes one particular night.

a What is the probability that at least one dish will be broken?

b Given that at least one dish is broken, what is the probability that the person washing was Michael?

Solution

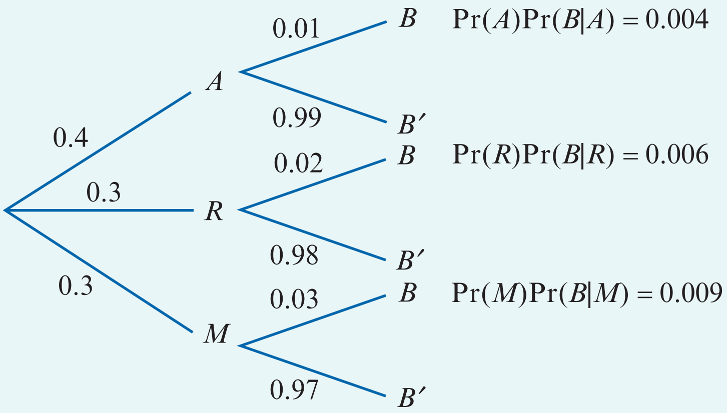
Let A be the event 'Adrienne washes the dishes', let R be the event 'Regan washes the dishes' and let M be the event 'Michael washes the dishes'. Then

$$\Pr(A) = 0.4, \qquad \Pr(R) = 0.3, \qquad \Pr(M) = 0.3$$

Let B be the event 'at least one dish is broken'. Then

$$\Pr(B \mid A) = 0.01, \qquad \Pr(B \mid R) = 0.02, \qquad \Pr(B \mid M) = 0.03$$

This information can be summarised in a tree diagram as shown:



a The probability of at least one dish being broken is

$$\begin{aligned}\Pr(B) &= \Pr(B \cap A) + \Pr(B \cap R) + \Pr(B \cap M)\\ &= \Pr(A)\Pr(B \mid A) + \Pr(R)\Pr(B \mid R) + \Pr(M)\Pr(B \mid M)\\ &= 0.004 + 0.006 + 0.009\\ &= 0.019\end{aligned}$$

b The required probability is

$$\begin{aligned}\Pr(M \mid B) &= \frac{\Pr(M \cap B)}{\Pr(B)}\\ &= \frac{0.009}{0.019} = \frac{9}{19}\end{aligned}$$

Example 9

As part of an evaluation of the school canteen, all students at a Senior Secondary College (Years 10–12) were asked to rate the canteen as poor, good or excellent. The results are shown in the table.

Rating	Year 10	Year 11	Year 12	Total
Poor	30	20	10	60
Good	80	65	35	180
Excellent	60	65	35	160
Total	170	150	80	400

What is the probability that a student chosen at random from this college:

a is in Year 12

b is in Year 12 and rates the canteen as excellent

c is in Year 12, given that they rate the canteen as excellent

d rates the canteen as excellent, given that they are in Year 12?

Solution

Let T be the event 'the student is in Year 12' and let E be the event 'the rating is excellent'.

a $\Pr(T) = \dfrac{80}{400} = \dfrac{1}{5}$

Explanation: From the table, we can see that there are 80 students in Year 12 and 400 students altogether.

b $\Pr(T \cap E) = \dfrac{35}{400} = \dfrac{7}{80}$

From the table, there are 35 students who are in Year 12 and also rate the canteen as excellent.

c $\Pr(T \mid E) = \dfrac{35}{160} = \dfrac{7}{32}$

From the table, a total of 160 students rate the canteen as excellent, and of these 35 are in Year 12.

d $\Pr(E \mid T) = \dfrac{35}{80} = \dfrac{7}{16}$

From the table, there are 80 students in Year 12, and of these 35 rate the canteen as excellent.

Note: The answers to parts **c** and **d** could also have been found using the rule for conditional probability, but here it is easier to determine the probability directly from the table.

Independent events

Two events A and B are **independent** if the probability of A occurring is the same, whether or not B has occurred.

Independent events

For events A and B with $\Pr(A) \neq 0$ and $\Pr(B) \neq 0$, the following three conditions are all equivalent conditions for the independence of A and B:

- $\Pr(A \mid B) = \Pr(A)$
- $\Pr(B \mid A) = \Pr(B)$
- $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$

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Notes:

- Sometimes this definition of independence is referred to as **pairwise independence**.
- In the special case that $\Pr(A) = 0$ or $\Pr(B) = 0$, the condition $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$ holds since both sides are zero, and so we say that A and B are independent.

Example 10

The probability that Monica remembers to do her homework is 0.7, while the probability that Patrick remembers to do his homework is 0.4. If these events are independent, then what is the probability that:

a both will do their homework

b Monica will do her homework but Patrick forgets?

Solution

Let M be the event 'Monica does her homework' and let P be the event 'Patrick does his homework'. Since these events are independent:

a $$\begin{aligned}\Pr(M \cap P) &= \Pr(M) \times \Pr(P)\\ &= 0.7 \times 0.4\\ &= 0.28\end{aligned}$$

b $$\begin{aligned}\Pr(M \cap P') &= \Pr(M) \times \Pr(P')\\ &= 0.7 \times 0.6\\ &= 0.42\end{aligned}$$

Section summary

- Conditional probability
  - The probability of an event A occurring when it is known that some event B has already occurred is called conditional probability and is written $\Pr(A \mid B)$.
  - In general, the **conditional probability** of an event A, given that event B has already occurred, is given by

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} \qquad \text{if } \Pr(B) \neq 0$$

This formula may be rearranged to give the **multiplication rule of probability**:

$$\Pr(A \cap B) = \Pr(A \mid B) \times \Pr(B)$$

- Law of total probability

The **law of total probability** states that, in the case of two events A and B,

$$\Pr(A) = \Pr(A \mid B)\Pr(B) + \Pr(A \mid B')\Pr(B')$$

- Independence

Two events A and B are **independent** if the occurrence of one event has no effect on the probability of the occurrence of the other, that is, if

$$\Pr(A \mid B) = \Pr(A)$$

Events A and B are independent if and only if

$$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$$

Exercise 13B

Skillsheet

Example 7

1 In a certain town, the probability that it rains on any Saturday is 0.25. If it rains on Saturday, then the probability of rain on Sunday is 0.8. If it does not rain on Saturday, then the probability of rain on Sunday is 0.1. For a given week, find the probability that:

a it rains on both Saturday and Sunday

b it rains on neither day

c it rains on Sunday.

2 Given that for two events A and B, $\Pr(A) = 0.6$, $\Pr(B) = 0.3$ and $\Pr(A \cap B) = 0.1$, find:

a $\Pr(B \mid A)$

b $\Pr(A \mid B)$

3 Given that for two events A and B, $\Pr(A) = 0.6$, $\Pr(B) = 0.3$ and $\Pr(B \mid A) = 0.1$, find:

a $\Pr(A \cap B)$

b $\Pr(A \mid B)$

4 In Alia's school, the probability that a student studies French is 0.5, and the probability that they study both French and Chinese is 0.3. Find the probability that a student studies Chinese, given that they study French.

Example 8

5 The chance that a harvest is poorer than average is 0.5, but if it is known that a certain disease D is present, this probability increases to 0.8. The disease D is present in 30% of harvests. Find the probability that, when a harvest is observed to be poorer than average, the disease D is present.

Example 9

6 A group of 1000 eligible voters were asked their age and their preference in an upcoming election, with the following results.

	Age			
Preference	18–25	26–40	Over 40	Total
Candidate *A*	200	100	85	385
Candidate *B*	250	230	50	530
No preference	50	20	15	85
Total	500	350	150	1000

What is the probability that a person chosen from this group at random:

a is 18–25 years of age

b prefers candidate A

c is 18–25 years of age, given that they prefer candidate A

d prefers candidate A, given that they are 18–25 years of age?

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7 The following data was derived from accident records on a highway noted for its above-average accident rate.

	Probable cause			
Type of accident	Speed	Alcohol	Other	Total
Fatal	42	61	12	115
Non-fatal	88	185	60	333
Total	130	246	72	448

Use the table to find:

a the probability that speed is the cause of the accident

b the probability that the accident is fatal

c the probability that the accident is fatal, given that speed is the cause

d the probability that the accident is fatal, given that alcohol is the cause.

Example 10

8 The probability of James winning a particular tennis match is independent of Sally winning another particular tennis match. If the probability of James winning is 0.8 and the probability of Sally winning is 0.3, find:

a the probability that they both win

b the probability that either or both of them win.

9 An experiment consists of drawing a number at random from $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Let $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{1, 3, 5, 7, 9\}$ and $C = \{4, 6, 8, 9\}$.

a Are A and B independent?

b Are A and C independent?

c Are B and C independent?

10 If A and B are independent events such that $\Pr(A) = 0.5$ and $\Pr(B) = 0.4$, find:

a $\Pr(A \mid B)$ **b** $\Pr(A \cap B)$ **c** $\Pr(A \cup B)$

11 Nathan knows that his probability of kicking more than four goals on a wet day is 0.3, while on a dry day it is 0.6. The probability that it will be wet on the day of the next game is 0.7. Calculate the probability that Nathan will kick more than four goals in the next game.

12 Find the probability that, in three tosses of a fair coin, there are three heads, given that there is at least one head.

13 The test used to determine if a person suffers from a particular disease is not perfect. The probability of a person with the disease returning a positive result is 0.95, while the probability of a person without the disease returning a positive result is 0.02. The probability that a randomly selected person has the disease is 0.03. What is the probability that a randomly selected person will return a positive result?

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14 Anya goes through three sets of traffic lights when she cycles to school each morning. The probability she stops at the first set is 0.6. If she stops at any one set, the probability that she has to stop at the next is 0.9. If she doesn't have to stop at any one set, the probability that she doesn't have to stop at the next is 0.7. Use a tree diagram to find the probability that:

a she stops at all three sets of lights
b she stops only at the second set of lights
c she stops at exactly one set of lights.

15 There are four red socks and two blue socks in a drawer. Two socks are removed at random. What is the probability of obtaining:

a two red socks
b two blue socks
c one of each colour?

16 A car salesperson was interested in the relationship between the size of the car a customer purchased and their marital status. From the sales records, the table on the right was constructed.

	Marital status		
Size of car	Married	Single	Total
Large	60	20	80
Medium	100	60	160
Small	90	70	160
Total	250	150	400

What is the probability that a person chosen at random from this group:

a drives a small car
b is single and drives a small car
c is single, given that they drive a small car
d drives a small car, given that they are single?

17 Jenny has two boxes of chocolates. Box A contains three white chocolates and four dark chocolates. Box B contains two white chocolates and five dark chocolates. Jenny first chooses a box at random and then selects a chocolate at random from it. Find the probability that:

a Jenny selects a white chocolate
b given that Jenny selects a white chocolate, it was chosen from box A.

18 At a particular petrol station, 30% of customers buy premium unleaded, 60% buy standard unleaded and 10% buy diesel. When a customer buys premium unleaded, there is a 25% chance they will fill the tank. Of the customers buying standard unleaded, 20% fill their tank. Of those buying diesel, 70% fill their tank.

a What is the probability that, when a car leaves the petrol station, it will not have a full tank?
b Given that a car leaving the petrol station has a full tank, what is the probability that the tank contains standard unleaded petrol?

19 A bag contains three red, four white and five black balls. If three balls are taken without replacement, what is the probability that they are all the same colour?

13C Discrete random variables

Suppose that three balls are drawn at random from a jar containing four white and six black balls, with replacement (i.e. each selected ball is replaced before the next draw). The sample space for this random experiment is as follows:

$$\varepsilon = \{WWW, WWB, WBW, BWW, WBB, BWB, BBW, BBB\}$$

Suppose the variable of interest is the number of white balls in the sample. This corresponds to a simpler sample space whose outcomes are numbers.

If X represents the number of white balls in the sample, then the possible values of X are 0, 1, 2 and 3. Since the actual value that X will take is the result of a random experiment, we say that X is a random variable.

A **random variable** is a function that assigns a number to each outcome in the sample space ε.

A random variable can be discrete or continuous:

- A **discrete random variable** is one that can take only a countable number of values. For example, the number of white balls in a sample of size three is a discrete random variable which may take one of the values 0, 1, 2, 3. Other examples include the number of children in a family, and a person's shoe size. (Note that discrete random variables do not have to take only whole-number values.)
- A **continuous random variable** is one that can take any value in an interval of the real number line, and is usually (but not always) generated by measuring. Height, weight, and the time taken to complete a puzzle are all examples of continuous random variables.

In this chapter we are interested in understanding more about discrete random variables.

Consider again the sample space for the random experiment described above. Each outcome in the sample space is associated with a value of X:

Experiment outcome	Value of X
WWW	$X = 3$
WWB	$X = 2$
WBW	$X = 2$
BWW	$X = 2$
WBB	$X = 1$
BWB	$X = 1$
BBW	$X = 1$
BBB	$X = 0$

Associated with each event is a probability. Since the individual draws of the ball from the jar are independent events, we can determine the probabilities by multiplying and adding appropriate terms.

Example 11

A jar contains four white and six black balls. What is the probability that, if three balls are drawn at random from the jar, with replacement, a white ball will be drawn exactly once (i.e. the situations where $X = 1$ in the table)?

Solution

$X = 1$ corresponds to the outcomes WBB, BWB and BBW.

Since there are 10 balls in total, $\Pr(W) = \dfrac{4}{10} = 0.4$ and $\Pr(B) = \dfrac{6}{10} = 0.6$.

$$\begin{aligned} \text{Thus } \Pr(X = 1) &= \Pr(WBB) + \Pr(BWB) + \Pr(BBW) \\ &= (0.4 \times 0.6 \times 0.6) + (0.6 \times 0.4 \times 0.6) + (0.6 \times 0.6 \times 0.4) \\ &= 0.432 \end{aligned}$$

Discrete probability distributions

The **probability distribution** for a discrete random variable consists of all the values that the random variable can take, together with the probability of each of these values. For example, if a fair die is rolled, then the probability distribution is:

x	1	2	3	4	5	6
$\Pr(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

The probability distribution of a discrete random variable X is described by a function

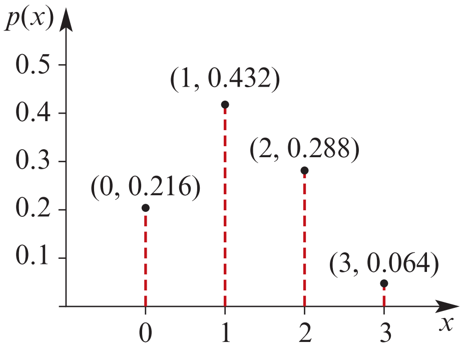
$$p(x) = \Pr(X = x)$$

This function is called a **discrete probability function** or a **probability mass function**.

Consider again the black and white balls from Example 11. The probability distribution for X, the number of white balls in the sample, is given by the following table:

x	0	1	2	3
$p(x)$	0.216	0.432	0.288	0.064

The probability distribution may also be given graphically, as shown on the right.



Note that the probabilities in the table sum to 1, which must occur if all values of the random variable have been listed.

We will use the following notation, which is discussed further in Appendix A:

- the sum of all the values of $p(x)$ is written as $\sum_{x} p(x)$
- the sum of the values of $p(x)$ for x between a and b inclusive is written as $\sum_{a \le x \le b} p(x)$

For *any* discrete probability function $p(x)$, the following two conditions must hold:

1. Each value of $p(x)$ belongs to the interval [0, 1]. That is,

$$0 \le p(x) \le 1 \quad \text{for all } x$$

2. The sum of all the values of $p(x)$ must be 1. That is,

$$\sum_x p(x) = 1$$

To determine the probability that X takes a value in the interval from a to b (including the values a and b), add the values of $p(x)$ from $x = a$ to $x = b$:

$$\Pr(a \le X \le b) = \sum_{a \le x \le b} p(x)$$

Example 12

Consider the table shown.

x	0	1	2	3
$p(x)$	0.2	0.3	0.1	0.4

a Does this meet the conditions to be a discrete probability distribution?

b Use the table to find $\Pr(X \le 2)$.

Solution

a Yes, each value of $p(x)$ is between 0 and 1, and the values add to 1.

b
$$\begin{aligned}\Pr(X \le 2) &= p(0) + p(1) + p(2)\\ &= 0.2 + 0.3 + 0.1\\ &= 0.6\end{aligned}$$

Example 13

Let X be the number of heads showing when a fair coin is tossed three times.

a Find the probability distribution of X and show that all the probabilities sum to 1.

b Find the probability that one or more heads show.

c Find the probability that more than one head shows.

Solution

a The sample space is $\varepsilon = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.

Now
$$\begin{aligned}p(0) &= \Pr(X = 0) = \Pr(\{TTT\}) &&= \frac{1}{8}\\ p(1) &= \Pr(X = 1) = \Pr(\{HTT, THT, TTH\}) &&= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}\\ p(2) &= \Pr(X = 2) = \Pr(\{HHT, HTH, THH\}) &&= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}\\ p(3) &= \Pr(X = 3) = \Pr(\{HHH\}) &&= \frac{1}{8}\end{aligned}$$

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Thus the probability distribution of X is:

x	0	1	2	3
$p(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

b The probability that one or more heads shows is

$$\Pr(X \geq 1) = p(1) + p(2) + p(3) = \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = \frac{7}{8}$$

c The probability that more than one head shows is

$$\Pr(X > 1) = \Pr(X \geq 2) = p(2) + p(3) = \frac{3}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$$

Example 14

The random variable X represents the number of chocolate chips in a certain brand of biscuit, and is known to have the following probability distribution.

x	2	3	4	5	6	7
$p(x)$	0.01	0.25	0.40	0.30	0.02	0.02

Find:

a $\Pr(X \geq 4)$ **b** $\Pr(X \geq 4 \mid X > 2)$ **c** $\Pr(X < 5 \mid X > 2)$

Solution

a
$$\begin{aligned}\Pr(X \geq 4) &= \Pr(X = 4) + \Pr(X = 5) + \Pr(X = 6) + \Pr(X = 7)\\ &= 0.4 + 0.3 + 0.02 + 0.02\\ &= 0.74\end{aligned}$$

b
$$\begin{aligned}\Pr(X \geq 4 \mid X > 2) &= \frac{\Pr(X \geq 4)}{\Pr(X > 2)}\\ &= \frac{0.74}{0.99} \qquad \text{since } \Pr(X > 2) = 1 - 0.01 = 0.99\\ &= \frac{74}{99}\end{aligned}$$

c
$$\begin{aligned}\Pr(X < 5 \mid X > 2) &= \frac{\Pr(2 < X < 5)}{\Pr(X > 2)}\\ &= \frac{\Pr(X = 3) + \Pr(X = 4)}{\Pr(X > 2)}\\ &= \frac{0.65}{0.99}\\ &= \frac{65}{99}\end{aligned}$$

Section summary

- For *any* discrete probability function $p(x)$, the following two conditions must hold:
  1. Each value of $p(x)$ belongs to the interval $[0, 1]$. That is,

     $$0 \leq p(x) \leq 1 \quad \text{for all } x$$

  2. The sum of all the values of $p(x)$ must be 1. That is,

     $$\sum_{x} p(x) = 1$$

- To determine the probability that X takes a value in the interval from a to b (including the values a and b), add the values of $p(x)$ from $x = a$ to $x = b$:

  $$\Pr(a \leq X \leq b) = \sum_{a \leq x \leq b} p(x)$$

Exercise 13C

Skillsheet

1 Which of the following random variables are discrete?

a the number of people in your family

b waist measurement

c shirt size

d the number of times a die is rolled before obtaining a six

2 Which of the following random variables are discrete?

a your age

b your height to the nearest centimetre

c the time you will wait to be served at the bank

d the number of people in the queue at the bank

Example 11

3 A fair coin is tossed three times and the number of heads noted.

a List the sample space.

b List the possible values of the random variable X, the number of heads, together with the corresponding outcomes.

c Find $\Pr(X \geq 2)$.

Example 12

4 Consider the following table:

x	0	1	2	3	4
$p(x)$	0.1	0.2	0.1	0.4	0.2

a Does this meet the conditions to be a discrete probability distribution?

b Use the table to find $\Pr(X \leq 3)$.

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Example 13 **5** A jar contains four red and five blue balls. A ball is withdrawn, its colour is observed, and it is then replaced. This is repeated three times. Let X be the number of red balls among the three balls withdrawn.

- **a** Find the probability distribution of X and show that all the probabilities sum to 1.
- **b** Find the probability that one or more red balls are obtained.
- **c** Find the probability that more than one red ball is obtained.

Example 14 **6** Two dice are rolled and the numbers noted.

- **a** List the sample space.
- **b** A random variable Y is defined as the total of the numbers showing on the two dice. List the possible values of Y, together with the corresponding outcomes.
- **c** Find:
  - **i** $\Pr(Y < 5)$
  - **ii** $\Pr(Y = 3 \mid Y < 5)$
  - **iii** $\Pr(Y \le 3 \mid Y < 7)$
  - **iv** $\Pr(Y \ge 7 \mid Y > 4)$
  - **v** $\Pr(Y = 7 \mid Y > 4)$
  - **vi** $\Pr(Y = 7 \mid Y < 8)$

7 A die is weighted as follows:

$$\Pr(2) = \Pr(3) = \Pr(4) = \Pr(5) = 0.2, \qquad \Pr(1) = \Pr(6) = 0.1$$

The die is rolled twice, and the smaller of the numbers showing is noted. Let Y represent this value.

- **a** List the sample space.
- **b** List the possible values of Y.
- **c** Find $\Pr(Y = 1)$.

8 Suppose that three balls are selected at random, with replacement, from a jar containing four white and six black balls. If X is the number of white balls in the sample, find:

- **a** $\Pr(X = 2)$
- **b** $\Pr(X = 3)$
- **c** $\Pr(X \ge 2)$
- **d** $\Pr(X = 3 \mid X \ge 2)$

9 A fair die is rolled twice and the numbers noted. Define the following events:

A = 'a four on the first roll'
B = 'a four on the second roll'
C = 'the sum of the two numbers is at least eight'
D = 'the sum of the two numbers is at least 10'

- **a** List the sample space obtained.
- **b** Find $\Pr(A)$, $\Pr(B)$, $\Pr(C)$ and $\Pr(D)$.
- **c** Find $\Pr(A \mid B)$, $\Pr(A \mid C)$ and $\Pr(A \mid D)$.
- **d** Which of the following pairs of events are independent?
  - **i** A and B
  - **ii** A and C
  - **iii** A and D

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10 Consider the table shown on the right.

x	0	1	2	3
$p(x)$	0.1	0.4	0.2	0.3

a Does this meet the conditions to be a discrete probability distribution?

b Use the table to find $\Pr(X \geq 2)$.

11 Which of the following is *not* a probability distribution?

a

x	1	3	5	7
$p(x)$	0.1	0.3	0.5	0.7

b

x	−1	0	1	2
$p(x)$	0.25	0.25	0.25	0.25

c

x	0.25	0.5	0.75	1.0
$p(x)$	−0.5	−0.25	0.25	0.5

d

x	10	20	30	40
$p(x)$	10%	20%	30%	40%

12 Three balls are selected from a jar containing four black and six red balls. Find the probability distribution of the number of black balls in the sample:

a if the ball chosen is replaced after each selection

b if the ball chosen is not replaced after each selection.

13 A coin is known to be biased such that the probability of obtaining a head on any toss is 0.4. Find the probability distribution of X, the number of heads observed when the coin is tossed twice.

14 A spinner is numbered from 1 to 5, and each of the five numbers is equally likely to come up. Find:

a the probability distribution of X, the number showing on the spinner

b $\Pr(X \geq 3)$, the probability that the number showing on the spinner is three or more

c $\Pr(X \leq 3 \mid X \geq 3)$

15 Two dice are rolled and the numbers noted.

a List the sample space for this experiment.

b Find the probability distribution of X, the sum of the numbers showing on the two dice.

c Draw a graph of the probability distribution of X.

d Find $\Pr(X \geq 9)$, the probability that the sum of the two numbers showing is nine or more.

e Find $\Pr(X \leq 10 \mid X \geq 9)$.

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16 Two dice are rolled and the numbers noted.

a List the sample space for this experiment.

b Find the probability distribution of Y, the remainder when the larger number showing is divided by the smaller number. (Note that, if the two numbers are the same, then $Y = 0$.)

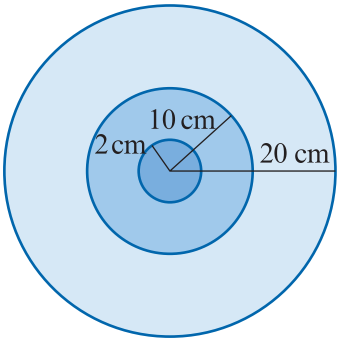
c Draw a graph of the probability distribution of Y.

17 Suppose that two socks are drawn without replacement from a drawer containing four red and six black socks. Let X represent the number of red socks obtained.

a Find the probability distribution for X.

b From the probability distribution, determine the probability that a pair of socks is obtained.

18 A dartboard consists of three circular sections, with radii of 2 cm, 10 cm and 20 cm respectively, as shown in the diagram.
When a dart lands in the centre circle the score is 100 points, in the middle circular section the score is 20 points and in the outer circular section the score is 10 points. Assume that all darts thrown hit the board, each dart is equally likely to land at any point on the dartboard, and none lands on the lines.



a Find the probability distribution for X, the number of points scored on one throw.

b Find the probability distribution for Y, the total score when two darts are thrown.

19 Erin and Nick are going to play a tennis match. Suppose that they each have an equal chance of winning any set (0.5) and that they plan to play until one player has won three sets. Let X be the number of sets played until the match is complete.

a Find $\Pr(X = 3)$.

b List the outcomes that correspond to $X = 4$, and use this to find $\Pr(X = 4)$.

c Hence, or otherwise, find $\Pr(X = 5)$.

13D Expected value (mean), variance and standard deviation

From your studies of statistics, you may already be familiar with the mean as a measure of centre and with the variance and the standard deviation as measures of spread. When these are calculated from a set of data, they are termed 'sample statistics'. It is also possible to use the probability distribution to determine the theoretically 'true' values of the mean, variance and standard deviation. When they are calculated from the probability distribution, they are called 'population parameters'. Determining the values of these parameters is the topic for this section.

Expected value (mean)

When the mean of a random variable is determined from the probability distribution, it is generally called the **expected value** of the random variable. Expected value has a wide variety of applications. The concept of expected value first arose in gambling problems, where gamblers wished to know how much they could expect to win or lose in the long run, in order to decide whether or not a particular game was a good investment.

Example 15

A person may buy a lucky ticket for \$1. They have a 20% chance of winning \$2, a 5% chance of winning \$11, and otherwise they lose. Is this a good game to play?

Solution

Let P be the amount the person will profit from each game. As it costs \$1 to play, the person can lose \$1 ($P = -1$), win \$1 ($P = 1$) or win \$10 ($P = 10$). Thus the amount that the person may win, \$$P$, has a probability distribution given by:

p	-1	1	10
$\Pr(P = p)$	0.75	0.20	0.05

Suppose you played the game 1000 times. You would expect to lose \$1 about 750 times, to win \$1 about 200 times and to win \$10 about 50 times. Thus, you would win about

$$\frac{-1 \times 750 + 1 \times 200 + 10 \times 50}{1000} = -\$0.05 \text{ per game}$$

Thus your 'expectation' is to lose 5 cents per game, and we write this as

$$\mathrm{E}(P) = -0.05$$

Note: This value gives an indication of the worth of the game: in the long run, you would expect to lose about 5 cents per game. This is called the **expected value** of P (or the **mean** of P). It is not the amount we expect to profit on any one game. (You cannot lose 5 cents in one game!) It is the amount that we expect to win on average per game in the long run.

Example 15 demonstrates how the expected value of a random variable X is determined.

The **expected value** of a discrete random variable X is determined by summing the products of each value of X and the probability that X takes that value.

That is,

$$\begin{aligned}\mathrm{E}(X) &= \sum_x x \cdot \Pr(X = x)\\ &= \sum_x x \cdot p(x)\end{aligned}$$

The expected value $\mathrm{E}(X)$ may be considered as the long-run average value of X. It is generally denoted by the Greek letter μ (mu), and is also called the **mean** of X.

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Example 16

A coin is biased in favour of heads such that the probability of obtaining a head on any single toss is 0.6. The coin is tossed three times and the results noted. If X is the number of heads obtained on the three tosses, find $\mathrm{E}(X)$, the expected value of X.

Solution

The following probability distribution can be found by listing the outcomes in the sample space and determining the value of X and the associated probability for each outcome.

x	0	1	2	3
$p(x)$	0.064	0.288	0.432	0.216

$$\begin{aligned}\mu = \mathrm{E}(X) &= \sum_x x \cdot p(x)\\ &= (0 \times 0.064) + (1 \times 0.288) + (2 \times 0.432) + (3 \times 0.216)\\ &= 0.288 + 0.864 + 0.648\\ &= 1.8\end{aligned}$$

Note: This means that, if the experiment were repeated many times, then an average of 1.8 heads per three tosses would be observed.

Sometimes we wish to find the expected value of a function of X. This is determined by calculating the value of the function for each value of X, and then summing the products of these values and the associated probabilities.

The expected value of $g(X)$ is given by

$$E[g(X)] = \sum_x g(x) \cdot p(x)$$

Example 17

For the random variable X defined in Example 16, find:

a $\mathrm{E}(3X + 1)$

b $\mathrm{E}(X^2)$

Solution

a
$$\begin{aligned}\mathrm{E}(3X + 1) &= \sum_x (3x + 1) \cdot p(x)\\ &= (1 \times 0.064) + (4 \times 0.288) + (7 \times 0.432) + (10 \times 0.216)\\ &= 6.4\end{aligned}$$

b
$$\begin{aligned}\mathrm{E}(X^2) &= \sum_x x^2 \cdot p(x)\\ &= (0^2 \times 0.064) + (1^2 \times 0.288) + (2^2 \times 0.432) + (3^2 \times 0.216)\\ &= 3.96\end{aligned}$$

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Let us compare the values found in Example 17 with the value of $\mathrm{E}(X)$ found in Example 16. In part **a**, we found that $\mathrm{E}(3X+1) = 6.4$. Since $3\mathrm{E}(X) + 1 = 3 \times 1.8 + 1 = 6.4$, we see that

$$\mathrm{E}(3X+1) = 3\mathrm{E}(X) + 1$$

In part **b**, we found that $\mathrm{E}(X^2) = 3.96$. Since $[\mathrm{E}(X)]^2 = 1.8^2 = 3.24$, we see that

$$\mathrm{E}(X^2) \neq [\mathrm{E}(X)]^2$$

These two examples illustrate an important point concerning expected values.

In general, the expected value of a function of X is not equal to that function of the expected value of X. That is,

$$\mathrm{E}[g(X)] \neq g[\mathrm{E}(X)]$$

An exception is when the function is linear:

Expected value of $aX + b$

$$\mathrm{E}(aX+b) = a\mathrm{E}(X) + b \qquad \text{(for } a, b \text{ constant)}$$

This is illustrated in Example 18.

Example 18

For a \$5 monthly fee, a TV repair company guarantees customers a complete service. The company estimates the probability that a customer will require one service call in a month as 0.05, the probability of two calls as 0.01 and the probability of three or more calls as 0.00. Each call costs the repair company \$40. What is the TV repair company's expected monthly gain from such a contract?

Solution

We may summarise the given information in the following table.

Calls	0	1	2	≥ 3
Gain, g	5	-35	-75	
$\Pr(G = g)$	0.94	0.05	0.01	0.00

$$\begin{aligned}\mathrm{E}(G) &= \sum_{g=0}^{2} g \cdot \Pr(G = g) \\ &= 5 \times 0.94 - 35 \times 0.05 - 75 \times 0.01 \\ &= 2.20\end{aligned}$$

Thus, the company can expect to gain \$2.20 per month on each contract sold.

Alternative solution

An alternative method of solution uses the formula for the expected value of $aX + b$, as follows.

Let X be the number of calls received. Then

$$G = 5 - 40X$$

and so $\mathrm{E}(G) = 5 - 40 \times \mathrm{E}(X)$

Since
$$\begin{aligned}\mathrm{E}(X) &= 1 \times 0.05 + 2 \times 0.01\\ &= 0.07\end{aligned}$$

we have
$$\begin{aligned}\mathrm{E}(G) &= 5 - 40 \times 0.07\\ &= 2.20 \quad \text{as previously determined.}\end{aligned}$$

Another useful property of expectation is that the expected value of the sum of two random variables is equal to the sum of their expected values. That is, if X and Y are two random variables, then

$$\mathrm{E}(X + Y) = \mathrm{E}(X) + \mathrm{E}(Y)$$

Measures of variability: variance and standard deviation

As well as knowing the long-run average value of a random variable (the mean), it is also useful to have a measure of how close to this mean are the possible values of the random variable — that is, a measure of how spread out the probability distribution is. The most useful measures of variability for a discrete random variable are the variance and the standard deviation.

The **variance** of a random variable X is a measure of the spread of the probability distribution about its mean or expected value μ. It is defined as

$$\mathrm{Var}(X) = \mathrm{E}[(X - \mu)^2]$$

and may be considered as the long-run average value of the square of the distance from X to μ. The variance is usually denoted by σ^2, where σ is the lowercase Greek letter *sigma*.

From the definition,

$$\begin{aligned}\mathrm{Var}(X) &= \mathrm{E}[(X - \mu)^2]\\ &= \sum_{x} (x - \mu)^2 \cdot \Pr(X = x)\end{aligned}$$

Since the variance is determined by squaring the distance from X to μ, it is no longer in the units of measurement of the original random variable X. A measure of spread in the appropriate unit is found by taking the square root of the variance.

The **standard deviation** of X is defined as

$$\mathrm{sd}(X) = \sqrt{\mathrm{Var}(X)}$$

The standard deviation is usually denoted by σ.

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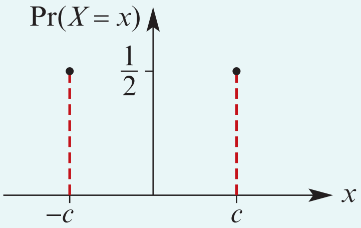
Example 19

Suppose that a discrete random variable X has the probability distribution shown in the following table, where $c > 0$.

x	$-c$	c
$\Pr(X = x)$	0.5	0.5

Find the standard deviation of X.

Solution



$$\begin{aligned}
\mu = \mathrm{E}(X) &= \sum_x x \cdot \Pr(X = x) \\
&= (-c \times 0.5) + (c \times 0.5) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\sigma^2 = \mathrm{Var}(X) &= \mathrm{E}[(X - \mu)^2] \\
&= \mathrm{E}(X^2) && \text{since } \mu = 0 \\
&= \sum_x x^2 \cdot \Pr(X = x) \\
&= (-c)^2 \times 0.5 + c^2 \times 0.5 \\
&= c^2
\end{aligned}$$

which is the average of (the distance from X to μ)^2.

Therefore $\sigma = \mathrm{sd}(X) = c$.

Using the definition is not always the easiest way to calculate the variance.

An alternative (computational) formula for variance is

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$$

Proof We already know that

$$\mathrm{E}(aX + b) = a\mathrm{E}(X) + b \qquad (1)$$

and $$\mathrm{E}(X + Y) = \mathrm{E}(X) + \mathrm{E}(Y) \qquad (2)$$

Hence
$$\begin{aligned}
\mathrm{Var}(X) &= \mathrm{E}[(X - \mu)^2] \\
&= \mathrm{E}(X^2 - 2\mu X + \mu^2) \\
&= \mathrm{E}(X^2) + \mathrm{E}(-2\mu X + \mu^2) && \text{using (2)} \\
&= \mathrm{E}(X^2) - 2\mu\mathrm{E}(X) + \mu^2 && \text{using (1)} \\
&= \mathrm{E}(X^2) - 2\mu^2 + \mu^2 && \text{since } \mu = \mathrm{E}(X) \\
&= \mathrm{E}(X^2) - \mu^2
\end{aligned}$$

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Example 20

For the probability distribution shown, find $E(X^2)$ and $[E(X)]^2$ and hence find the variance of X.

x	0	1	2	3
$\Pr(X = x)$	0.08	0.18	0.4	0.34

Solution

We have

$$\begin{aligned} E(X) &= 1 \times 0.18 + 2 \times 0.4 + 3 \times 0.34 = 2 \\ [E(X)]^2 &= \mu^2 = 4 \\ E(X^2) &= 1 \times 0.18 + 4 \times 0.4 + 9 \times 0.34 = 4.84 \end{aligned}$$

Hence

$$\text{Var}(X) = E(X^2) - \mu^2 = 4.84 - 4 = 0.84$$

Variance of $aX + b$

$$\text{Var}(aX + b) = a^2\text{Var}(X) \qquad \text{(for } a, b \text{ constant)}$$

Example 21

If X is a random variable such that $\text{Var}(X) = 9$, find:

a $\text{Var}(3X + 2)$

b $\text{Var}(-X)$

Solution

a $$\begin{aligned} \text{Var}(3X + 2) &= 3^2\text{Var}(X) \\ &= 9 \times 9 \\ &= 81 \end{aligned}$$

b $$\begin{aligned} \text{Var}(-X) &= \text{Var}(-1 \times X) \\ &= (-1)^2\text{Var}(X) \\ &= \text{Var}(X) \\ &= 9 \end{aligned}$$

Interpretation of standard deviation

We can make the standard deviation more meaningful by giving it an interpretation that relates to the probability distribution.

Example 22

The number of chocolate bars, X, sold by a manufacturer in any month has the following distribution:

x	100	150	200	250	300	400
$p(x)$	0.05	0.15	0.35	0.25	0.15	0.05

What is the probability that X takes a value in the interval $\mu - 2\sigma$ to $\mu + 2\sigma$?

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Solution

First we must find the values of μ and σ.

$$\begin{aligned}\mu = \mathrm{E}(X) &= \sum_x x \cdot p(x)\\ &= 5 + 22.5 + 70 + 62.5 + 45 + 20\\ &= 225\end{aligned}$$

Before determining the standard deviation σ, we need to find the variance σ^2.

Now $\quad \mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$

and

$$\begin{aligned}\mathrm{E}(X^2) &= \sum_x x^2 \cdot p(x)\\ &= 500 + 3375 + 14\,000 + 15\,625 + 13\,500 + 8000\\ &= 55\,000\end{aligned}$$

Thus

$$\begin{aligned}\mathrm{Var}(X) &= 55\,000 - (225)^2\\ &= 4375\end{aligned}$$

and so

$$\begin{aligned}\sigma &= \mathrm{sd}(X)\\ &= \sqrt{4375}\\ &= 66.14 \quad \text{(correct to two decimal places)}\end{aligned}$$

Hence

$$\begin{aligned}&\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma)\\ &= \Pr(92.72 \le X \le 357.28)\\ &= \Pr(100 \le X \le 300) \qquad \text{since } X \text{ only takes the values in the table}\\ &= 0.95 \qquad \text{from the probability distribution of } X\end{aligned}$$

In this example, 95% of the distribution lies within two standard deviations either side of the mean. While this is not always true, in many circumstances it is approximately true.

For many random variables X,

$$\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma) \approx 0.95$$

Example 23

A manufacturer knows that the mean number of faulty light bulbs in a batch of 10 000 is 12, with a standard deviation of 3. He wishes to claim to his clients that 95% of batches will contain between c_1 and c_2 faulty light bulbs. What are two possible values of c_1 and c_2?

Solution

Since $\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma) \approx 0.95$, we can say

$$c_1 = \mu - 2\sigma = 6 \quad \text{and} \quad c_2 = \mu + 2\sigma = 18$$

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Section summary

- The **expected value** (or **mean**) of a discrete random variable X may be considered as the long-run average value of X. It is found by summing the products of each value of X and the probability that X takes that value. That is,

$$\begin{aligned}\mu = \mathrm{E}(X) &= \sum_x x \cdot \Pr(X = x)\\ &= \sum_x x \cdot p(x)\end{aligned}$$

- The **variance** of a random variable X is a measure of the spread of the probability distribution about its mean μ. It is defined as

$$\sigma^2 = \mathrm{Var}(X) = \mathrm{E}[(X - \mu)^2]$$

  An alternative (computational) formula for variance is

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$$

- The **standard deviation** of a random variable X is defined as

$$\sigma = \mathrm{sd}(X) = \sqrt{\mathrm{Var}(X)}$$

- In general, for many random variables X,

$$\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma) \approx 0.95$$

Exercise 13D

Example 15

1 Tickets in a game of chance can be purchased for \$2. Each ticket has a 30% chance of winning \$2, a 10% chance of winning \$20, and otherwise loses. How much might you expect to win or lose if you play the game 100 times?

Example 16

2 For each of the following probability distributions, find the mean (expected value):

a

x	1	3	5	7
$p(x)$	0.1	0.3	0.3	0.3

b

x	−1	0	1	2
$p(x)$	0.25	0.25	0.25	0.25

c

x	0	1	2	3	4	5	6	7
$p(x)$	0.09	0.22	0.26	0.21	0.13	0.06	0.02	0.01

d

x	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$p(x)$	0.08	0.13	0.09	0.19	0.20	0.03	0.10	0.18

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e

x	2	3	4	5	6	7	8	9	10	11	12
$p(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

f

x	-3	-2	-1	0	1	2	3
$p(x)$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{10}$

3 A business consultant evaluates a proposed venture as follows. A company stands to make a profit of \$10 000 with probability 0.15, to make a profit of \$5000 with probability 0.45, to break even with probability 0.25, and to lose \$5000 with probability 0.15. Find the expected profit.

4 A spinner is numbered from 0 to 5, and each of the six numbers has an equal chance of coming up. A player who bets \$1 on any number wins \$5 if that number comes up; otherwise the \$1 is lost. What is the player's expected profit on the game?

5 Suppose that the probability of having a female child is not as high as that of having a male child, and that the probability distribution for the number of male children in a three-child family has been determined from past records as follows.

Number of males, x	0	1	2	3
$p(x)$	0.12	0.36	0.38	0.14

What is the mean number of males in a three-child family?

6 A player throws a die with faces numbered from 1 to 6 inclusive. If the player obtains a 6, she throws the die a second time, and in this case her score is the sum of 6 and the second number; otherwise her score is the number first obtained. The player has no more than two throws.

Let X be the random variable denoting the player's score. Write down the probability distribution of X, and determine the mean of X.

Example 17

7 The random variable X represents the number of chocolate chips in a certain brand of biscuit, and is known to have the following probability distribution.

x	2	3	4	5	6
$p(x)$	0.01	0.25	0.40	0.30	0.04

Calculate:

a $E(X)$

b $E(X^3)$

c $E(5X - 4)$

d $E\left(\frac{1}{X}\right)$

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Example 18 **8** Manuel is a car salesperson. In any week his probability of making sales is as follows:

Number of cars sold, x	2	3	4	5	6
$\Pr(X = x)$	0.45	0.25	0.20	0.08	0.02

If he is paid \$2000 commission on each car sold, what is his expected weekly income?

Example 20 **9** A discrete random variable X takes values 0, 1, 2, 4, 8 with probabilities as shown in the table.

x	0	1	2	4	8
$\Pr(X = x)$	p	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$

a Find p. **b** Find $E(X)$. **c** Find $\mathrm{Var}(X)$.

10 A biased die is such that the probability of any face landing uppermost is proportional to the number on that face. Thus, if X denotes the score obtained in one throw of this die, then $\Pr(X = r) = kr$ for $r = 1, 2, 3, 4, 5, 6$, where k is a constant.

a Find the value of k. **b** Find $E(X)$. **c** Find $\mathrm{Var}(X)$.

11 An unbiased die is in the form of a regular tetrahedron and has its faces numbered 1, 2, 3 and 4. When the die is thrown onto a horizontal table, the number on the face in contact with the table is noted. Two such dice are thrown and the score, X, is found by multiplying these numbers together.

a Give the probability distribution of X.

b Determine the values of:

i $\Pr(X > 8)$ **ii** $E(X)$ **iii** $\mathrm{Var}(X)$

12 A coin and a six-sided die are thrown simultaneously. The random variable X is defined as follows: If the coin shows a head, then X is the score on the die; if the coin shows a tail, then X is twice the score on the die.

a Find the expected value, μ, of X.

b Find $\Pr(X < \mu)$.

c Find $\mathrm{Var}(X)$.

Example 21 **13** If $\mathrm{Var}(X) = 16$, find:

a $\mathrm{Var}(2X)$ **b** $\mathrm{Var}(X + 2)$ **c** $\mathrm{Var}(1 - X)$ **d** $\mathrm{sd}(3X)$

Example 22 **14** A random variable X has the probability distribution shown. Find:

x	1	2	3	4	5
$\Pr(X = x)$	c	0.3	0.1	0.2	0.05

a the constant c

b $E(X)$, the mean of X

c $\mathrm{Var}(X)$, the variance of X, and hence the standard deviation of X

d $\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma)$

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15 A random variable X has the probability distribution shown.

x	1	2	3	4	5
$\Pr(X = x)$	k	$2k$	$3k$	$4k$	$5k$

Find:

a the constant k

b $\mathrm{E}(X)$, the expectation of X

c $\mathrm{Var}(X)$, the variance of X

d $\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma)$

16 Two dice are rolled. If X is the sum of the numbers showing on the two dice, find:

a $\mathrm{E}(X)$, the mean of X

b $\mathrm{Var}(X)$, the variance of X

c $\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma)$

17 The number of heads, X, obtained when a fair coin is tossed six times has the following probability distribution.

x	0	1	2	3	4	5	6
$p(x)$	0.0156	0.0937	0.2344	0.3126	0.2344	0.0937	0.0156

Find:

a $\mathrm{E}(X)$, the mean of X

b $\mathrm{Var}(X)$, the variance of X

c $\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma)$

Example 23

18 The random variable X, the number of heads observed when a fair coin is tossed 100 times, has a mean of 50 and a standard deviation of 5. If $\Pr(c_1 \le X \le c_2) \approx 0.95$, give possible values of c_1 and c_2.

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Chapter summary

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Probability

- Probability is a numerical measure of the chance of a particular event occurring and may be determined experimentally or by symmetry.
- Whatever method is used to determine the probability, the following rules will hold:
  - $0 \le \Pr(A) \le 1$ for all events $A \subseteq \varepsilon$
  - $\Pr(\varnothing) = 0$ and $\Pr(\varepsilon) = 1$
  - $\Pr(A') = 1 - \Pr(A)$, where A' is the complement of A
  - $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$, the **addition rule**.
- Probabilities associated with combined events are sometimes able to be calculated more easily from a probability table.
- Two events A and B are **mutually exclusive** if $A \cap B = \varnothing$. In this case, we have $\Pr(A \cap B) = 0$ and therefore $\Pr(A \cup B) = \Pr(A) + \Pr(B)$.
- The **conditional probability** of event A occurring, given that event B has already occurred, is

  $$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} \qquad \text{if } \Pr(B) \neq 0$$

  giving $\Pr(A \cap B) = \Pr(A \mid B) \times \Pr(B)$ (the **multiplication rule**)
- The probabilities associated with multi-stage experiments can be calculated by constructing an appropriate tree diagram and multiplying along the relevant branches (from the multiplication rule).
- The **law of total probability** states that, in the case of two events A and B,

  $$\Pr(A) = \Pr(A \mid B)\Pr(B) + \Pr(A \mid B')\Pr(B')$$
- Two events A and B are **independent** if

  $$\Pr(A \mid B) = \Pr(A)$$

  so whether or not B has occurred has no effect on the probability of A occurring.
- Events A and B are independent if and only if $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$.

Discrete random variables

- A **discrete** random variable X is one which can take only a countable number of values. Often these values are whole numbers, but not necessarily.
- The **probability distribution** of X is a function $p(x) = \Pr(X = x)$ that assigns a probability to each value of X. It can be represented by a rule, a table or a graph, and must give a probability $p(x)$ for every value x that X can take.
- For *any* discrete probability distribution, the following two conditions must hold:
  1. Each value of $p(x)$ belongs to the interval $[0, 1]$. That is,

     $$0 \le p(x) \le 1 \quad \text{for all } x$$
  2. The sum of all the values of $p(x)$ must be 1. That is,

     $$\sum_x p(x) = 1$$

- To determine the probability that X takes a value in the interval from a to b (including the values a and b), add the values of $p(x)$ from $x = a$ to $x = b$:

$$\Pr(a \le X \le b) = \sum_{a \le x \le b} p(x)$$

- The **expected value** (or **mean**) of a discrete random variable X may be considered as the long-run average value of X. It is found by summing the products of each value of X and the probability that X takes that value. That is,

$$\begin{aligned}\mu = \mathrm{E}(X) &= \sum_x x \cdot \Pr(X = x)\\ &= \sum_x x \cdot p(x)\end{aligned}$$

- The expected value of a function of X is given by

$$\mathrm{E}[g(X)] = \sum_x g(x) \cdot p(x)$$

- The **variance** of a random variable X is a measure of the spread of the probability distribution about its mean μ. It is defined as

$$\sigma^2 = \mathrm{Var}(X) = \mathrm{E}[(X - \mu)^2]$$

  An alternative (computational) formula for variance is

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$$

- The **standard deviation** of a random variable X is defined as

$$\sigma = \mathrm{sd}(X) = \sqrt{\mathrm{Var}(X)}$$

- Linear function of a discrete random variable:

$$\begin{aligned}\mathrm{E}(aX + b) &= a\mathrm{E}(X) + b\\ \mathrm{Var}(aX + b) &= a^2\mathrm{Var}(X)\end{aligned}$$

- In general, for many random variables X,

$$\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma) \approx 0.95$$

Technology-free questions

1 If $\Pr(A) = 0.5$, $\Pr(B) = 0.2$ and $\Pr(A \cup B) = 0.7$, are the events A and B mutually exclusive? Explain.

2 Show, using a diagram or otherwise, that $\Pr(A \cup B) = 1 - \Pr(A' \cap B')$. How would you describe this relationship in words?

3 A box contains five black and four white balls. Find the probability that two balls drawn at random are of different colours if:

a the first ball drawn is replaced before the second is drawn

b the balls are drawn without replacement.

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4 A gambler has two coins, A and B; the probabilities of their turning up heads are 0.8 and 0.4 respectively. One coin is selected at random and tossed twice, and a head and a tail are observed. Find the probability that the coin selected was A.

5 The probability distribution of a discrete random variable X is given by the following table. Show that $p = 0.5$ or $p = 1$.

x	0	1	2	3
$\Pr(X = x)$	$0.4p^2$	0.1	0.1	$1 - 0.6p$

6 A random variable X has the following probability distribution.

x	-1	0	1	2	3	4
$\Pr(X = x)$	k	$2k$	$3k$	$2k$	k	k

Find:

a the constant k **b** $\mathrm{E}(X)$, the mean of X **c** $\mathrm{Var}(X)$, the variance of X

7 If X has a probability function given by

$$p(x) = \frac{1}{4}, \quad x = 2, 4, 16, 64$$

find:

a $\mathrm{E}(X)$ **b** $\mathrm{E}\left(\frac{1}{X}\right)$ **c** $\mathrm{Var}(X)$ **d** $\mathrm{sd}(X)$

8 A manufacturer sells cylinders for \$$x$ each; the cost of the manufacture of each cylinder is \$2. If a cylinder is defective, it is returned and the purchase money refunded. A returned cylinder is regarded as a total loss to the manufacturer. The probability that a cylinder is returned is $\frac{1}{5}$.

a Let P be the profit per cylinder. Find the probability function of P.

b Find the mean of P in terms of x.

c How much should the manufacturer sell the cylinders for in order to make a profit in the long term?

9 A group of 1000 drivers were classified according to their age and the number of accidents they had been involved in during the previous year. The results are shown in the table.

	Age < 30	Age ≥ 30
At most one accident	130	170
More than one accident	470	230

a Calculate the probability that, if a driver is chosen at random from this group, the driver is aged less than 30 and has had more than one accident.

b Calculate the probability that a randomly chosen driver is aged less than 30, given that he or she has had more than one accident.

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10 This year, 70% of the population have been immunised against a certain disease. Records indicate that an immunised person has a 5% chance of contracting the disease, whereas for a non-immunised person the chance is 60%. Calculate the overall percentage of the population who are expected to contract the disease.

11 Given $\Pr(A) = \frac{1}{2}$, $\Pr(B) = \frac{1}{4}$ and $\Pr(A \mid B) = \frac{1}{6}$, find:

a $\Pr(A \cap B)$ **b** $\Pr(A \cup B)$ **c** $\Pr(A' \mid B)$ **d** $\Pr(A \mid B')$

Multiple-choice questions

1 Consider the following table:

x	-2	-1	0	1	2
$\Pr(X = x)$	$2k$	$3k$	0.1	$3k$	$2k$

For the table to represent a probability function, the value of k is

A 0.09 **B** 0.9 **C** 0.01 **D** 0.2 **E** 1

2 Suppose that the random variable X has the probability distribution given in the following table:

x	-3	-2	-1	0	1	2	3
$\Pr(X = x)$	0.07	0.15	0.22	0.22	0.17	0.12	0.05

$\Pr(-3 \le X < 0)$ is equal to

A 0.59 **B** 0.37 **C** 0.22 **D** 0.44 **E** 0.66

3

x	1	2	3	4	5
$\Pr(X = x)$	0.46	0.24	0.14	0.09	0.07

For this probability distribution, the expected value $\mathrm{E}(X)$ is

A 2 **B** 1 **C** 1.59 **D** 2.07 **E** 5.87

4 A random variable X is such that $\mathrm{E}(X) = 1.20$ and $\mathrm{E}(X^2) = 1.69$. The standard deviation of X is equal to

A 1.3 **B** $\sqrt{3.13}$ **C** 0.25 **D** 0.7 **E** 0.5

5 Suppose that a random variable X is such that $\mathrm{E}(X) = 100$ and $\mathrm{Var}(X) = 100$. Suppose further that Y is a random variable such that $Y = 3X + 10$. Then

A $\mathrm{E}(Y) = 300$ and $\mathrm{Var}(Y) = 900$ **B** $\mathrm{E}(Y) = 310$ and $\mathrm{Var}(Y) = 300$

C $\mathrm{E}(Y) = 310$ and $\mathrm{Var}(Y) = 900$ **D** $\mathrm{E}(Y) = 300$ and $\mathrm{Var}(Y) = 30$

E $\mathrm{E}(Y) = 310$ and $\mathrm{Var}(Y) = 100\sqrt{3}$

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6 The random variable X has the probability distribution shown, where $0 < p < \frac{1}{3}$. The mean of X is

x	-1	0	1
$\Pr(X = x)$	p	$2p$	$1 - 3p$

A 1 **B** 0 **C** $1 - 4p$ **D** $4p$ **E** $1 + 4p$

7 The random variable X has the probability distribution shown on the right. If the mean of X is 0.2, then

x	-2	0	2
$\Pr(X = x)$	a	b	0.2

A $a = 0.2,\ b = 0.6$ **B** $a = 0.1,\ b = 0.7$ **C** $a = 0.4,\ b = 0.4$

D $a = 0.1,\ b = 0.7$ **E** $a = 0.5,\ b = 0.3$

Extended-response questions

1 Given the following probability function:

x	1	2	3	4	5	6	7
$\Pr(X = x)$	c	$2c$	$2c$	$3c$	c^2	$2c^2$	$7c^2 + c$

a Find c.

b Evaluate $\Pr(X \geq 5)$.

c If $\Pr(X \leq k) > 0.5$, find the minimum value of k.

2 Janet and Alan are going to play a tennis match. The probability of Janet winning the first set is 0.3. After that, Janet's probability of winning a set is 0.6 if she has won the previous set, but only 0.4 if she has lost it. The match will continue until either Janet or Alan has won two sets.

a Construct a tree diagram to show the possible course of the match.

b Find the probability that:

i Janet will win

ii Alan will win.

c Let X be the number of sets played until the match is complete.

i Find the probability distribution of X.

ii Find the expected number of sets that the match will take, $E(X)$.

d Given that the match lasted three sets, find the probability that Alan won.

3 Five identical cards are placed face down on the table. Three of the cards are marked \$5 and the remaining two are marked \$10. A player picks two cards at random (without replacement) and is paid an amount equal to the sum of the values on the two cards. How much should the player pay to play if this is to be a fair game? (A fair game is considered to be one for which $E(X) = 0$, where X is the profit from the game.)

4 A manufacturing company has three assembly lines: A, B and C. It has been found that 95% of the products produced on assembly line A will be free from faults, 98% from assembly line B will be free from faults and 99% from assembly line C will be free from faults. Assembly line A produces 50% of the day's output, assembly line B produces 30% of the day's output, and the rest is produced on assembly line C. If an item is chosen at random from the company's stock, find the probability that it:

- **a** was produced on assembly line A
- **b** is defective, given that it came from assembly line A
- **c** is defective
- **d** was produced on assembly line A, given that it was found to be defective.

5 A recent study found that P, the number of passengers per car entering a city on the freeway on a workday morning, is given by:

p	0	1	2	3	4	5
$\Pr(P = p)$	0.39	0.27	0.16	0.12	0.04	0.02

- **a** **i** Compute $E(P)$, the mean number of passengers per car.
  - **ii** Compute $\text{Var}(P)$ and hence find the standard deviation of P.
  - **iii** Find $\Pr(\mu - 2\sigma \le P \le \mu + 2\sigma)$.
- **b** The fees for cars at a toll booth on the freeway are as follows:
  - Cars carrying no passengers pay \$1 toll.
  - Cars carrying one passenger pay \$0.40 toll.
  - Cars carrying two or more passengers pay no toll.

  Let T be the toll paid by a randomly selected car on the freeway.
  - **i** Construct the probability distribution of T.
  - **ii** Find $E(T)$, the mean toll paid per car.
  - **iii** Find $\Pr(\mu - 2\sigma \le T \le \mu + 2\sigma)$.

6 The random variable Y, the number of cars sold in a week by a car salesperson, has the following probability distribution:

y	0	1	2	3	4	5	6	7	8
$\Pr(Y = y)$	0.135	0.271	0.271	0.180	0.090	0.036	0.012	0.003	0.002

- **a** Compute $E(Y)$, the mean number of sales per week.
- **b** Compute $\text{Var}(Y)$ and hence find the standard deviation of Y.
- **c** The car salesperson is given a bonus as follows: If fewer than three cars are sold in the week, no bonus is given; if three or four cars are sold, a \$100 bonus is given; for more than four cars, the bonus is \$200. Let B be the bonus paid to the salesperson.
  - **i** Construct the probability distribution for B.
  - **ii** Find $E(B)$, the mean bonus paid.

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7 A given investment scheme is such that there is a 10% chance of receiving a profit of 40% of the amount invested, a 15% chance of a 30% profit, a 25% chance of a 20% profit, a 20% chance of a 10% profit, a 15% chance of breaking even, a 10% chance of a 10% loss and a 5% chance of 20% loss.

- **a** Find the mean and standard deviation of the percentage return on the amount invested.
- **b** Find the probability that the percentage return on the amount invested is within two standard deviations of the mean.
- **c** An investor investing in the scheme pays a brokerage fee of 2% on the amount invested and a tax of 40% on the return (= profit − brokerage) of the investment. (Assume that a loss results in a tax refund for this investment.)
  Express the percentage gain in terms of the percentage return on the amount invested, and hence find the mean and standard deviation of the percentage gain.

8 A concert featuring a popular singer is scheduled to be held in a large open-air theatre. The promoter is concerned that rain will cause people to stay away. A weather forecaster predicts that the probability of rain on any day at that particular time of the year is 0.33. If it does not rain, the promoter will make a profit of \$250 000 on the concert. If it does rain, the profit will be reduced to \$20 000. An insurance company agrees to insure the concert for \$250 000 against rain for a premium of \$60 000. Should the promoter buy the insurance?

9 A game is devised as follows: On two rolls of a single die, you will lose \$10 if the sum showing is 7, and win \$11 if the sum showing is either 11 or 12. How much should you win or lose if any other sum comes up in order for the game to be fair?

10 A new machine is to be developed by a manufacturing company. Prototypes are to be made until one satisfies the specifications of the company. Only then will it go into production. However, if after three prototypes are made none is satisfactory, then the project will be abandoned.

It is estimated that the probability a prototype will fail to produce a satisfactory model is 0.35, independent of any other already tested.

- **a** Find the probability that:
  - **i** the first prototype is successful
  - **ii** the first is not successful but the second is
  - **iii** the first two are not successful but the third is
  - **iv** the project is abandoned.
- **b** It is estimated that the cost of developing and testing the first prototype is \$7 million and that each subsequent prototype developed costs half of the one before. Find the expected cost of the project.
- **c** If a machine is developed, then it is estimated that the income will be \$20 million. (If the project is abandoned, there is no income.) Find the expected profit.

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11 Alfred and Bertie play a game, each starting with \$100. Two dice are thrown. If the total score is 5 or more, then Alfred pays \$$x$ to Bertie, where $0 < x \le 8$. If the total score is 4 or less, then Bertie pays \$$(x + 8)$ to Alfred.

a Find the expected value of Alfred's cash after the first game.

b Find the value of x for the game to be fair.

c Given that $x = 3$, find the variance of Alfred's cash after the first game.

12 A die is loaded such that the chance of throwing a 1 is $\frac{x}{4}$, the chance of a 2 is $\frac{1}{4}$ and the chance of a 6 is $\frac{1}{4}(1 - x)$. The chance of a 3, 4 or 5 is $\frac{1}{6}$. The die is thrown twice.

a Prove that the chance of throwing a total of 7 is $\dfrac{9x - 9x^2 + 10}{72}$.

b Find the value of x which will make this chance a maximum and find this maximum probability.

13 A game of chance consists of rolling a disc of diameter 2 cm on a horizontal square board. The board is divided into 25 small squares, each of side length 4 cm. A player wins a prize if, when the disc settles, it lies entirely within any one small square. There is a ridge around the outside edge of the board so that the disc always bounces back, cannot fall off and lies entirely within the boundary of the large square.
Prizes are awarded as follows:

Centre	(the middle square)	50c
Inner	(the eight squares surrounding the centre)	25c
Corner	(the four corner squares)	12c
Outer	(any other smaller square)	5c

When no skill is involved, the centre of the disc may be assumed to be randomly distributed over the accessible region.

a Calculate the probability in any one throw of winning:

i 50c **ii** 25c **iii** 12c **iv** 5c **v** no prize

b The proprietor wishes to make a profit in the long run, but is anxious to charge as little as possible to attract customers. He charges C cents, where C is an integer. Find the lowest value of C that will yield a profit.

Chapter 14

14

The binomial distribution

Objectives

- To define a **Bernoulli sequence**.
- To review the **binomial probability distribution**.
- To investigate the shape of the graph of the binomial probability distribution for different values of the parameters.
- To calculate and interpret the **mean**, **variance** and **standard deviation** for the binomial probability distribution.
- To use the binomial probability distribution to solve problems.

The binomial distribution is important because it has very wide application. It is concerned with situations where there are two possible outcomes, and many ‘real life’ scenarios of interest fall into this category.

For example:

- A political poll of voters is carried out. Each polled voter is asked whether or not they would vote for the present government.
- A poll of Year 12 students in Victoria is carried out. Each student is asked whether or not they watch the ABC on a regular basis.
- The effectiveness of a medical procedure is tested by selecting a group of patients and recording whether or not it is successful for each patient in the group.
- Components for an electronic device are tested to see if they are defective or not.

The binomial distribution has application in each of these examples.

We will use the binomial distribution again in Chapter 17, where we further develop our understanding of sampling.

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14A Bernoulli sequences and the binomial probability distribution

An experiment often consists of repeated trials, each of which may be considered as having only two possible outcomes. For example, when a coin is tossed, the two possible outcomes are ‘head’ and ‘tail’. When a die is rolled, the two possible outcomes are determined by the random variable of interest for the experiment. If the event of interest is a ‘six’, then the two outcomes are ‘six’ and ‘not a six’.

A **Bernoulli sequence** is the name used to describe a sequence of repeated trials with the following properties:

- Each trial results in one of two outcomes, which are usually designated as either a success, S, or a failure, F.
- The probability of success on a single trial, p, is constant for all trials (and thus the probability of failure on a single trial is $1 - p$).
- The trials are independent (so that the outcome of any trial is not affected by the outcome of any previous trial).

Example 1

Suppose that a netball player has a probability of $\frac{1}{3}$ of scoring a goal each time she attempts to goal. She repeatedly has shots for goal. Is this a Bernoulli sequence?

Solution

In this example:

- Each trial results in one of two outcomes, goal or miss.
- The probability of scoring a goal $(\frac{1}{3})$ is constant for all attempts, as is the probability of a miss $(\frac{2}{3})$.
- The trials are independent (so that the outcome of any trial is not affected by the outcome of any previous trial).

Thus, the player’s shots at goal can be considered a Bernoulli sequence.

The binomial probability distribution

The number of successes in a Bernoulli sequence of n trials is called a **binomial random variable** and is said to have a **binomial probability distribution**.

For example, consider rolling a fair six-sided die three times. Let the random variable X be the number of 3s observed.

Let T represent a 3, and let N represent not a 3. Each roll meets the conditions of a Bernoulli trial. Thus X is a binomial random variable.

Now consider all the possible outcomes from the three rolls and their probabilities.

Outcome	Number of 3s	Probability	
TTT	$X = 3$	$\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6}$	$\Pr(X = 3) = (\frac{1}{6})^3$
TTN	$X = 2$	$\frac{1}{6} \times \frac{1}{6} \times \frac{5}{6}$	$\Pr(X = 2) = 3 \times (\frac{1}{6})^2 \times \frac{5}{6}$
TNT	$X = 2$	$\frac{1}{6} \times \frac{5}{6} \times \frac{1}{6}$	
NTT	$X = 2$	$\frac{5}{6} \times \frac{1}{6} \times \frac{1}{6}$	
TNN	$X = 1$	$\frac{1}{6} \times \frac{5}{6} \times \frac{5}{6}$	$\Pr(X = 1) = 3 \times \frac{1}{6} \times (\frac{5}{6})^2$
NTN	$X = 1$	$\frac{5}{6} \times \frac{1}{6} \times \frac{5}{6}$	
NNT	$X = 1$	$\frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}$	
NNN	$X = 0$	$\frac{5}{6} \times \frac{5}{6} \times \frac{5}{6}$	$\Pr(X = 0) = (\frac{5}{6})^3$

Thus the probability distribution of X is given by the following table.

x	0	1	2	3
$\Pr(X = x)$	$\frac{125}{216}$	$\frac{75}{216}$	$\frac{15}{216}$	$\frac{1}{216}$

Instead of listing all the outcomes to find the probability distribution, we can use our knowledge of selections from Mathematical Methods Units 1 & 2 (revised in Appendix A).

Consider the probability that $X = 1$, that is, when exactly one 3 is observed. We can see from the table that there are three ways this can occur. Since the 3 could occur on the first, second or third roll of the die, we can consider this as selecting one object from a group of three, which can be done in $\binom{3}{1}$ ways.

Consider the probability that $X = 2$, that is, when exactly two 3s are observed. Again from the table there are three ways this can occur. Since the two 3s could occur on any two of the three rolls of the die, we can consider this as selecting two objects from a group of three, which can be done in $\binom{3}{2}$ ways.

This leads us to a general formula for this probability distribution:

$$\Pr(X = x) = \binom{3}{x} \left(\frac{1}{6}\right)^x \left(\frac{5}{6}\right)^{3-x} \qquad x = 0, 1, 2, 3$$

This is an example of the binomial distribution.

If the random variable X is the number of successes in n independent trials, each with probability of success p, then X has a **binomial distribution** and the rule is

$$\Pr(X = x) = \binom{n}{x} p^x (1 - p)^{n-x} \qquad x = 0, 1, \dots, n$$

where $\binom{n}{x} = \dfrac{n!}{x!\,(n-x)!}$

Example 2

Find the probability of obtaining exactly three heads when a fair coin is tossed seven times, correct to four decimal places.

Solution

Obtaining a head is considered a success here, and the probability of success on each of the seven independent trials is 0.5.

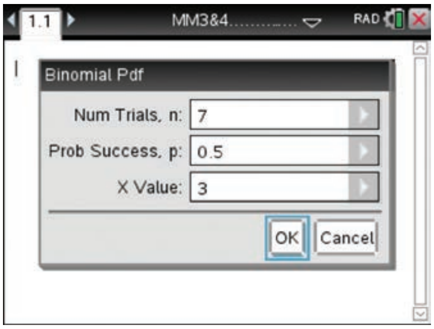
Let X be the number of heads obtained. In this case, the parameters are $n = 7$ and $p = 0.5$.

$$\begin{aligned}\Pr(X = 3) &= \binom{7}{3}(0.5)^3(1 - 0.5)^{7-3}\\ &= 35 \times (0.5)^7\\ &= 0.2734\end{aligned}$$

Using the TI-Nspire

Use (menu) > **Probability** > **Distributions** > **Binomial Pdf** and complete as shown.

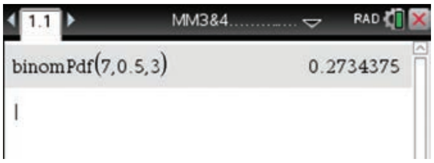
Use (tab) or ▼ to move between cells.



Num Trials, n: 7
Prob Success, p: 0.5
X Value: 3

The result is as shown.

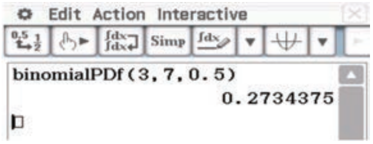
Note: You can also type in the command and the parameter values directly if preferred.



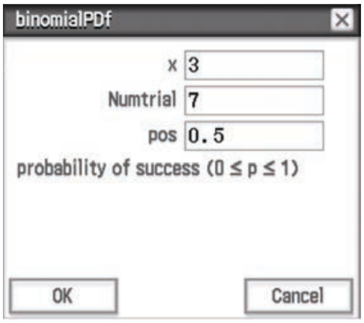
```
binomPdf(7,0.5,3)          0.2734375
```

Using the Casio ClassPad

- In $\overset{\text{Main}}{\sqrt{\alpha}}$, go to **Interactive** > **Distribution** > **Discrete** > **binomialPDf**.
- Enter the number of successes and the parameters as shown. Tap OK.



```
binomialPDf(3,7,0.5)
                0.2734375
```



x 3
Numtrial 7
pos 0.5
probability of success (0 ≤ p ≤ 1)

Example 3

The probability that a person currently in prison has ever been imprisoned before is 0.72. Find the probability that of five prisoners chosen at random at least three have been imprisoned before, correct to four decimal places.

Solution

If X is the number of prisoners who have been imprisoned before, then

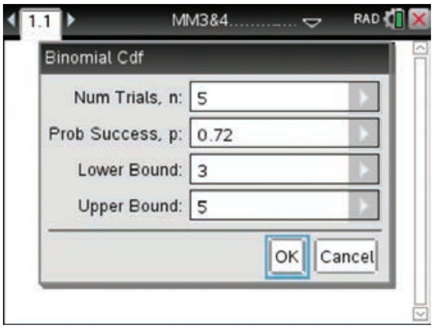
$$\Pr(X = x) = \binom{5}{x}(0.72)^x\,(0.28)^{5-x} \qquad x = 0, 1, \dots, 5$$

and so

$$\begin{aligned}\Pr(X \geq 3) &= \Pr(X = 3) + \Pr(X = 4) + \Pr(X = 5)\\ &= \binom{5}{3}(0.72)^3(0.28)^2 + \binom{5}{4}(0.72)^4(0.28)^1 + \binom{5}{5}(0.72)^5(0.28)^0\\ &= 0.8624\end{aligned}$$

Using the TI-Nspire

Use menu > **Probability** > **Distributions** > **Binomial Cdf** and complete as shown.



Binomial Cdf

Num Trials, n: 5

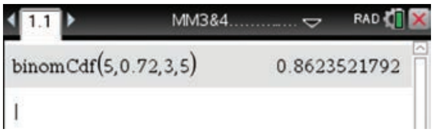
Prob Success, p: 0.72

Lower Bound: 3

Upper Bound: 5

Use tab or ▼ to move between cells.

The result is shown below.

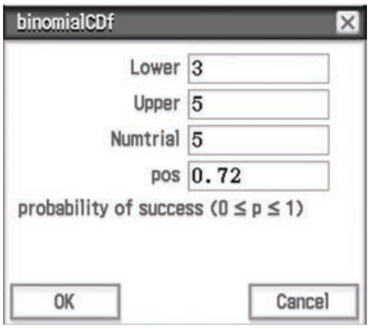


```
binomCdf(5,0.72,3,5)        0.8623521792
```

Note: You can also type in the command and the parameter values directly if preferred.

Using the Casio ClassPad

- In $\overset{\text{Main}}{\sqrt{\alpha}}$, go to **Interactive** > **Distribution** > **Discrete** > **binomialCDf**.
- Enter lower and upper bounds for the number of successes and the parameters as shown. Tap OK.


binomialCDf

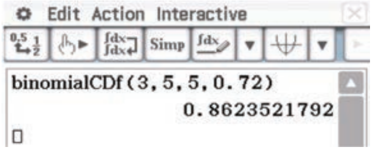
Lower 3

Upper 5

Numtrial 5

pos 0.72

probability of success (0 ≤ p ≤ 1)



```
binomialCDf(3,5,5,0.72)
                0.8623521792
```

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The binomial distribution and conditional probability

We can also use the binomial distribution to solve problems involving conditional probabilities.

Example 4

The probability of a netballer scoring a goal is 0.3. Find the probability that out of six attempts the netballer scores a goal:

a four times

b four times, given that she scores at least one goal.

Solution

Let X be the number of goals scored.
Then X has a binomial distribution with $n = 6$ and $p = 0.3$.

a
$$\begin{aligned}\Pr(X = 4) &= \binom{6}{4}(0.3)^4(0.7)^2\\ &= 15 \times 0.0081 \times 0.49\\ &= 0.059535\end{aligned}$$

b
$$\begin{aligned}\Pr(X = 4 \mid X \geq 1) &= \frac{\Pr(X = 4 \cap X \geq 1)}{\Pr(X \geq 1)}\\ &= \frac{\Pr(X = 4)}{\Pr(X \geq 1)}\\ &= \frac{0.059535}{1 - 0.7^6} \qquad \text{since } \Pr(X \geq 1) = 1 - \Pr(X = 0)\\ &= 0.0675\end{aligned}$$

Section summary

- A **Bernoulli sequence** is a sequence of trials with the following properties:
  - Each trial results in one of two outcomes, which are usually designated as either a success, S, or a failure, F.
  - The probability of success on a single trial, p, is constant for all trials (and thus the probability of failure on a single trial is $1 - p$).
  - The trials are independent (so that the outcome of any trial is not affected by the outcome of any previous trial).
- The number of successes, X, in a Bernoulli sequence of n trials is called a **binomial random variable** and has a **binomial probability distribution**:

$$\Pr(X = x) = \binom{n}{x} p^x (1 - p)^{n-x} \qquad x = 0, 1, \dots, n$$

where $\binom{n}{x} = \dfrac{n!}{x!\,(n - x)!}$

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Exercise 14A

Skillsheet
Example 1

1 Which of the following describes a Bernoulli sequence?

a tossing a fair coin many times

b drawing balls from an urn containing five red and three black balls, replacing the chosen ball each time

c selecting people at random from the population and noting their age

d selecting people at random from the population and noting their sex, male or female

Example 2

2 Find the probability of obtaining exactly four heads when a fair coin is tossed seven times, correct to four decimal places.

3 For a binomial distribution with $n = 4$ and $p = 0.2$, find the probability of:

a three successes **b** four successes.

4 For a binomial distribution with $n = 5$ and $p = 0.4$, find the probability of:

a no successes **b** three successes **c** five successes.

5 Suppose that a fair coin is tossed three times, and the number of heads observed.

a Write down a general rule for the probability distribution of the number of heads.

b Use the rule to calculate the probability of observing two heads.

6 Suppose that X is the number of male children born into a family of six children. Assume that the distribution of X is binomial, with probability of success 0.48.

a Write down a general rule for the probability distribution of the number of male children.

b Use the rule to calculate the probability that a family with six children will have exactly two male children.

Example 3

7 A fair die is rolled six times and the number of 2s noted. Find the probability of:

a exactly three 2s **b** more than three 2s **c** at least three 2s.

8 Jo knows that each ticket has a probability of 0.1 of winning a prize in a lucky ticket competition. Suppose she buys 10 tickets.

a Write down a general rule for the probability distribution of the number of winning tickets.

b Use the rule to calculate the probability that Jo has:

i no wins **ii** at least one win.

9 Suppose that the probability that a person selected at random is left-handed is always 0.2. If 11 people are selected at random for the cricket team:

a Write down a general rule for the probability distribution of the number left-handed people on the team.

b Use the rule to calculate the probability of selecting:

i exactly two left-handers **ii** no left-handers **iii** at least one left-hander.

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10 In a particular city, the probability of rain falling on any given day is $\frac{1}{5}$.

a Write down a general rule for the probability distribution of the number of days of rain in a week.

b Use the rule to calculate the probability that in a particular week rain will fall:

i every day **ii** not at all **iii** on two or three days.

11 The probability of a particular drug causing side effects in a person is 0.2. What is the probability that at least two people in a random sample of 10 people will experience side effects?

12 Records show that $x\%$ of people will pass their driver's license on the first attempt. If six students attempt their driver's license, write down in terms of x the probability that:

a all six students pass **b** only one fails **c** no more than two fail.

13 A supermarket has four checkouts. A customer in a hurry decides to leave without making a purchase if all the checkouts are busy. At that time of day the probability of each checkout being free is 0.25. Assuming that whether or not a checkout is busy is independent of any other checkout, calculate the probability that the customer will make a purchase.

14 A fair die is rolled 50 times. Find the probability of observing:

a exactly 10 sixes **b** no more than 10 sixes **c** at least 10 sixes.

15 Find the probability of getting at least nine successes in 100 trials for which the probability of success is $p = 0.1$.

16 A fair coin is tossed 50 times. If X is the number of heads observed, find:

a $\Pr(X = 25)$ **b** $\Pr(X \leq 25)$ **c** $\Pr(X \leq 10)$ **d** $\Pr(X \geq 40)$

17 A survey of the population in a particular city found that 40% of people regularly participate in sport. What is the probability that fewer than half of a random sample of six people regularly participate in sport?

18 An examination consists of six multiple-choice questions. Each question has four possible answers. At least three correct answers are required to pass the examination. Suppose that a student guesses the answer to each question.

a What is the probability the student guesses every question correctly?

b What is the probability the student will pass the examination?

Example 4 **19** The manager of a shop knows from experience that 60% of her customers will use a credit card to pay for their purchases. Find the probability that:

a the next three customers will use a credit card, and the three after that will not

b three of the next six customers will use a credit card

c at least three of the next six customers will use a credit card

d exactly three of the next six customers will use a credit card, given that at least three of the next six customers use a credit card.

20 A multiple-choice test has eight questions, each with five possible answers, only one of which is correct. Find the probability that a student who guesses the answer to every question will have:

a no correct answers

b six or more correct answers

c every question correct, given they have six or more correct answers.

21 The probability that a full forward in Australian Rules football will kick a goal from outside the 50-metre line is 0.15. If the full forward has 10 kicks at goal from outside the 50-metre line, find the probability that he will:

a kick a goal every time

b kick at least one goal

c kick more than one goal, given that he kicks at least one goal.

22 A multiple-choice test has 20 questions, each with five possible answers, only one of which is correct. Find the probability that a student who guesses the answer to every question will have:

a no correct answers

b 10 or more correct answers

c at least 12 correct answers, given they have 10 or more correct answers.

14B The graph, expectation and variance of a binomial distribution

We looked at the properties of discrete probability distributions in Chapter 13. We now consider these properties for the binomial distribution.

▶ The graph of a binomial probability distribution

As discussed in Chapter 13, a probability distribution may be represented as a rule, a table or a graph. We now investigate the shape of the graph of a binomial probability distribution for different values of the parameters n and p.

A method for plotting a binomial distribution with a CAS calculator can be found in the calculator appendices in the Interactive Textbook.

Example 5

Construct and compare the graph of the binomial probability distribution for 20 trials ($n = 20$) with probability of success:

a $p = 0.2$

b $p = 0.5$

c $p = 0.8$

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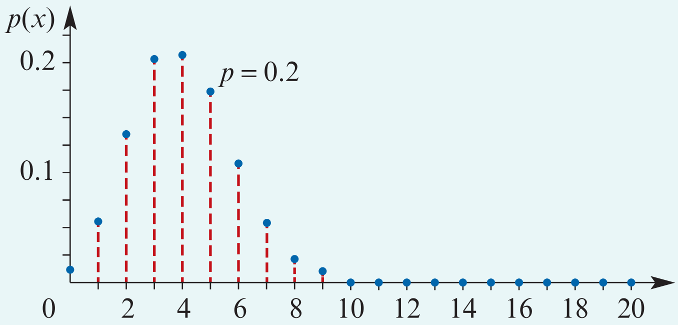
Solution

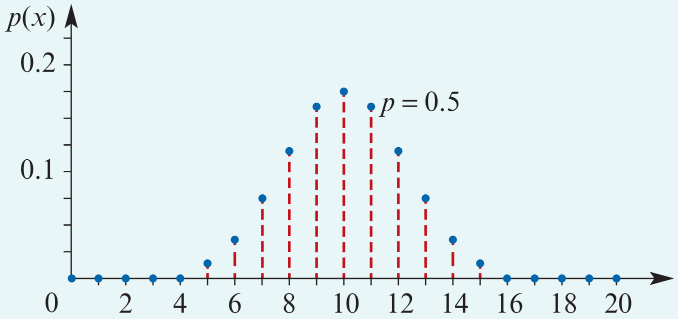
a For $p = 0.2$, the graph is positively skewed.

Mostly from 1 to 8 successes will be observed in 20 trials.



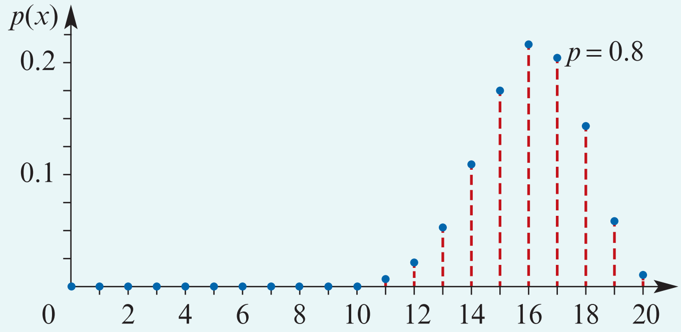
b For $p = 0.5$, the graph is symmetrical (as the probability of success is the same as the probability of failure).

Mostly from 6 to 14 successes will be observed in 20 trials.



c For $p = 0.8$, the graph is negatively skewed.

Mostly from 12 to 19 successes will be observed in 20 trials.



Expectation and variance

How many heads would you expect to obtain, on average, if a fair coin was tossed 10 times?

While the exact number of heads in the 10 tosses would vary, and could theoretically take values from 0 to 10, it seems reasonable that the long-run average number of heads would be 5. It turns out that this is correct. That is, for a binomial random variable X with $n = 10$ and $p = 0.5$,

$$\mathrm{E}(X) = \sum_x x \cdot \Pr(X = x) = 5$$

In general, the expected value of a binomial random variable is equal to the number of trials multiplied by the probability of success. The variance can also be calculated from the parameters n and p.

If X is the number of successes in n trials, each with probability of success p, then the expected value and the variance of X are given by

$$\begin{aligned} \mathrm{E}(X) &= np \\ \mathrm{Var}(X) &= np(1-p) \end{aligned}$$

While it is not necessary in this course to be familiar with the derivations of these formulas, they are included for completeness in the final section of this chapter.

Example 6

An examination consists of 30 multiple-choice questions, each question having three possible answers. A student guesses the answer to every question. Let X be the number of correct answers.

a How many will she expect to get correct? That is, find $\mathrm{E}(X) = \mu$.

b Find $\mathrm{Var}(X)$.

Solution

The number of correct answers, X, is a binomial random variable with parameters $n = 30$ and $p = \frac{1}{3}$.

a The student has an expected result of $\mu = np = 10$ correct answers. (This is not enough to pass if the pass mark is 50%.)

b
$$\begin{aligned} \mathrm{Var}(X) &= np(1-p) \\ &= 30 \times \frac{1}{3} \times \frac{2}{3} = \frac{20}{3} \end{aligned}$$

Example 7

The probability of contracting influenza this winter is known to be 0.2. Of the 100 employees at a certain business, how many would the owner expect to get influenza? Find the standard deviation of the number who will get influenza and calculate $\mu \pm 2\sigma$. Interpret the interval $[\mu - 2\sigma, \mu + 2\sigma]$ for this example.

Solution

The number of employees who get influenza is a binomial random variable, X, with parameters $n = 100$ and $p = 0.2$.

The owner will expect $\mu = np = 20$ of the employees to contract influenza.

The variance is

$$\begin{aligned} \sigma^2 &= np(1-p) \\ &= 100 \times 0.2 \times 0.8 \\ &= 16 \end{aligned}$$

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Hence the standard deviation is

$$\sigma = \sqrt{16} = 4$$

Thus

$$\begin{aligned}\mu \pm 2\sigma &= 20 \pm (2 \times 4)\\ &= 20 \pm 8\end{aligned}$$

The owner of the business knows there is a probability of about 0.95 that from 12 to 28 of the employees will contract influenza this winter.

Section summary

If X is the number of successes in n trials, each with probability of success p, then the expected value and the variance of X are given by

- $\mathrm{E}(X) = np$
- $\mathrm{Var}(X) = np(1-p)$

Exercise 14B

Example 5

1 Plot the graph of the probability distribution

$$\Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad x = 0, 1, \dots, n$$

for $n = 8$ and $p = 0.25$.

2 Plot the graph of the probability distribution

$$\Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad x = 0, 1, \dots, n$$

for $n = 12$ and $p = 0.35$.

3 **a** Plot the graph of the probability distribution

$$\Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad x = 0, 1, \dots, n$$

for $n = 10$ and $p = 0.2$.

b On the same axes, plot the graph of

$$\Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad x = 0, 1, \dots, n$$

for $n = 10$ and $p = 0.8$, using a different plotting symbol.

c Compare the two distributions.

d Comment on the effect of the value of p on the shape of the distribution.

Example 6

4 Find the mean and variance of the binomial random variables with parameters:

a $n = 25,\ p = 0.2$

b $n = 10,\ p = 0.6$

c $n = 500,\ p = \frac{1}{3}$

d $n = 40,\ p = 20\%$

5 A fair die is rolled six times.

a Find the expected value for the number of sixes obtained.

b Find the probability that more than the expected number of sixes is obtained.

6 The survival rate for a certain disease is 75%. Of the next 50 people who contract the disease, how many would you expect would survive?

7 A binomial random variable X has mean 12 and variance 9. Find the parameters n and p, and hence find $\Pr(X = 7)$.

8 A binomial random variable X has mean 30 and variance 21. Find the parameters n and p, and hence find $\Pr(X = 20)$.

Example 7

9 A fair coin is tossed 20 times. Find the mean and standard deviation of the number of heads obtained and calculate $\mu \pm 2\sigma$. Interpret the interval $[\mu - 2\sigma, \mu + 2\sigma]$ for this example.

10 Records show that 60% of the students in a certain state attend government schools. If a group of 200 students are to be selected at random, find the mean and standard deviation of the number of students in the group who attend government schools, and calculate $\mu \pm 2\sigma$. Interpret the interval $[\mu - 2\sigma, \mu + 2\sigma]$ for this sample.

14C Finding the sample size

While we can never be absolutely certain about the outcome of a random experiment, sometimes we are interested in knowing what size sample would be required to observe a certain outcome. For example, how many times do you need to roll a die to be reasonably sure of observing a six, or how many lotto tickets must you buy to be reasonably sure that you will win a prize?

Example 8

The probability of winning a prize in a game of chance is 0.48.

a What is the least number of games that must be played to ensure that the probability of winning at least once is more than 0.95?

b What is the least number of games that must be played to ensure that the probability of winning at least twice is more than 0.95?

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Solution

Since the probability of winning each game is the same each time the game is played, this is an example of a binomial distribution, with the probability of success $p = 0.48$.

a The required answer is the smallest value of n such that $\Pr(X \geq 1) > 0.95$.

$$\begin{aligned} & \Pr(X \geq 1) > 0.95 \\ \Leftrightarrow \quad & 1 - \Pr(X = 0) > 0.95 \\ \Leftrightarrow \quad & \Pr(X = 0) < 0.05 \\ \Leftrightarrow \quad & 0.52^n < 0.05 \quad \text{since } \Pr(X = 0) = 0.52^n \end{aligned}$$

This can be solved by taking logarithms of both sides:

$$\begin{aligned} \log_e(0.52^n) &< \log_e(0.05) \\ n \log_e(0.52) &< \log_e(0.05) \\ \therefore \quad n &> \frac{\log_e(0.05)}{\log_e(0.52)} \approx 4.58 \end{aligned}$$

Thus the game must be played at least five times to ensure that the probability of winning at least once is more than 0.95.

b The required answer is the smallest value of n such that $\Pr(X \geq 2) > 0.95$, or equivalently, such that

$$\Pr(X < 2) < 0.05$$

We have

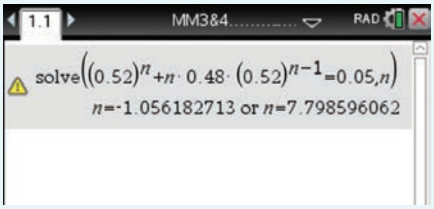
$$\begin{aligned} \Pr(X < 2) &= \Pr(X = 0) + \Pr(X = 1) \\ &= \binom{n}{0} 0.48^0 0.52^n + \binom{n}{1} 0.48^1 0.52^{n-1} \\ &= 0.52^n + 0.48n(0.52)^{n-1} \end{aligned}$$

So the answer is the smallest value of n such that

$$0.52^n + 0.48n(0.52)^{n-1} < 0.05$$

This equation cannot be solved algebraically; but a CAS calculator can be used to find the solution $n > 7.7985\ldots$.

Thus the game must be played at least eight times to ensure that the probability of winning at least twice is more than 0.95.



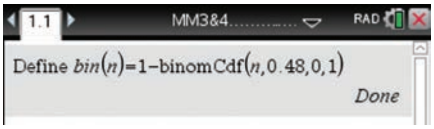
$\text{solve}\left((0.52)^n + n \cdot 0.48 \cdot (0.52)^{n-1} = 0.05, n\right)$

$n=-1.056182713$ or $n=7.798596062$

The following calculator inserts give a solution to part **b** of Example 8. Similar techniques can be used for part **a**. For further explanation, refer to the calculator appendices in the Interactive Textbook.

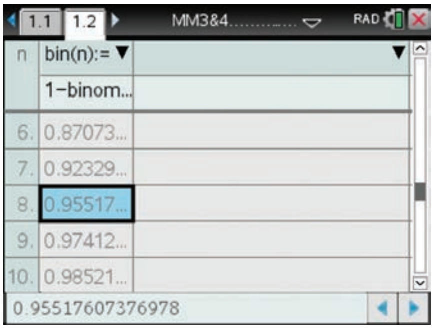
Using the TI-Nspire

To find the smallest value of n such that $\Pr(X \geq 2) > 0.95$, where $p = 0.48$:



Define bin(n)=1−binomCdf(n,0.48,0,1)
Done

- Define the binomial CDF as shown.
  The last two parameters are the lower and upper bounds (inclusive) of the X value.



n	bin(n):=
	1−binom...
6.	0.87073...
7.	0.92329...
8.	0.95517...
9.	0.97412...
10.	0.98521...

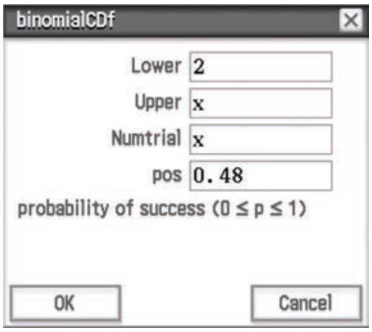
0.95517607376978

- Insert a **Lists & Spreadsheet** page. Press ctrl T to show the table of values.
- Scroll through the table to find where the probability is greater than 0.95. Hence $n = 8$.

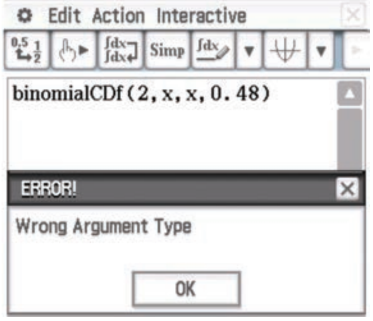
Using the Casio ClassPad

To find the smallest value of n such that $\Pr(X \geq 2) > 0.95$, where $p = 0.48$:

- In Main $\sqrt{\alpha}$, go to **Interactive** > **Distribution** > **Discrete** > **binomialCDf**.
- Enter bounds for the number of successes and the parameters as shown below.



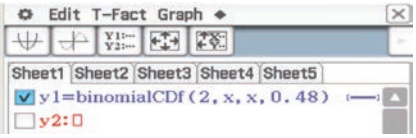
Lower: 2
Upper: x
Numtrial: x
pos: 0.48
probability of success (0 ≤ p ≤ 1)



binomialCDf(2, x, x, 0.48)

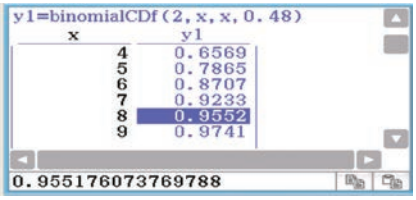
Wrong Argument Type

- Highlight and copy the expression.
- Go to the main menu Menu and select the **Graph & Table** application.
- Paste the expression in $y1$.



☑ y1=binomialCDf(2, x, x, 0.48)
☐ y2:□

- Tap on the table icon.
- Scroll down the table of values until the probability first exceeds 0.95: the answer is $n = 8$.



y1=binomialCDf(2, x, x, 0.48)

x	y1
4	0.6569
5	0.7865
6	0.8707
7	0.9233
8	0.9552
9	0.9741

0.955176073769788

Note: To view larger values of x in the table, tap the icon and set **End** at a larger value.

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Exercise 14C

Skillsheet

Example 8

1 The probability of a target shooter hitting the bullseye on any one shot is 0.2.

  a If the shooter takes five shots at the target, find the probability of:

    i missing the bullseye every time

    ii hitting the bullseye at least once.

  b What is the smallest number of shots the shooter should make to ensure a probability of more than 0.95 of hitting the bullseye at least once?

  c What is the smallest number of shots the shooter should make to ensure a probability of more than 0.95 of hitting the bullseye at least twice?

2 The probability of winning a prize with a lucky ticket on a wheel of fortune is 0.1.

  a If a person buys 10 lucky tickets, find the probability of:

    i winning twice

    ii winning at least once.

  b What is the smallest number of tickets that should be bought to ensure a probability of more than 0.7 of winning at least once?

3 Rex is shooting at a target. His probability of hitting the target is 0.6. What is the minimum number of shots needed for the probability of Rex hitting the target exactly five times to be more than 25%?

4 Janet is selecting chocolates at random out of a box. She knows that 20% of the chocolates have hard centres. What is the minimum number of chocolates she needs to select to ensure that the probability of choosing exactly three hard centres is more than 10%?

5 The probability of winning a prize in a game of chance is 0.35. What is the fewest number of games that must be played to ensure that the probability of winning at least twice is more than 0.9?

6 Geoff has determined that his probability of hitting ‘4’ off any ball when playing cricket is 0.07. What is the fewest number of balls he must face to ensure that the probability of hitting more than one ‘4’ is more than 0.8?

7 Monique is practising goaling for netball. She knows from past experience that her chance of making any one shot is about 70%. Her coach has asked her to keep practising until she scores 50 goals. How many shots would she need to attempt to ensure that the probability of scoring at least 50 goals is more than 0.99?

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14D Proofs for the expectation and variance

In this section we give proofs of three important results on the binomial distribution.

The probabilities of a binomial distribution sum to 1.

Proof The binomial theorem, discussed in Appendix A, states that

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Now, using the binomial theorem, the sum of the probabilities for a binomial random variable X with parameters n and p is given by

$$\begin{aligned}\sum_{x=0}^{n} \Pr(X=x) &= \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} \\ &= ((1-p)+p)^n = (1)^n = 1\end{aligned}$$

Expected value

If X is a binomial random variable with parameters n and p, then $\mathrm{E}(X) = np$.

Proof By the definition of expected value:

$$\begin{aligned}\mathrm{E}(X) &= \sum_{x=0}^{n} x \cdot \binom{n}{x} p^x (1-p)^{n-x} && \text{by the distribution formula} \\ &= \sum_{x=0}^{n} x \cdot \left(\frac{n!}{x!\,(n-x)!}\right) p^x (1-p)^{n-x} && \text{expanding } \tbinom{n}{x} \\ &= \sum_{x=1}^{n} x \cdot \left(\frac{n}{x!\,(n-x)!}\right) p^x (1-p)^{n-x} && \text{since the } x = 0 \text{ term is zero} \\ &= \sum_{x=1}^{n} x \cdot \left(\frac{n!}{x(x-1)!\,(n-x)!}\right) p^x (1-p)^{n-x} && \text{since } x! = x(x-1)! \\ &= \sum_{x=1}^{n} \left(\frac{n!}{(x-1)!\,(n-x)!}\right) p^x (1-p)^{n-x} && \text{cancelling the } x\text{s}\end{aligned}$$

This expression is very similar to the probability function for a binomial random variable, and we know the probabilities sum to 1. Taking out factors of n and p from the expression and letting $z = x - 1$ gives

$$\begin{aligned}\mathrm{E}(X) &= np \sum_{x=1}^{n} \binom{n-1}{x-1} p^{x-1} (1-p)^{n-x} \\ &= np \sum_{z=0}^{n-1} \binom{n-1}{z} p^{z} (1-p)^{n-1-z}\end{aligned}$$

Note that this sum corresponds to the sum of all the values of the probability function for a binomial random variable Z, which is the number of successes in $n - 1$ trials each with probability of success p. Therefore the sum equals 1, and so

$$\mathrm{E}(X) = np$$

Variance

If X is a binomial random variable with parameters n and p, then $\mathrm{Var}(X) = np(1-p)$.

Proof The variance of the binomial random variable X may be found using

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - \mu^2, \qquad \text{where } \mu = np$$

Thus, to find the variance, we need to determine $\mathrm{E}(X^2)$:

$$\begin{aligned}\mathrm{E}(X^2) &= \sum_{x=0}^{n} x^2 \binom{n}{x} p^x (1-p)^{n-x} \\ &= \sum_{x=0}^{n} x^2 \left(\frac{n!}{x!\,(n-x)!}\right) p^x (1-p)^{n-x}\end{aligned}$$

But x^2 is not a factor of $x!$ and so we cannot proceed as in the previous proof for expected value.

The strategy used here is to determine $\mathrm{E}[X(X-1)]$:

$$\begin{aligned}\mathrm{E}[X(X-1)] &= \sum_{x=0}^{n} x(x-1) \binom{n}{x} p^x (1-p)^{n-x} \\ &= \sum_{x=0}^{n} x(x-1) \left(\frac{n!}{x!\,(n-x)!}\right) p^x (1-p)^{n-x} \\ &= \sum_{x=2}^{n} x(x-1) \left(\frac{n!}{x!\,(n-x)!}\right) p^x (1-p)^{n-x}\end{aligned}$$

since the first and second terms of the sum equal zero (when $x = 0$ and $x = 1$).

Taking out a factor of $n(n-1)p^2$ and letting $z = x - 2$ gives

$$\begin{aligned}\mathrm{E}[X(X-1)] &= n(n-1)p^2 \sum_{x=2}^{n} \left(\frac{(n-2)!}{(x-2)!\,(n-x)!}\right) p^{x-2} (1-p)^{n-x} \\ &= n(n-1)p^2 \sum_{z=0}^{n-2} \binom{n-2}{z} p^z (1-p)^{n-2-z}\end{aligned}$$

Now the sum corresponds to the sum of all the values of the probability function for a binomial random variable Z, which is the number of successes in $n-2$ trials each with probability of success p, and is thus equal to 1. Hence

$$\begin{aligned}\mathrm{E}[X(X-1)] &= n(n-1)p^2 \\ \therefore \quad \mathrm{E}(X^2) - \mathrm{E}(X) &= n(n-1)p^2 \\ \therefore \quad \mathrm{E}(X^2) &= n(n-1)p^2 + \mathrm{E}(X) \\ &= n(n-1)p^2 + np\end{aligned}$$

This is an expression for $\mathrm{E}(X^2)$ in terms of n and p, as required. Thus

$$\begin{aligned}\mathrm{Var}(X) &= \mathrm{E}(X^2) - \mu^2 \\ &= n(n-1)p^2 + np - (np)^2 \\ &= np(1-p)\end{aligned}$$

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Chapter summary

- A **Bernoulli sequence** is a sequence of trials with the following properties:
  - Each trial results in one of two outcomes, which are usually designated as either a success, S, or a failure, F.
  - The probability of success on a single trial, p, is constant for all trials (and thus the probability of failure on a single trial is $1 - p$).
  - The trials are independent (so that the outcome of any trial is not affected by the outcome of any previous trial).

- If X is the number of successes in n Bernoulli trials, each with probability of success p, then X is called a **binomial random variable** and is said to have a **binomial probability distribution** with parameters n and p. The probability of observing x successes in the n trials is given by

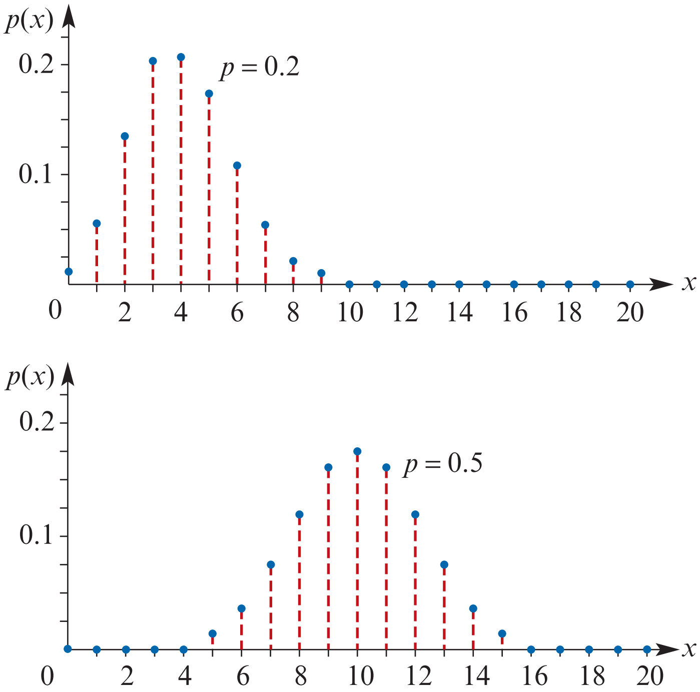
$$\Pr(X = x) = \binom{n}{x} p^x (1 - p)^{n-x} \qquad x = 0, 1, \dots, n$$

where $\binom{n}{x} = \dfrac{n!}{x!\,(n - x)!}$

- If X has a binomial probability distribution with parameters n and p, then

$$\mathrm{E}(X) = np$$
$$\mathrm{Var}(X) = np(1 - p)$$

- The shape of the graph of a binomial probability function depends on the values of n and p.



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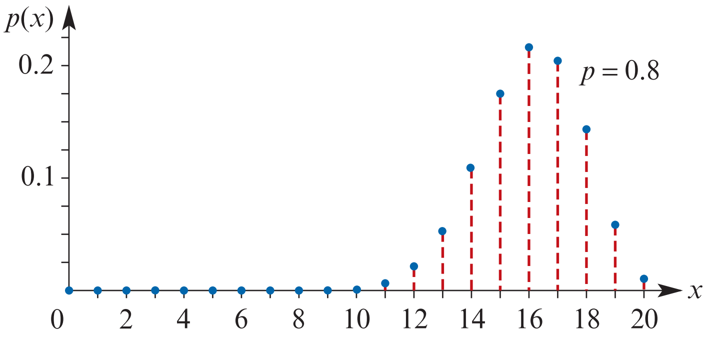

Technology-free questions

1 If X is a binomial random variable with parameters $n = 4$ and $p = \frac{1}{3}$, find:

a $\Pr(X = 0)$ **b** $\Pr(X = 1)$ **c** $\Pr(X \leq 1)$ **d** $\Pr(X \geq 1)$

2 A salesperson knows that 60% of the people who enter a particular shop will make a purchase. What is the probability that of the next three people who enter the shop exactly two will make a purchase?

3 If 10% of patients fail to improve on a certain medication, find the probability that of five patients selected at random one or more will fail to show improvement.

4 A machine has a probability of 0.1 of manufacturing a defective part.

a What is the expected number of defective parts in a random sample of 20 parts manufactured by the machine?

b What is the standard deviation of the number of defective parts?

5 An experiment consists of four independent trials. Each trial results in either a success or a failure. The probability of success in a trial is p. Find the probability of each of the following in terms of p:

a no successes **b** one success **c** at least one success

d four successes **e** at least two successes.

6 A coin is tossed 10 times. The probability of three heads is $m \times (\frac{1}{2})^{10}$. State the value of m.

7 An experiment consists of five independent trials. Each trial results in either a success or a failure. The probability of success in a trial is p. Find, in terms of p, the probability of exactly one success given at least one success.

8 A die is rolled five times. What is the probability of obtaining an even number on the uppermost face on exactly three of the rolls?

9 In a particular city, the probability of rain on any day in June is $\frac{1}{5}$. What is the probability of it raining on three of five days?

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Multiple-choice questions

1 A coin is biased such that the probability of a head is 0.6. The probability that exactly three heads will be observed when the coin is tossed five times is

A 0.6×3 **B** $(0.6)^3$ **C** $(0.6)^3(0.4)^2$ **D** $10 \times (0.6)^3(0.4)^2$ **E** $\binom{5}{3}(0.6)^5$

2 The probability that the 8:25 train arrives on time is 0.35. What is the probability that the train is on time at least once during a working week (Monday to Friday)?

A $1 - (0.65)^5$ **B** $(0.35)^5$ **C** $1 - (0.35)^5$
D $5 \times (0.35)^1(0.65)^4$ **E** $(0.65)^5$

3 A fair die is rolled four times. The probability that a number greater than 4 is observed on two occasions is

A $\frac{1}{4}$ **B** $\frac{16}{81}$ **C** $\frac{1}{9}$ **D** $\frac{1}{81}$ **E** $\frac{8}{27}$

4 The probability that a person in a certain town has a tertiary education is 0.4. What is the probability that, if 80 people are chosen at random from this town, less than 30 will have a tertiary education?

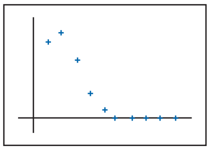
A 0.7139 **B** 0.2861 **C** 0.0827 **D** 0.3687 **E** 0.3750

5 If X is a binomial random variable with parameters $n = 18$ and $p = \frac{1}{3}$, then the mean and variance of X are closest to

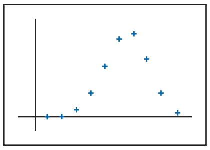
A $\mu = 6, \ \sigma^2 = 4$ **B** $\mu = 9, \ \sigma^2 = 4$ **C** $\mu = 6, \ \sigma^2 = 2$
D $\mu = 6, \ \sigma^2 = 16$ **E** $\mu = 18, \ \sigma^2 = 6$

6 Which one of the following best represents the shape of the probability distribution of a binomial random variable X with 10 independent trials and probability of success 0.7?

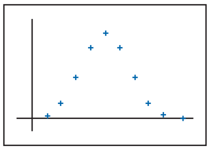
A



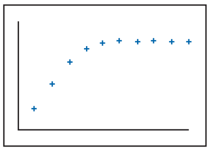
B



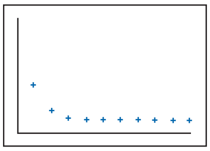
C



D



E



7 Suppose that X is a binomial random variable with mean $\mu = 10$ and standard deviation $\sigma = 2$. The probability of success, p, in any trial is

A 0.4 **B** 0.5 **C** 0.6 **D** 0.7 **E** 0.8

8 Suppose that X is the number of heads observed when a coin known to be biased towards heads is tossed 10 times. If $\text{Var}(X) = 1.875$, then the probability of a head on any one toss is

A 0.25 **B** 0.55 **C** 0.75 **D** 0.65 **E** 0.80

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Questions 9 and 10 refer to the following information.

The probability of Thomas beating William in a set of tennis is 0.24, and Thomas and William decide to play a set of tennis every day for n days.

9 What is the fewest number of days on which they should play to ensure that the probability of Thomas winning at least one set is more than 0.95?

A 7 **B** 8 **C** 9 **D** 10 **E** 11

10 What is the fewest number of days on which they should play to ensure that the probability of Thomas winning at least two sets is more than 0.95?

A 12 **B** 18 **C** 17 **D** 21 **E** 14

Extended-response questions

1 In a test to detect learning disabilities, a child is asked 10 questions, each of which has possible answers labelled A, B and C. Children with a disability of type 1 almost always answer A or B on every question, while children with a disability of type 2 almost always answer C on every question. Children without either disability have an equal chance of answering A, B or C for each question.

- **a** What is the probability that the answers given by a child without either disability will be all As and Bs, thereby indicating a type 1 disability?
- **b** A child is further tested for type 2 disability if he or she answers C five or more times. What is the probability that a child without either disability will test positive for type 2 disability?

2 An inspector takes a random sample of 10 items from a very large batch. If none of the items is defective, he accepts the batch; otherwise, he rejects the batch. What is the probability that a batch is accepted if the fraction of defective items is 0, 0.01, 0.02, 0.05, 0.1, 0.2, 0.5, 1? Plot these probabilities against the corresponding fraction defective. Is the inspection method a good one or not?

3 It has been found in the past that 4% of the CDs produced in a certain factory are defective. A sample of 10 CDs is drawn randomly from each hour's production and the number of defective CDs is noted.

- **a** What percentage of these hourly samples would contain at least two defective CDs?
- **b** Find the mean and standard deviation of the number of defective CDs in a sample, and calculate $\mu \pm 2\sigma$.
- **c** A particular sample is found to contain three defective CDs. Would this cause you to have doubts about the production process?

4 A pizza company claims that they deliver 90% of orders within 30 minutes. In a particular 2-hour period, the supervisor notes that there are 67 orders, and of these 12 orders are delivered late. If the company claim is correct, and 90% of orders are delivered on time, what is the probability that at least 12 orders are delivered late?

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5 **a** A sample of six objects is to be drawn from a large population in which 20% of the objects are defective. Find the probability that the sample contains:

i three defectives **ii** fewer than three defectives.

b Another large population contains a proportion p of defective items.

i Write down an expression in terms of p for P, the probability that a sample of six items contains exactly two defectives.

ii By differentiating to find $\frac{dP}{dp}$, show that P is greatest when $p = \frac{1}{3}$.

6 Groups of six people are chosen at random and the number, x, of people in each group who normally wear glasses is recorded. The table gives the results from 200 groups.

Number wearing glasses, x	0	1	2	3	4	5	6
Number of occurrences	17	53	65	45	18	2	0

a Calculate, from the above data, the mean value of x.

b Assuming that the situation can be modelled by a binomial distribution having the same mean as the one calculated above, state the appropriate values for the binomial parameters n and p.

c Calculate the theoretical frequencies corresponding to those in the table.

7 A sampling inspection scheme is devised as follows. A sample of size 10 is drawn at random from a large batch of articles and all 10 articles are tested. If the sample contains fewer than two faulty articles, the batch is accepted; if the sample contains three or more faulty articles, the batch is rejected; but if the sample contains exactly two faulty articles, a second sample of size 10 is taken and tested. If this second sample contains no faulty articles, the batch is accepted; but if it contains any faulty articles, the batch is rejected. Previous experience has shown that 5% of the articles in a batch are faulty.

a Find the probability that the batch is accepted after the first sample is taken.

b Find the probability that the batch is rejected.

c Find the expected number of articles to be tested.

8 Assume that dates of birth in a large population are distributed such that the probability of a randomly chosen person's birthday being in any particular month is $\frac{1}{12}$.

a Find the probability that of six people chosen at random exactly two will have a birthday in January.

b Find the probability that of eight people at least one will have a birthday in January.

c N people are chosen at random. Find the least value of N such that the probability that at least one will have a birthday in January exceeds 0.9.

9 Suppose that, in flight, aeroplane engines fail with probability q, independently of each other, and that a plane will complete the flight successfully if at least half of its engines are still working. For what values of q is a two-engine plane to be preferred to a four-engine one?

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15

Continuous random variables and their probability distributions

Objectives

- To introduce **continuous random variables**.
- To use **probability density functions** to specify the distributions for continuous random variables.
- To relate the probability for an interval to an area under the graph of a probability density function.
- To use calculus to find probabilities for intervals from a probability density function.
- To use technology to find probabilities for intervals from a probability density function.
- To calculate and interpret the **expectation** (**mean**), **median**, **variance** and **standard deviation** for a continuous random variable.

In Chapter 13 we studied discrete random variables, that is, random variables that take only a countable number of values. Most of the examples in that chapter involved the natural numbers: for example, the number of heads observed when tossing a coin several times.

In this chapter we extend our knowledge to include continuous random variables, which can take any value in an interval of the real number line. Examples include the time taken to complete a puzzle and the height of an adult. When considering the heights of adults, the range of values could be from 56 cm to 251 cm, and in principle the measurement could be any value in this interval.

We also introduce the concept of the probability density function to describe the distribution of a continuous random variable. We shall see that probabilities associated with a continuous random variable are described by areas under the probability density function, and thus integration is an important skill required to determine these probabilities.

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15A Continuous random variables

A **continuous random variable** is one that can take any value in an interval of the real number line. For example, if X is the random variable which takes its values as 'distance in metres that a parachutist lands from a marker', then X is a continuous random variable, and here the values which X may take are the non-negative real numbers.

▶ An example of a continuous random variable

A continuous random variable has no limit as to the accuracy with which it can be measured. For example, let W be the random variable with values 'a person's weight in kilograms' and let W_i be the random variable with values 'a person's weight in kilograms measured to the ith decimal place'.

$$\begin{aligned}
\text{Then}\quad & W_0 = 83 && \text{implies} && 82.5 \le W < 83.5 \\
& W_1 = 83.3 && \text{implies} && 83.25 \le W < 83.35 \\
& W_2 = 83.28 && \text{implies} && 83.275 \le W < 83.285 \\
& W_3 = 83.281 && \text{implies} && 83.2805 \le W < 83.2815
\end{aligned}$$

and so on. Thus, the random variable W cannot take an exact value, since it is always rounded to the limits imposed by the method of measurement used. Hence, the probability of W being exactly equal to a particular value is zero, and this is true for all continuous random variables.

That is,

$$\Pr(W = w) = 0 \quad \text{for all } w$$

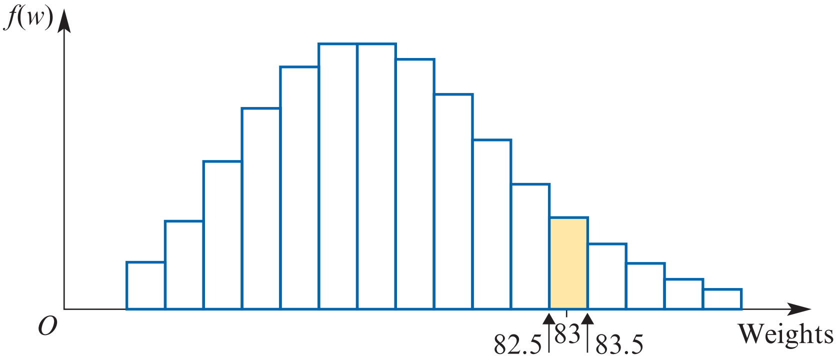
In practice, considering W_i taking a particular value is equivalent to W taking a value in an appropriate interval.

Thus, from above:

$$\Pr(W_0 = 83) = \Pr(82.5 \le W < 83.5)$$

To determine the value of this probability, you could begin by measuring the weight of a large number of randomly chosen people, and determine the proportion of the people in the group who have weights in this interval.

Suppose after doing this a histogram of weights was obtained as shown.



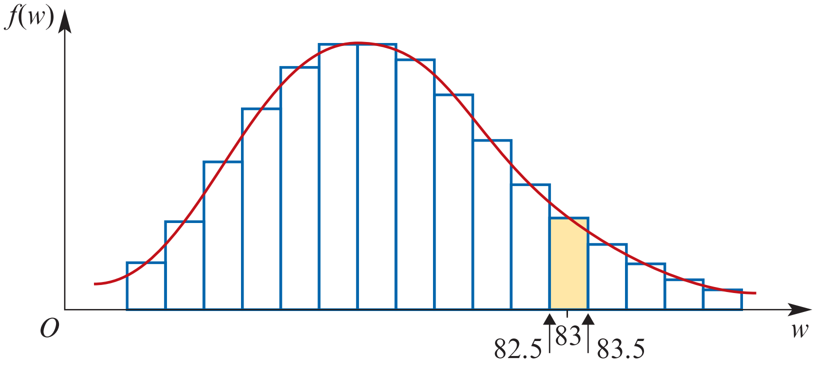
From this histogram:

$$\begin{aligned}\Pr(W_0 = 83) &= \Pr(82.5 \le W < 83.5)\\ &= \frac{\text{shaded area from 82.5 to 83.5}}{\text{total area}}\end{aligned}$$

If the histogram is scaled so that the total area under the blocks is 1, then

$$\begin{aligned}\Pr(W_0 = 83) &= \Pr(82.5 \le W < 83.5)\\ &= \text{area under block from 82.5 to 83.5}\end{aligned}$$

Now suppose that the sample size gets larger and that the class interval width gets smaller. If theoretically this process is continued so that the intervals are arbitrarily small, then the histogram can be modelled by a smooth curve, as shown in the following diagram.



The curve obtained here is of great importance for a continuous random variable.

The function f whose graph models the histogram as the number of intervals is increased is called the **probability density function**. The probability density function f is used to describe the probability distribution of a continuous random variable X.

Now, the probability of interest is no longer represented by the area under the histogram, but by the area under the curve. That is,

$$\begin{aligned}\Pr(W_0 = 83) &= \Pr(82.5 \le W < 83.5)\\ &= \text{area under the graph of the function with rule } f(w) \text{ from 82.5 to 83.5}\\ &= \int_{82.5}^{83.5} f(w)\,dw\end{aligned}$$

Probability density functions

In general, a **probability density function** f is a function with domain some interval (e.g. domain $[c, d]$ or $\mathbb{R}$) such that:

1. $f(x) \ge 0$ for all x in the interval, and
2. the area under the graph of the function is equal to 1.

If the domain of f is $[c, d]$, then the second condition corresponds to $\int_c^d f(x)\,dx = 1$.

In many cases, however, the domain of f will be an 'unbounded' interval such as $[1, \infty)$ or $\mathbb{R}$. Therefore, some new notation is necessary.

- If the probability density function f has domain $[1, \infty)$, then $\int_1^\infty f(x)\,dx = 1$. This integral is computed as $\lim_{k\to\infty} \int_1^k f(x)\,dx$.
- If the probability density function f has domain $\mathbb{R}$, then $\int_{-\infty}^\infty f(x)\,dx = 1$. This integral is computed as $\lim_{k\to\infty} \int_{-k}^k f(x)\,dx$.

Note: Definite integrals which have one or both limits infinite are called **improper integrals**. There are possible complications with such integrals which we avoid in this course; you will only need the methods of evaluation illustrated in Examples 1 and 3.

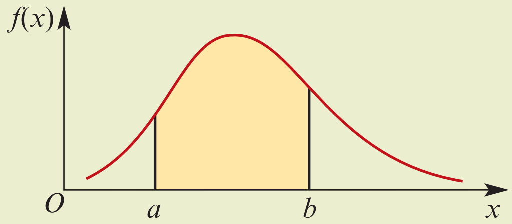
The probability density function of a random variable

Now consider a continuous random variable X with range $[c, d]$. (Alternatively, the range of X may be an unbounded interval such as $(-\infty, d]$, $[c, \infty)$ or $\mathbb{R}$.) Let f be a probability density function with domain $[c, d]$. Then:

We say that f is the **probability density function of X** if

$$\Pr(a < X < b) = \int_a^b f(x)\,dx$$

for all $a < b$ in the range of X.



Notes:

- The values of a probability density function f are not probabilities, and $f(x)$ may take values greater than 1.
- The probability of any specific value of X is 0. That is, $\Pr(X = a) = 0$. It follows that all of the following expressions have the same numerical value:
  - $\Pr(a < X < b)$
  - $\Pr(a \le X < b)$
  - $\Pr(a < X \le b)$
  - $\Pr(a \le X \le b)$
- If f has domain $[c, d]$ and $a \in [c, d]$, then $\Pr(X < a) = \Pr(X \le a) = \int_c^a f(x)\,dx$.

The natural extension of a probability density function

Any probability density function f with domain $[c, d]$ (or any other interval) may be extended to a function f^* with domain $\mathbb{R}$ by defining

$$f^*(x) = \begin{cases} f(x) & \text{if } x \in [c, d] \\ 0 & \text{if } x \notin [c, d] \end{cases}$$

This leads to the following:

A probability density function f (or its natural extension) must satisfy the following two properties:

1 $f(x) \ge 0$ for all x **2** $\int_{-\infty}^\infty f(x)\,dx = 1$

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Example 1

Suppose that the random variable X has the probability density function with rule:

$$f(x) = \begin{cases} cx & \text{if } 0 \le x \le 2 \\ 0 & \text{if } x > 2 \text{ or } x < 0 \end{cases}$$

a Find the value of c that makes f a probability density function.

b Find $\Pr(X > 1.5)$.

Solution

a Since f is a probability density function, we know that $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

Now $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^2 cx\,dx$ since $f(x) = 0$ elsewhere

$$= \left[\frac{cx^2}{2}\right]_0^2$$

$$= 2c$$

Therefore $2c = 1$ and so $c = 0.5$.

b $\Pr(X > 1.5) = \int_{1.5}^{2} 0.5x\,dx$

$$= 0.5\left[\frac{x^2}{2}\right]_{1.5}^2$$

$$= 0.5\left(\frac{4}{2} - \frac{2.25}{2}\right)$$

$$= 0.4375$$

Example 2

Consider the function f with the rule:

$$f(x) = \begin{cases} 1.5(1 - x^2) & \text{if } 0 \le x \le 1 \\ 0 & \text{if } x > 1 \text{ or } x < 0 \end{cases}$$

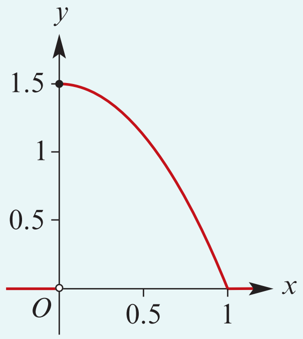
a Sketch the graph of f.

b Show that f is a probability density function.

c Find $\Pr(X > 0.5)$, where the random variable X has probability density function f.

Solution

a For $0 \le x \le 1$, the graph of $y = f(x)$ is part of a parabola with intercepts at $(0, 1.5)$ and $(1, 0)$.



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b From the graph, we can see that $f(x) \geq 0$ for all x, and so the first condition holds.

The second condition to check is that $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

$$\begin{aligned} \text{Now } \int_{-\infty}^{\infty} f(x)\, dx &= \int_0^1 1.5(1 - x^2)\, dx \qquad \text{since } f(x) = 0 \text{ elsewhere} \\ &= 1.5\left[x - \frac{x^3}{3}\right]_0^1 \\ &= 1.5\left(1 - \frac{1}{3}\right) \\ &= 1 \end{aligned}$$

Thus the second condition holds, and hence f is a probability density function.

c
$$\begin{aligned} \Pr(X > 0.5) &= \int_{0.5}^{1} 1.5(1 - x^2)\, dx \\ &= 1.5\left[x - \frac{x^3}{3}\right]_{0.5}^1 \\ &= 1.5\left[\left(1 - \frac{1}{3}\right) - \left(0.5 - \frac{0.125}{3}\right)\right] \\ &= 0.3125 \end{aligned}$$

Probability density functions with unbounded domain

Some intervals for which definite integrals need to be evaluated are of the form $(-\infty, a]$ or $[a, \infty)$ or $(-\infty, \infty)$. For a function f with non-negative values, such integrals are defined as follows (provided the limits exist):

- To integrate over the interval $(-\infty, a]$, find $\lim\limits_{k \to -\infty} \int_k^a f(x)\, dx$.
- To integrate over the interval $[a, \infty)$, find $\lim\limits_{k \to \infty} \int_a^k f(x)\, dx$.
- To integrate over the interval $(-\infty, \infty)$, find $\lim\limits_{k \to \infty} \int_{-k}^k f(x)\, dx$.

Example 3

Consider the exponential probability density function f with the rule:

$$f(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

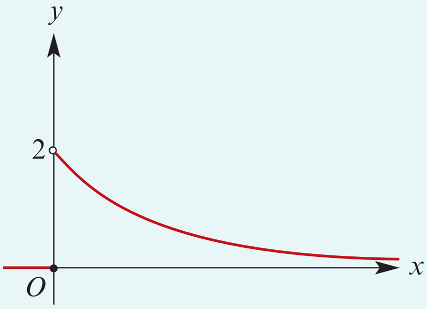
a Sketch the graph of f.

b Show that f is a probability density function.

c Find $\Pr(X > 1)$, where the random variable X has probability density function f.

Solution

a For $x > 0$, the graph of $y = f(x)$ is part of the graph of an exponential function with y-axis intercept 2. As $x \to \infty$, $y \to 0$.



b Since $f(x) \geq 0$ for all x, the first condition holds.

The second condition to check is that $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

Now $\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{\infty} 2e^{-2x}\,dx$ since $f(x) = 0$ elsewhere

$$\begin{aligned}
&= \lim_{k\to\infty} \int_0^k 2e^{-2x}\,dx \\
&= \lim_{k\to\infty} \left[\frac{2e^{-2x}}{-2}\right]_0^k \\
&= \lim_{k\to\infty} \left[-e^{-2x}\right]_0^k \\
&= \lim_{k\to\infty} \left((-e^{-2k}) - (-e^{-0})\right) \\
&= 0 + e^0 \\
&= 1
\end{aligned}$$

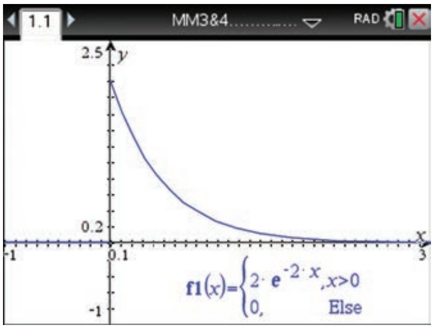
Thus f satisfies the two conditions for a probability density function.

c
$$\begin{aligned}
\Pr(X > 1) &= \lim_{k\to\infty} \int_1^k 2e^{-2x}\,dx \\
&= \lim_{k\to\infty} \left[\frac{2e^{-2x}}{-2}\right]_1^k \\
&= \lim_{k\to\infty} \left[-e^{-2x}\right]_1^k \\
&= \lim_{k\to\infty} \left((-e^{-2k}) - (-e^{-2})\right) \\
&= 0 + e^{-2} \\
&= \frac{1}{e^2} \\
&= 0.1353 \quad \text{correct to four decimal places}
\end{aligned}$$

Using the TI-Nspire

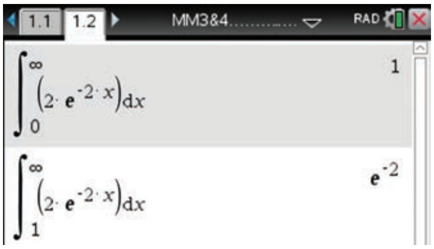
This is an application of integration.

a The graph is as shown. The piecewise function template has been used in this example; access the template using .



$f1(x)=\begin{cases}2\cdot e^{-2\cdot x}, & x>0\\ 0, & \text{Else}\end{cases}$

b, c The two required integrations are shown. The symbol ∞ can be found using (π▸) or (ctrl).



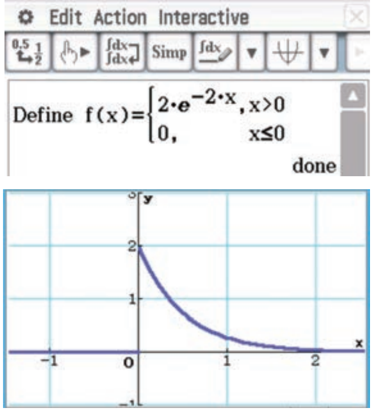
$\int_0^\infty \left(2\cdot e^{-2\cdot x}\right)dx \quad 1$

$\int_1^\infty \left(2\cdot e^{-2\cdot x}\right)dx \quad e^{-2}$

Using the Casio ClassPad

a To sketch the graph:

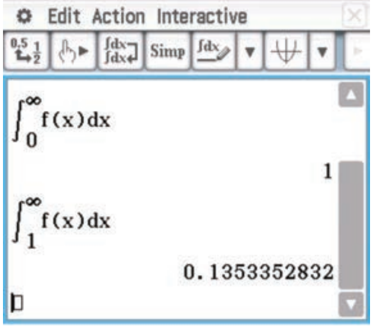
- Select the Math3 keyboard and tap on the piecewise template .
- Enter the function as shown, highlight and go to **Interactive** > **Define**.
- Now select , highlight $f(x)$ and drag into the graph screen.
- Adjust the window using .



$\text{Define } f(x)=\begin{cases}2\cdot e^{-2\cdot x}, & x>0\\ 0, & x\le 0\end{cases}$

done

b, c Find the definite integrals as shown.



$\int_0^\infty f(x)dx$

1

$\int_1^\infty f(x)dx$

0.1353352832

Conditional probability

Next is an example involving conditional probability with continuous random variables.

Example 4

The time (in seconds) that it takes a student to complete a puzzle is a random variable X with a density function given by

$$f(x) = \begin{cases} \dfrac{5}{x^2} & x \ge 5 \\ 0 & x < 5 \end{cases}$$

a Find the probability that a student takes less than 12 seconds to complete the puzzle.

b Find the probability that a student takes between 8 and 10 seconds to complete the puzzle, given that he takes less than 12 seconds.

Solution

a
$$\begin{aligned}\Pr(X < 12) &= \int_5^{12} f(x)\,dx \\ &= \int_5^{12} \frac{5}{x^2}\,dx \\ &= \left[-\frac{5}{x}\right]_5^{12} \\ &= -\frac{5}{12} + 1 \\ &= \frac{7}{12}\end{aligned}$$

b
$$\begin{aligned}&\Pr(8 < X < 10 \mid X < 12) \\ &= \frac{\Pr(8 < X < 10 \cap X < 12)}{\Pr(X < 12)} \\ &= \frac{\Pr(8 < X < 10)}{\Pr(X < 12)} \\ &= \frac{\int_8^{10} f(x)\,dx}{\int_5^{12} f(x)\,dx} \\ &= \frac{-\frac{1}{2} + \frac{5}{8}}{\frac{7}{12}} = \frac{3}{14}\end{aligned}$$

Section summary

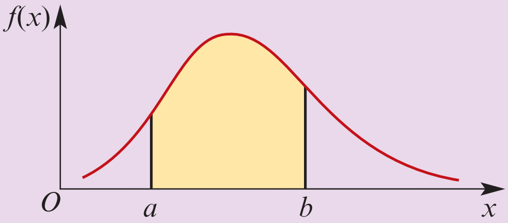
- A probability density function f (or its natural extension) must satisfy the following two properties:

  1 $f(x) \ge 0$ for all x **2** $\int_{-\infty}^{\infty} f(x)\,dx = 1$

- If X is a continuous random variable with density function f, then

  $$\Pr(a < X < b) = \int_a^b f(x)\,dx$$

  which is the area of the shaded region.



- Definite integrals may need to be evaluated over unbounded intervals:
  - To integrate over the interval $(-\infty, a]$, find $\lim\limits_{k\to-\infty} \int_k^a f(x)\,dx$.
  - To integrate over the interval $[a, \infty)$, find $\lim\limits_{k\to\infty} \int_a^k f(x)\,dx$.
  - To integrate over the interval $(-\infty, \infty)$, find $\lim\limits_{k\to\infty} \int_{-k}^k f(x)\,dx$.

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Exercise 15A

Skillsheet **1** Show that the function f with the following rule is a probability density function:

$$f(x) = \begin{cases} \dfrac{24}{x^3} & 3 \le x \le 6 \\ 0 & x < 3 \text{ or } x > 6 \end{cases}$$

Example 1 **2** Let X be a continuous random variable with the following probability density function:

$$f(x) = \begin{cases} x^2 + kx + 1 & 0 \le x \le 2 \\ 0 & x < 0 \text{ or } x > 2 \end{cases}$$

Determine the constant k such that f is a valid probability density function.

Example 2 **3** Consider the random variable X having the probability density function with the rule:

$$f(x) = \begin{cases} 12x^2(1 - x) & 0 \le x \le 1 \\ 0 & x < 0 \text{ or } x > 1 \end{cases}$$

a Sketch the graph of $y = f(x)$. **b** Find $\Pr(X < 0.5)$.

c Shade the region which represents this probability on your sketch graph.

Example 3 **4** Consider the random variable Y with the probability density function:

$$f(y) = \begin{cases} ke^{-y} & y \ge 0 \\ 0 & y < 0 \end{cases}$$

a Find the constant k. **b** Find $\Pr(Y \le 2)$.

Example 4 **5** The quarantine period for a certain disease is between 5 and 11 days after contact. The probability of showing the first symptoms at various times during the quarantine period is described by the probability density function:

$$f(t) = \frac{1}{36}(t - 5)(11 - t)$$

a Sketch the graph of the function.

b Find the probability that the symptoms appear within 7 days.

c Find the probability that the symptoms appear within 7 days, given that they appear after 5.5 days.

d Find the probability that the symptoms appear within 7 days, given that they appear within 10 days.

6 A probability model for the mass, X kg, of a 2-year-old child is given by

$$f(x) = k\sin\left(\frac{\pi(x - 7)}{10}\right), \quad 7 \le x \le 17$$

a Show that $k = \dfrac{\pi}{20}$.

b Hence find the percentage of 2-year-old children whose mass is:

i greater than 16 kg **ii** between 12 kg and 13 kg.

7 A probability density function for the lifetime, T hours, of Electra light bulbs has rule

$$f(t) = ke^{\left(\frac{-t}{200}\right)}, \quad t > 0$$

a Find the value of the constant k.

b Find the probability that an Electra light bulb will last more than 1000 hours.

8 A random variable X has a probability density function given by

$$f(x) = \begin{cases} k(1 + x) & -1 \le x \le 0 \\ k(1 - x) & 0 < x \le 1 \\ 0 & x < -1 \text{ or } x > 1 \end{cases}$$

where $k > 0$.

a Sketch the graph of the probability density function. **b** Evaluate k.

c Find the probability that X lies between -0.5 and 0.5.

9 Let X be a continuous random variable with probability density function given by

$$f(x) = \begin{cases} 3x^2 & 0 \le x \le 1 \\ 0 & x < 0 \text{ or } x > 1 \end{cases}$$

a Sketch the graph of $y = f(x)$.

b Find $\Pr(0.25 < X < 0.75)$ and illustrate this on your graph.

10 A random variable X has a probability density function f with the rule:

$$f(x) = \begin{cases} \dfrac{1}{100}(10 + x) & \text{if } -10 < x \le 0 \\ \dfrac{1}{100}(10 - x) & \text{if } 0 < x \le 10 \\ 0 & \text{if } x \le -10 \text{ or } x > 10 \end{cases}$$

a Sketch the graph of f. **b** Find $\Pr(-1 \le X < 1)$.

11 The life, X hours, of a type of light bulb has a probability density function with the rule:

$$f(x) = \begin{cases} \dfrac{k}{x^2} & x > 1000 \\ 0 & x \le 1000 \end{cases}$$

a Evaluate k. **b** Find the probability that a bulb will last at least 2000 hours.

12 The weekly demand for petrol, X (in thousands of litres), at a particular service station is a random variable with probability density function:

$$f(x) = \begin{cases} 2\left(1 - \dfrac{1}{x^2}\right) & 1 \le x \le 2 \\ 0 & x < 1 \text{ or } x > 2 \end{cases}$$

a Determine the probability that more than 1.5 thousand litres are bought in one week.

b Determine the probability that the demand for petrol in one week is less than 1.8 thousand litres, given than it is more than 1.5 thousand litres.

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13 The length of time, X minutes, between the arrival of customers at an ATM is a random variable with probability density function:

$$f(x) = \begin{cases} \frac{1}{5}e^{-\frac{x}{5}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

a Find the probability that more than 8 minutes elapses between successive customers.

b Find the probability that more than 12 minutes elapses between successive customers, given that more than 8 minutes has passed.

14 A random variable X has density function given by

$$f(x) = \begin{cases} 0.2 & -1 < x \leq 0 \\ 0.2 + 1.2x & 0 < x \leq 1 \\ 0 & x \leq -1 \text{ or } x > 1 \end{cases}$$

a Find $\Pr(X \leq 0.5)$.

b Hence find $\Pr(X > 0.5 \mid X > 0.1)$.

15 The continuous random variable X has probability density function f given by

$$f(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

a Sketch the graph of f.

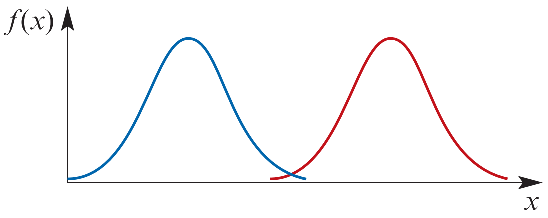
b Find:

i $\Pr(X < 0.5)$ **ii** $\Pr(X \geq 1)$ **iii** $\Pr(X \geq 1 \mid X > 0.5)$

15B Mean and median for a continuous random variable

The centre is an important summary feature of a probability distribution.

The following diagram shows two probability distributions which are identical except for their centres.



More than one measure of centre may be determined for a continuous random variable, and each gives useful information about the random variable under consideration. The most generally useful measure of centre is the mean.

▶ Mean

We defined the mean for a discrete random variable in Section 13D. We can also define the mean for a continuous random variable.

For a continuous random variable X with probability density function f, the **mean** or **expected value** of X is given by

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

provided the integral exists. The mean is denoted by the Greek letter μ (mu).

If $f(x) = 0$ for all $x \notin [c, d]$, then $\mathrm{E}(X) = \int_c^d x f(x)\, dx$.

This definition is consistent with the definition of the expected value for a discrete random variable. As in the case of a discrete random variable, the expected value of a continuous random variable is the long-run average value of the variable. For example, consider the daily demand for petrol at a service station. The mean of this variable tells us the average daily demand for petrol over a very long period of time.

Example 5

Find the expected value of the random variable X which has probability density function with rule:

$$f(x) = \begin{cases} 0.5x & 0 \le x \le 2 \\ 0 & x < 0 \text{ or } x > 2 \end{cases}$$

Solution

By definition,

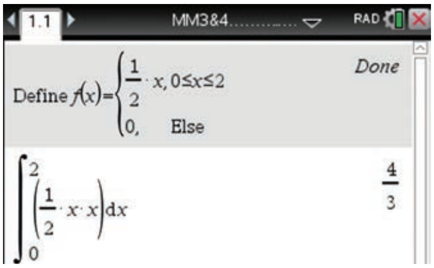
$$\begin{aligned} \mathrm{E}(X) = \int_{-\infty}^{\infty} x f(x)\, dx &= \int_0^2 x \times 0.5x\, dx \qquad \text{since } f(x) = 0 \text{ elsewhere} \\ &= 0.5 \int_0^2 x^2\, dx \\ &= 0.5 \left[\frac{x^3}{3} \right]_0^2 \\ &= \frac{4}{3} \end{aligned}$$

Using the TI-Nspire

Define the function f as shown; access the piecewise function template using .

Notes:

- Leave the domain for the last function piece blank; it will autofill as ‘Else’.
- To obtain an exact answer, enter $\frac{1}{2}x$ instead of $0.5x$.



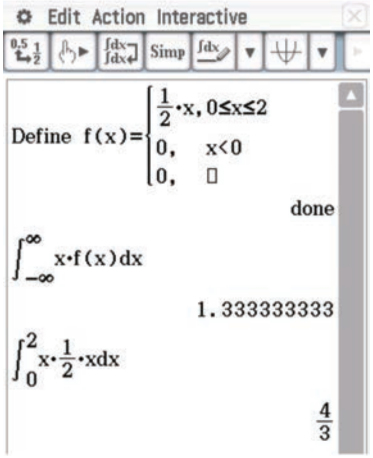
$$\text{Define } f(x) = \begin{cases} \frac{1}{2}\cdot x, & 0 \le x \le 2 \\ 0, & \text{Else} \end{cases} \qquad \textit{Done}$$

$$\int_0^2 \left(\frac{1}{2}\cdot x \cdot x \right) dx \qquad \frac{4}{3}$$

Using the Casio ClassPad

- Tap the piecewise template ▦ twice.
- Define the function f as shown.
- Find E(X) by evaluating the definite integral as shown.

Note: Using the defined function to find E(X) gives the decimal answer only.



Define $f(x)=\begin{cases}\frac{1}{2}\cdot x, & 0\le x\le 2\\ 0, & x<0\\ 0, & \square\end{cases}$

done

$\int_{-\infty}^{\infty} x\cdot f(x)dx$

1.333333333

$\int_0^2 x\cdot\frac{1}{2}\cdot x dx$

$\frac{4}{3}$

The mean of a function of X is calculated as follows. (In this case, the function of X is denoted by $g(X)$ and is the composition of the random variable X followed by the function g.)

The expected value of $g(X)$ is given by

$$\mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)\,dx$$

provided the integral exists.

Generally, as in the case of a discrete random variable, the expected value of a function of X is not equal to that function of the expected value of X. That is,

$$\mathrm{E}[g(X)] \neq g[\mathrm{E}(X)]$$

Example 6

Let X be a random variable with probability density function f given by

$$f(x) = \begin{cases} 0.5x & 0 \le x \le 2 \\ 0 & x < 0 \text{ or } x > 2 \end{cases}$$

Find:

a the expected value of X^2

b the expected value of e^X.

Solution

a
$$\begin{aligned}\mathrm{E}(X^2) &= \int_{-\infty}^{\infty} x^2 f(x)\,dx\\ &= \int_0^2 x^2 \times 0.5x\,dx\\ &= 0.5\int_0^2 x^3\,dx\\ &= 0.5\left[\frac{x^4}{4}\right]_0^2\\ &= 2\end{aligned}$$

b
$$\begin{aligned}\mathrm{E}(e^X) &= \int_{-\infty}^{\infty} e^x f(x)\,dx\\ &= \int_0^2 e^x f(x)\,dx\\ &= \int_0^2 e^x \times 0.5x\,dx\\ &= 4.195\end{aligned}$$

correct to three decimal places.

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A case where the equality does hold is where g is a linear function:

$$\mathrm{E}(aX + b) = a\mathrm{E}(X) + b \qquad \text{(for } a, b \text{ constant)}$$

Percentiles and the median

Another value of interest is the value of X which bounds a particular area under the probability density function. For example, a teacher may wish determine the mark, p, below which lie 75% of all students' marks. This is called the 75th percentile of the population, and is found by solving

$$\int_{-\infty}^{p} f(x)\, dx = 0.75$$

This can be stated more generally:

Percentiles

The value p of X which is the solution of an equation of the form

$$\int_{-\infty}^{p} f(x)\, dx = q$$

is called a **percentile** of the distribution.

For example, the 75th percentile is the value p found by taking $q = 75\% = 0.75$.

Example 7

The duration of telephone calls to the order department of a large company is a random variable, X minutes, with probability density function:

$$f(x) = \begin{cases} \frac{1}{3}e^{-\frac{x}{3}} & x > 0 \\ 0 & x \le 0 \end{cases}$$

Find the value of a such that 90% of phone calls last less than a minutes.

Solution

To find the value of a, solve the equation:

$$\int_0^a \frac{1}{3}e^{-\frac{x}{3}}\, dx = 0.9$$

$$\left[-e^{-\frac{x}{3}}\right]_0^a = 0.9$$

$$1 - e^{-\frac{a}{3}} = 0.9$$

$$-\frac{a}{3} = \log_e 0.1$$

$$\therefore \quad a = 3\log_e 10$$

$$= 6.908 \qquad \text{(correct to three decimal places)}$$

So 90% of the calls to this company last less than 6.908 minutes.

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A percentile of special interest is the **median**, or 50th percentile. The median is the middle value of the distribution. That is, the probability of X taking a value below the median is 0.5, and the probability of X taking a value above the median is 0.5. Thus, if m is the median value of the distribution, then

$$\Pr(X \le m) = \Pr(X > m) = 0.5$$

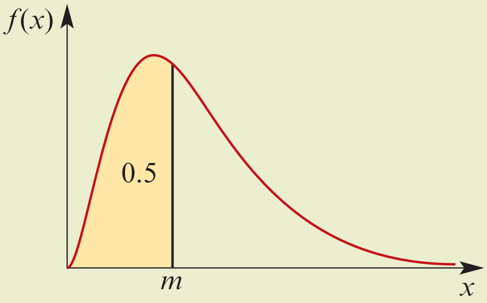
Graphically, the median is the value of the random variable which divides the area under the probability density function in half.

The median

The median is another measure of centre for a continuous probability distribution.

The median, m, of a continuous random variable X is the value of X such that

$$\int_{-\infty}^{m} f(x)\,dx = 0.5$$



Example 8

Suppose the probability density function of weekly sales of topsoil, X (in tonnes), is given by the rule:

$$f(x) = \begin{cases} 2(1-x) & 0 \le x \le 1 \\ 0 & x < 0 \text{ or } x > 1 \end{cases}$$

Find the median value of X, and interpret.

Solution

The median m is such that

$$\int_0^m 2(1-x)\,dx = 0.5$$

$$2\left[x - \frac{x^2}{2}\right]_0^m = 0.5$$

$$2m - m^2 = 0.5$$

$$m^2 - 2m + 0.5 = 0$$

$$\therefore \quad m = 0.293 \text{ or } m = 1.707$$

But since $0 \le x \le 1$, the median is $m = 0.293$ tonnes.

This means that, in the long run, 50% of weekly sales will be less than 0.293 tonnes, and 50% will be more.

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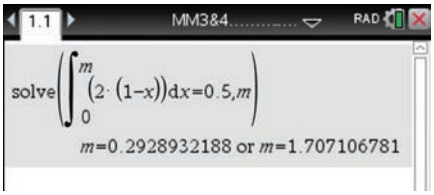
Using the TI-Nspire

This is an application of integration.

- Solve the definite integral equal to 0.5 as shown to find m (the median value).
- Since $0 \le x \le 1$, the median is $m = 0.293$.

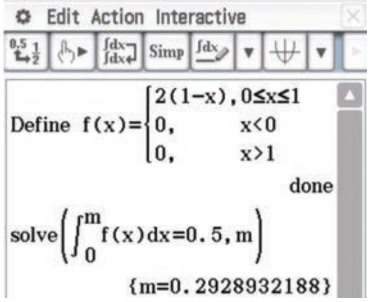


solve$\left(\int_0^m (2\cdot(1-x))dx=0.5, m\right)$

$m=0.2928932188$ or $m=1.707106781$

Using the Casio ClassPad

- Define the function f.
- Solve the definite integral equal to 0.5 as shown to find m (the median value).



Define $f(x)=\begin{cases}2(1-x), & 0\le x\le 1\\ 0, & x<0\\ 0, & x>1\end{cases}$

done

solve$\left(\int_0^m f(x)dx=0.5, m\right)$

$\{m=0.2928932188\}$

Section summary

For a continuous random variable X with probability density function f:

- the **mean** or **expected value** of X is given by $\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$
- the expected value of $g(X)$ is given by $E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\,dx$
- the **median** of X is the value m such that $\int_{-\infty}^{m} f(x)\,dx = 0.5$

Exercise 15B

Skillsheet

Example 5

1 Find the mean, E(X), of the continuous random variables with the following probability density functions:

a $f(x) = 2x, \ 0 < x < 1$

b $f(x) = \dfrac{1}{2\sqrt{x}}, \ 0 < x < 1$

c $f(x) = 6x(1-x), \ 0 < x < 1$

d $f(x) = \dfrac{1}{x^2}, \ x \ge 1$

2 For each of the following, use your calculator to check that f is a probability density function and then to find the mean, E(X), of the corresponding continuous random variable:

a $f(x) = \sin x, \ 0 < x < \dfrac{\pi}{2}$

b $f(x) = \log_e x, \ 1 < x < e$

c $f(x) = \dfrac{1}{\sin^2 x}, \ \dfrac{\pi}{4} < x < \dfrac{\pi}{2}$

d $f(x) = -4x\log_e x, \ 0 < x < 1$

3 A continuous random variable X has the probability density function given by

$$f(x) = \begin{cases} 2x^3 - x + 1 & 0 \le x \le 1 \\ 0 & x < 0 \text{ or } x > 1 \end{cases}$$

a Find μ, the mean value of X.

b Find the probability that X takes a value less than or equal to the mean.

4 Consider the probability density function given by

$$f(x) = \frac{1}{2\pi}(1 + \cos x), \quad -\pi \le x \le \pi$$

Find the expected value of X.

5 A random variable Y has the probability density function:

$$f(y) = \begin{cases} Ay & 0 \le y \le B \\ 0 & y < 0 \text{ or } y > B \end{cases}$$

Find A and B if the mean of Y is 2.

Example 6 **6** A random variable X has the probability density function given by

$$f(x) = \begin{cases} 12x^2(1 - x) & 0 \le x \le 1 \\ 0 & x < 0 \text{ or } x > 1 \end{cases}$$

a Find $\mathrm{E}\left(\dfrac{1}{X}\right)$. **b** Find $\mathrm{E}(e^X)$.

Example 7 **7** The time, X seconds, between arrivals of particles at a radiation counter has been found to have a probability density function f with the rule:

$$f(x) = \begin{cases} 0 & x < 0 \\ e^{-x} & x \ge 0 \end{cases}$$

a Find $\Pr(X \le 1)$. **b** Find $\Pr(1 \le X \le 2)$. **c** Find the median, m, of X.

8 The random variable X has a probability density function given by

$$f(x) = \begin{cases} k & 0 \le x \le 1 \\ 0 & x < 0 \text{ or } x > 1 \end{cases}$$

a Find the value of k. **b** Find the median, m, of X.

9 A continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} 5(1 - x)^4 & 0 \le x \le 1 \\ 0 & x < 0 \text{ or } x > 1 \end{cases}$$

Find the median, m, of X correct to four decimal places.

10 Suppose that the time (in minutes) between telephone calls received at a pizza restaurant has the probability density function:

$$f(x) = \begin{cases} \dfrac{1}{4}e^{-\frac{x}{4}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Find the median time between calls.

Example 8 **11** A continuous random variable X has the probability density function given by

$$f(x) = \begin{cases} x & 0 \leq x < 1 \\ 2 - x & 1 \leq x < 2 \\ 0 & x < 0 \text{ or } x \geq 2 \end{cases}$$

a Find μ, the expected value of X. **b** Find m, the median value of X.

12 Let the probability density function of X be given by

$$f(x) = \begin{cases} 30x^4(1 - x) & 0 < x < 1 \\ 0 & x \leq 0 \text{ or } x \geq 1 \end{cases}$$

a Find the expected value, μ, of X.

b Find the median value, m, of X and hence show the mean is less than the median.

13 A probability model for the mass, X kg, of a 2-year-old child is given by

$$f(x) = \frac{\pi}{20}\sin\left(\frac{\pi(x - 7)}{10}\right), \quad 7 \leq x \leq 17$$

Find the median value, m, of X.

14 A random variable X has density function given by

$$f(x) = \begin{cases} 0.2 & -1 \leq x \leq 0 \\ 0.2 + 1.2x & 0 < x \leq 1 \\ 0 & x < -1 \text{ or } x > 1 \end{cases}$$

a Find μ, the expected value of X. **b** Find m, the median value of X.

15 The exponential probability distribution describes the distribution of the time between random events, such as phone calls. The general form of the exponential distribution with parameter λ is

$$f(x) = \begin{cases} \dfrac{1}{\lambda}e^{-\frac{x}{\lambda}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

a Differentiate kxe^{-kx} and hence find an antiderivative of kxe^{-kx}.

b Show that the mean of an exponential random variable is λ.

c On the same axes, sketch the graphs of the distribution for $\lambda = \frac{1}{2}$, $\lambda = 1$ and $\lambda = 2$.

d Describe the effect of varying the value of λ on the graph of the distribution.

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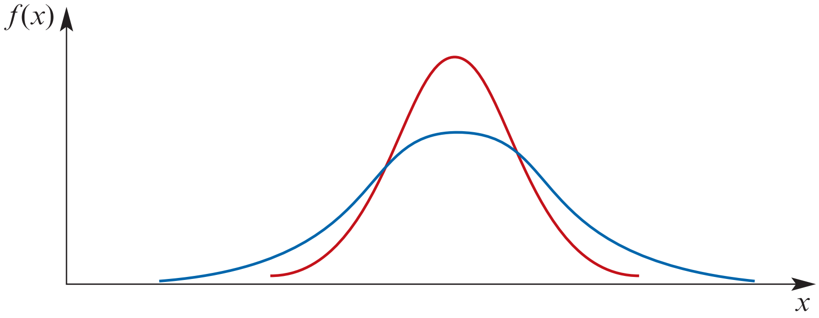
15C Measures of spread

Another important summary feature of a distribution is variation or spread. The following diagram shows two distributions that are identical except for their spreads.



As in the case of centre, there is more than one measure of spread. The most commonly used is the variance, together with its companion measure, the standard deviation. Others that you may be familiar with are the range and the interquartile range.

Variance and standard deviation

The **variance** of a random variable X is a measure of the spread of the probability distribution about its mean or expected value μ. It is defined as:

$$\begin{aligned}\mathrm{Var}(X) &= \mathrm{E}[(X-\mu)^2]\\ &= \int_{-\infty}^{\infty}(x-\mu)^2 f(x)\,dx\end{aligned}$$

As for discrete random variables, the variance is usually denoted by σ^2, where σ is the lowercase Greek letter *sigma*.

Variance may be considered as the long-run average value of the square of the distance from X to μ. This means that the variance is not in the same units of measurement as the original random variable X. A measure of spread in the appropriate unit is found by taking the square root of the variance.

The **standard deviation** of X is defined as:

$$\mathrm{sd}(X) = \sqrt{\mathrm{Var}(X)}$$

The standard deviation is usually denoted by σ.

As in the case of discrete random variables, an alternative (computational) formula for variance is generally used.

To calculate variance, use

$$\mathrm{Var}(\mathrm{X}) = \mathrm{E}(X^2) - \mu^2$$

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Proof The computational form of the expression for variance is derived as follows:

$$\begin{aligned}\mathrm{Var}(X) &= \int_{-\infty}^{\infty}(x-\mu)^2 f(x)\,dx\\ &= \int_{-\infty}^{\infty}(x^2-2\mu x+\mu^2)f(x)\,dx\\ &= \int_{-\infty}^{\infty}x^2 f(x)\,dx - \int_{-\infty}^{\infty}2\mu x f(x)\,dx + \int_{-\infty}^{\infty}\mu^2 f(x)\,dx\\ &= \mathrm{E}(X^2) - 2\mu\int_{-\infty}^{\infty}x f(x)\,dx + \mu^2\int_{-\infty}^{\infty}f(x)\,dx\end{aligned}$$

Since $\int_{-\infty}^{\infty}x f(x)\,dx = \mu$ and $\int_{-\infty}^{\infty}f(x)\,dx = 1$, we obtain

$$\begin{aligned}\mathrm{Var}(X) &= \mathrm{E}(X^2) - 2\mu^2 + \mu^2\\ &= \mathrm{E}(X^2) - \mu^2\end{aligned}$$

Example 9

Find the variance and standard deviation of the random variable X which has the probability density function f with rule:

$$f(x) = \begin{cases} 0.5x & 0 \le x \le 2\\ 0 & x < 0 \text{ or } x > 2\end{cases}$$

Solution

Use the computational formula $\mathrm{Var}(X) = \mathrm{E}(X^2) - \mu^2$.

First evaluate $\mathrm{E}(X^2)$:

$$\begin{aligned}\mathrm{E}(X^2) &= \int_{-\infty}^{\infty}x^2 f(x)\,dx\\ &= \int_0^2 x^2 \times 0.5x\,dx\\ &= 0.5\int_0^2 x^3\,dx\\ &= 0.5\left[\frac{x^4}{4}\right]_0^2\\ &= 0.5\times 4\\ &= 2\end{aligned}$$

Since $\mathrm{E}(X) = \dfrac{4}{3}$ from Example 5, we now have

$$\mathrm{Var}(X) = 2 - \left(\frac{4}{3}\right)^2 = \frac{2}{9}$$

and $\mathrm{sd}(X) = \sqrt{\dfrac{2}{9}} = \dfrac{\sqrt{2}}{3} = 0.471$ (correct to three decimal places)

It helps to make the standard deviation more meaningful to give it an interpretation which relates to the probability distribution. As already stated for discrete random variables, it is also the case for many continuous random variables that about 95% of the distribution lies within two standard deviations either side of the mean.

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In general, for many continuous random variables X,

$$\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma) \approx 0.95$$

Example 10

The life of a certain brand of battery, X hours, is a continuous random variable with mean 50 and variance 16. Find an (approximate) interval for the time period for which 95% of the batteries would be expected to last.

Solution

$\Pr(\mu - 2\sigma \le X \le \mu + 2\sigma) \approx 0.95$

Since $\mu = 50$ and $\sigma = \sqrt{16} = 4$, we expect 95% of the batteries to last between 42 hours and 58 hours.

Interquartile range

The **interquartile range** is the range of the middle 50% of the distribution; it is the difference between the 75th percentile (also known as Q3) and the 25th percentile (also known as Q1).

Example 11

Determine the interquartile range of the random variable X which has the probability density function:

$$f(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & x < 0 \text{ or } x > 1 \end{cases}$$

Solution

To find the 25th percentile a, solve:

$$\begin{aligned} \int_0^a 2x\,dx &= 0.25 \\ [x^2]_0^a &= 0.25 \\ a^2 &= 0.25 \\ \therefore \quad a &= \sqrt{0.25} = 0.5 \end{aligned}$$

To find the 75th percentile b, solve:

$$\begin{aligned} \int_0^b 2x\,dx &= 0.75 \\ [x^2]_0^b &= 0.75 \\ b^2 &= 0.75 \\ \therefore \quad b &= \sqrt{0.75} \approx 0.866 \end{aligned}$$

Thus the interquartile range is $0.866 - 0.5 = 0.366$, correct to three decimal places.

Note that the negative solutions to these equations were not appropriate, as $0 \le x \le 1$.

Exercise 15C

Example 9

1 A random variable X has probability density function:

$$f(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & x \le 0 \text{ or } x \ge 1 \end{cases}$$

Find the variance of X, and hence find the standard deviation of X.

Example 10 **2** The life of a certain brand of light bulb, X hours, is a continuous random variable with mean 400 and variance 64. Find an (approximate) interval for the time period for which 95% of the light bulbs would be expected to last.

Example 11 **3** A continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} 3x^2 & 0 \le x \le 1 \\ 0 & x < 0 \text{ or } x > 1 \end{cases}$$

a Find a such that $\Pr(X \le a) = 0.25$. **b** Find b such that $\Pr(X \le b) = 0.75$.

c Find the interquartile range of X.

4 A random variable X has the probability density function given by

$$f(x) = \begin{cases} 0.5e^{x} & x \le 0 \\ 0.5e^{-x} & x > 0 \end{cases}$$

a Sketch the graph of $y = f(x)$.

b Find the interquartile range of X, giving your answer correct to three decimal places.

5 A continuous random variable X has probability density function given by

$$f(x) = \begin{cases} \dfrac{k}{x} & 1 \le x \le 9 \\ 0 & x < 1 \text{ or } x > 9 \end{cases}$$

a Find the value of k.

b Find the mean and variance of X, giving your answer correct to three decimal places.

6 A continuous random variable X has density function f given by

$$f(x) = \begin{cases} 0 & x < 0 \\ 2 - 2x & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$$

a Find the interquartile range of X. **b** Find the mean and variance of X.

7 A random variable X has probability density function f with the rule:

$$f(x) = \begin{cases} 0 & x < 0 \\ 2xe^{-x^2} & x \ge 0 \end{cases}$$

Find the interquartile range of X.

8 A random variable X has a probability density function given by

$$f(x) = \begin{cases} 0 & x < 0 \\ \dfrac{x}{2} & 0 \le x < 2 \\ 0 & x \ge 2 \end{cases}$$

a Find the interquartile range of X. **b** Find the mean and variance of X.

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9 The queuing time, X minutes, of a traveller at the ticket office of a large railway station has probability density function f defined by

$$f(x) = \begin{cases} kx(100 - x^2) & 0 \le x \le 10 \\ 0 & x > 10 \text{ or } x < 0 \end{cases}$$

- **a** Find the value of k.
- **b** Find the mean of the distribution.
- **c** Find the standard deviation of the distribution, correct to two decimal places.

10 A probability density function is given by

$$f(x) = \begin{cases} k(a^2 - x^2) & -a \le x \le a \\ 0 & x > a \text{ or } x < -a \end{cases}$$

- **a** Find k in terms of a.
- **b** Find the value of a which gives a standard deviation of 2.

11 A continuous random variable X has probability density function f given by

$$f(x) = \begin{cases} k(3 - x) & 0 \le x \le 3 \\ k(x - 3) & 3 < x \le 6 \\ 0 & x > 6 \text{ or } x < 0 \end{cases}$$

where k is a constant.

- **a** Sketch the graph of f.
- **b** Hence, or otherwise, find the value of k.
- **c** Verify that the mean of X is 3.
- **d** Find $\mathrm{Var}(X)$.

15D Properties of mean and variance*

It has already been stated that the expected value of a function of X is not necessarily equal to that function of the expected value of X. That is, in general,

$$\mathrm{E}[g(X)] \ne g[\mathrm{E}(X)]$$

An exception is the case where the function g is linear: the mean of a linear function of X is equal to the linear function of the mean of X.

▶ The mean and variance of $aX + b$

For any continuous random variable X,

$$\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$$

---

* This section is not required for Mathematical Methods Units 3 & 4.

Proof The validity of this statement can be readily demonstrated:

$$\begin{aligned}\mathrm{E}(aX+b) &= \int_{-\infty}^{\infty}(ax+b)f(x)\,dx\\ &= \int_{-\infty}^{\infty}axf(x)\,dx + \int_{-\infty}^{\infty}bf(x)\,dx\\ &= a\int_{-\infty}^{\infty}xf(x)\,dx + b\int_{-\infty}^{\infty}f(x)\,dx\\ &= a\mathrm{E}(X)+b \qquad \left(\text{since } \int_{-\infty}^{\infty}f(x)\,dx = 1\right)\end{aligned}$$

We can also obtain a formula for the variance of a linear function of X.

> For any continuous random variable X,
>
> $\mathrm{Var}(aX+b) = a^2\mathrm{Var}(X)$

Proof Consider the variance of a linear function of X:

$$\mathrm{Var}(aX+b) = \mathrm{E}[(aX+b)^2] - [\mathrm{E}(aX+b)]^2$$

Now $[\mathrm{E}(aX+b)]^2 = [a\mathrm{E}(X)+b]^2 = (a\mu+b)^2 = a^2\mu^2 + 2ab\mu + b^2$

and
$$\begin{aligned}\mathrm{E}[(aX+b)^2] &= \mathrm{E}(a^2X^2+2abX+b^2)\\ &= a^2\mathrm{E}(X^2)+2ab\mu+b^2\end{aligned}$$

Thus
$$\begin{aligned}\mathrm{Var}(aX+b) &= a^2\mathrm{E}(X^2)+2ab\mu+b^2-a^2\mu^2-2ab\mu-b^2\\ &= a^2\mathrm{E}(X^2)-a^2\mu^2\\ &= a^2\mathrm{Var}(X)\end{aligned}$$

Although initially the absence of b in the variance may seem surprising, on reflection it makes sense that adding a constant merely translates the probability density function, and has no effect on its spread.

▶ The probability density function of $aX + b$

The random variable $X + b$ If the probability density function of X has rule $f(x)$, then the probability density function of $X + b$ is obtained by the translation $(x, y) \to (x + b, y)$ and so has rule $f(x - b)$.

The random variable aX Similarly, multiplying by a is similar to a dilation of factor a from the y-axis. However, there has to be an adjustment to determine the rule for the probability density function of aX, as the transformation must be area-preserving. The rule is $\frac{1}{a}f\left(\frac{x}{a}\right)$.

The random variable $aX + b$ Thus, if the probability density function of X has rule $f(x)$, then the probability density function of $aX + b$ has rule $\frac{1}{a}f\left(\frac{x-b}{a}\right)$. The transformation is described by

$$(x, y) \to \left(ax + b, \frac{y}{a}\right)$$

In the case that a and b are positive, this is a dilation of factor a from the y-axis and factor $\frac{1}{a}$ from the x-axis, followed by a translation of b units in the positive direction of the x-axis.

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Example 12

Suppose that X is a continuous random variable with mean $\mu = 10$ and variance $\sigma^2 = 2$.

a Find $\mathrm{E}(2X + 1)$.

b Find $\mathrm{Var}(1 - 3X)$.

c If X has probability density function f, describe the rule of a probability density function g for $2X + 1$.

Solution

a
$$\begin{aligned}\mathrm{E}(2X + 1) &= 2\mathrm{E}(X) + 1\\ &= 2 \times 10 + 1\\ &= 21\end{aligned}$$

b
$$\begin{aligned}\mathrm{Var}(1 - 3X) &= (-3)^2\mathrm{Var}(X)\\ &= 9 \times 2\\ &= 18\end{aligned}$$

c The rule is $g(x) = \dfrac{1}{a}f\left(\dfrac{x-b}{a}\right)$ where $a = 2$ and $b = 1$. Therefore $g(x) = \dfrac{1}{2}f\left(\dfrac{x-1}{2}\right)$.

Exercise 15D

1 The amount of flour used each day in a bakery is a continuous random variable X with a mean of 4 tonnes. The cost of the flour is $C = 300X + 100$. Find $\mathrm{E}(C)$.

2 For certain glass ornaments, the proportion of impurities per ornament, X, is a random variable with a density function given by

$$f(x) = \begin{cases} \dfrac{3x^2}{2} + x & \text{if } 0 \le x \le 1 \\ 0 & \text{if } x < 0 \text{ or } x > 1 \end{cases}$$

The value of each ornament (in dollars) is $V = 100 - 1.5X$.

a Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

b Hence find the mean and standard deviation of V.

Example 12

3 Let X be a random variable with probability density function:

$$f(x) = \begin{cases} \dfrac{3x^2}{2} & \text{if } -1 \le x \le 1 \\ 0 & \text{if } x < -1 \text{ or } x > 1 \end{cases}$$

Find:

a $\mathrm{E}(3X)$ and $\mathrm{Var}(3X)$

b $\mathrm{E}(3 - X)$ and $\mathrm{Var}(3 - X)$

c $\mathrm{E}(3X + 1)$ and $\mathrm{Var}(3X + 1)$

d the rule of a probability density function for $3X$

e the rule of a probability density function for $3X + 1$.

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15E Cumulative distribution functions*

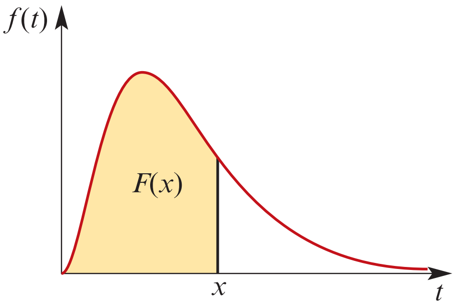
Another function of importance in describing a continuous random variable is the **cumulative distribution function** or **CDF**. For a continuous random variable X, with probability density function f defined on the interval $[c, d]$, the cumulative distribution function F is given by

$$\begin{aligned} F(x) &= \Pr(X \le x) \\ &= \int_c^x f(t)\,dt \end{aligned}$$

where t is the variable of integration. The cumulative distribution function at a particular value x gives the probability that the random variable X takes a value less than or equal to x.

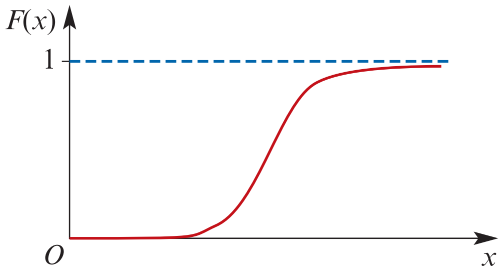
The diagram on the right shows the relationship between the probability density function f and the cumulative distribution function F.

The function F describes the area under the graph of the probability density function between the lower bound of the domain of f and x. (In the diagram, the lower bound is 0.)



For every continuous random variable X, the cumulative distribution function F is continuous.

Using the general version of the fundamental theorem of calculus, it can be shown that the derivative of the cumulative distribution function is the density function. More precisely, we have $F'(x) = f(x)$, for each value of x at which f is continuous.



There are three important properties of a cumulative distribution function. For a continuous random variable X with range $[c, d]$:

1. The probability that X takes a value less than or equal to c is 0. That is, $F(c) = 0$.
2. The probability that X takes a value less than or equal to d is 1. That is, $F(d) = 1$.
3. If x_1 and x_2 are values of X with $x_1 \le x_2$, then $\Pr(X \le x_1) \le \Pr(X \le x_2)$. That is,

   $$x_1 \le x_2 \quad \text{implies} \quad F(x_1) \le F(x_2)$$

   The function F is a **non-decreasing** function.

For a probability density function f defined on $\mathbb{R}$, the cumulative distribution is given by

$$\begin{aligned} F(x) &= \Pr(X \le x) \\ &= \int_{-\infty}^x f(t)\,dt \end{aligned}$$

In this case, we have $F(x) \to 0$ as $x \to -\infty$, and $F(x) \to 1$ as $x \to \infty$.

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* Cumulative distribution functions are not mentioned explicitly in the Mathematical Methods Units 3 & 4 study design, but are useful in our study of continuous probability distributions.

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Example 13

The time, X seconds, that it takes a student to complete a puzzle is a random variable with density function given by

$$f(x) = \begin{cases} \dfrac{5}{x^2} & x \geq 5 \\ 0 & x < 5 \end{cases}$$

Find $F(x)$, the cumulative distribution function of X.

Solution

$$\begin{aligned} F(x) = \int_5^x f(t)\, dt = \int_5^x \frac{5}{t^2}\, dt &= \left[\frac{-5}{t}\right]_5^x \\ &= \frac{-5}{x} + 1 \end{aligned}$$

Thus $F(x) = 1 - \dfrac{5}{x}$ for $x \geq 5$.

The importance of the cumulative distribution function is that probabilities for various intervals can be computed directly from $F(x)$.

Exercise 15E

1 The probability density function for a random variable X is given by

$$f(x) = \begin{cases} \dfrac{1}{5} & 0 < x < 5 \\ 0 & x \leq 0 \text{ or } x \geq 5 \end{cases}$$

a Find $F(x)$, the cumulative distribution function of X. **b** Hence find $\Pr(X \leq 3)$.

2 A random variable X has the cumulative distribution function with rule:

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x^2} & x \geq 0 \end{cases}$$

a Sketch the graph of $y = F(x)$. **b** Find $\Pr(X \geq 2)$. **c** Find $\Pr(X \geq 2 \mid X < 3)$.

3 The continuous random variable X has cumulative distribution function F given by

$$F(x) = \begin{cases} 0 & x < 0 \\ kx^2 & 0 \leq x \leq 6 \\ 1 & x > 6 \end{cases}$$

a Determine the value of the constant k. **b** Calculate $\Pr\left(\dfrac{1}{2} \leq X \leq 1\right)$.

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Chapter summary

Spreadsheet AS Nrich

- A **continuous random variable** is one that can take any value in an interval of the real number line.
- A continuous random variable can be described by a **probability density function** f. There are many different probability density functions with different shapes and properties. However, they all have the following two fundamental properties:

  1 For any value of x, the value of $f(x)$ is non-negative. That is,

  $$f(x) \geq 0 \text{ for all } x$$

  2 The total area enclosed by the graph of f and the x-axis is equal to 1. That is,

  $$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

  The probability of X taking a value in the interval (a, b) is found by determining the area under the probability density curve between a and b. That is,

  $$\Pr(a < X < b) = \int_a^b f(x)\, dx$$

- The **mean** or **expected value** of a continuous random variable X with probability density function f is given by

  $$\mu = \mathrm{E}(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

  provided the integral exists.
- If $g(X)$ is a function of X, then the expected value of $g(X)$ is given by

  $$\mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx$$

  provided the integral exists. In general, $\mathrm{E}[g(X)] \neq g[\mathrm{E}(X)]$.
- The **median** of a continuous random variable X is the value m such that

  $$\int_{-\infty}^{m} f(x)\, dx = 0.5$$

- The **variance** of a continuous random variable X with probability density function f is defined by

  $$\begin{aligned}\sigma^2 = \mathrm{Var}(X) &= \mathrm{E}[(X - \mu)^2] \\ &= \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\, dx\end{aligned}$$

  provided the integral exists. To calculate the variance, use

  $$\mathrm{Var}(X) = \mathrm{E}(X^2) - \mu^2$$

- The **standard deviation** of X is defined by

  $$\sigma = \mathrm{sd}(X) = \sqrt{\mathrm{Var}(X)}$$

- Linear function of a continuous random variable:

  $$\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$$

  $$\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$$

- In general, for many continuous random variables X,

  $$\Pr(\mu - 2\sigma \leq X \leq \mu + 2\sigma) \approx 0.95$$

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- The **interquartile range** of X is

$$\text{IQR} = b - a$$

where a and b are such that

$$\int_{-\infty}^{a} f(x)\,dx = 0.25 \quad \text{and} \quad \int_{-\infty}^{b} f(x)\,dx = 0.75$$

and where f is the probability density function of X.

Technology-free questions

1 The probability density function of X is given by

$$f(x) = \begin{cases} kx & \text{if } 1 \le x \le \sqrt{2} \\ 0 & \text{otherwise} \end{cases}$$

a Find k.

b Find $\Pr(1 < X < 1.1)$.

c Find $\Pr(1 < X < 1.2)$.

2 If the probability density function of X is given by

$$f(x) = \begin{cases} a + bx^2 & 0 \le x \le 1 \\ 0 & x > 1 \text{ or } x < 0 \end{cases}$$

and $\text{E}(X) = \dfrac{2}{3}$, find a and b.

3 The probability density function of X is given by

$$f(x) = \begin{cases} \dfrac{\sin x}{2} & 0 \le x \le \pi \\ 0 & x > \pi \text{ or } x < 0 \end{cases}$$

Find the median of X.

4 The probability density function of X is given by

$$f(x) = \begin{cases} \dfrac{1}{4} & 1 \le x < 5 \\ 0 & x < 1 \text{ or } x \ge 5 \end{cases}$$

a Find $\Pr(1 < X < 3)$.

b Find $\Pr(X > 2 \mid 1 < X < 3)$.

c Find $\Pr(X > 4 \mid X > 2)$.

5 Consider the random variable X having the probability density function given by

$$f(x) = \begin{cases} 12x^2(1 - x) & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a Sketch the graph of $y = f(x)$.

b Find $\Pr(X < 0.5)$ and illustrate this probability on your sketch graph.

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6 The probability density function of a random variable X is

$$f(x) = \begin{cases} kx^2(1-x) & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a Determine k.

b Find the probability that X is less than $\frac{2}{3}$.

c Find the probability that X is less than $\frac{1}{3}$, given that X is less than $\frac{2}{3}$.

7 Let X be a continuous random variable with probability density function:

$$f(x) = \begin{cases} 3x^2 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a Find $\Pr(X < 0.2)$.

b Find $\Pr(X < 0.2 \mid X < 0.3)$.

8 A continuous random variable X has probability density function:

$$f(x) = \begin{cases} \dfrac{\pi}{4}\cos\left(\dfrac{\pi x}{4}\right) & \text{if } 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Find the median value, m, of X.

9 The probability density function f of a random variable X is given by

$$f(x) = \begin{cases} \dfrac{x+2}{16} & \text{if } 0 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

a Find $\mathrm{E}(X)$.

b Find a such that $\Pr(X \le a) = \dfrac{5}{32}$.

10 The probability density function f of a random variable X is given by

$$f(x) = \begin{cases} c(1-x^2) & \text{if } -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a Find c.

b Find $\mathrm{E}(X)$.

11 Show that

$$f(x) = \begin{cases} n(1-x)^{n-1} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

is a probability function, where the constant n is a natural number.

12 The probability density function of X is given by

$$f(x) = \begin{cases} \dfrac{1}{x} & 1 \le x \le e \\ 0 & x > e \text{ or } x < 1 \end{cases}$$

a Find the median value of X.

b Find the interquartile range of X.

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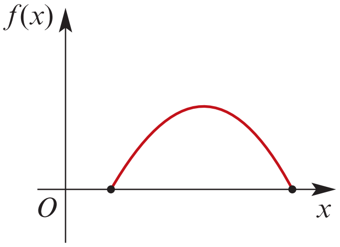
13 The amount of fluid, X mL, in a can of soft drink is a continuous random variable with mean 330 and standard deviation 5. Find an (approximate) interval for the amount of soft drink contained in 95% of the cans.

14 The weight, X g, of cereal in a packet is a continuous random variable with mean 250 and variance 4. Find an (approximate) interval for the weight of cereal contained in 95% of the packets.

Multiple-choice questions

1 Which of the following graphs could *not* represent a probability density function f?

A



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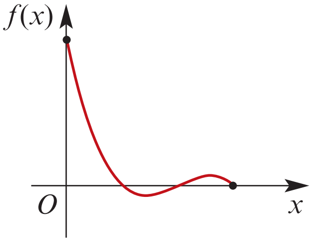

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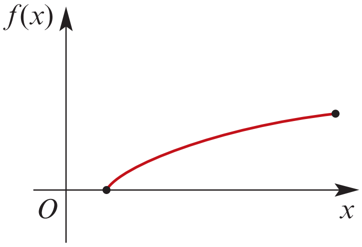
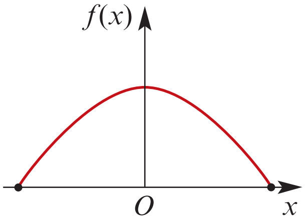
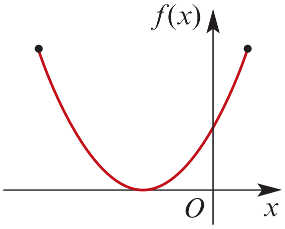


D



E



2 If the function $f(x) = 4x$ represents a probability density function, then which of the following could be the domain of f?

A $0 \leq x \leq 0.25$ **B** $0 \leq x \leq 0.5$ **C** $0 \leq x \leq 1$

D $0 \leq x \leq \dfrac{1}{\sqrt{2}}$ **E** $\dfrac{1}{\sqrt{2}} \leq x \leq \dfrac{2}{\sqrt{2}}$

3 If a random variable X has probability density function given by

$$f(x) = \begin{cases} \frac{1}{2}\sin x & 0 < x < k \\ 0 & x \geq k \text{ or } x \leq 0 \end{cases}$$

then k is equal to

A 1 **B** $\dfrac{\pi}{2}$ **C** 2 **D** π **E** 2π

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The following information relates to Questions 4, 5 and 6.

A random variable X has probability density function:

$$f(x) = \begin{cases} \frac{3}{4}(x^2 - 1) & 1 < x < 2 \\ 0 & x \le 1 \text{ or } x \ge 2 \end{cases}$$

4 $\Pr(X \le 1.3)$ is closest to

A 0.0743 **B** 0.4258 **C** 0.3 **D** 0.25 **E** 0.9258

5 The mean, $\mathrm{E}(X)$, of X is equal to

A 1 **B** $\frac{3}{2}$ **C** $\frac{9}{4}$ **D** $\frac{27}{32}$ **E** $\frac{27}{16}$

6 The variance of X is

A $\frac{27}{16}$ **B** $\frac{67}{1280}$ **C** $\frac{81}{16}$ **D** $\frac{81}{256}$ **E** $\frac{729}{256}$

7 If a random variable X has a probability density function given by

$$f(x) = \begin{cases} \frac{x^3}{4} & 0 \le x \le 2 \\ 0 & x > 2 \text{ or } x < 0 \end{cases}$$

then the median of X is closest to

A 1.5 **B** 1.4142 **C** 1.6818 **D** 1.2600 **E** 1

8 If a random variable X has a probability density function given by

$$f(x) = \begin{cases} \frac{3}{2}(x-1)(x-2)^2 & 1 \le x \le 3 \\ 0 & x < 1 \text{ or } x > 3 \end{cases}$$

then the mean of X is

A 1 **B** 1.333 **C** 2 **D** 2.6 **E** 3

9 If the consultation time (in minutes) at a surgery is represented by a random variable X which has probability density function

$$f(x) = \begin{cases} \frac{x}{40\,000}(400 - x^2) & 0 \le x \le 20 \\ 0 & x < 0 \text{ or } x > 20 \end{cases}$$

then the expected consultation time (in minutes) for three patients is

A $10\frac{2}{3}$ **B** 30 **C** 32 **D** 42 **E** $43\frac{2}{3}$

10 The top 10% of students in an examination will be awarded an 'A'. If the distribution of scores on the examination is a random variable X with probability density function

$$f(x) = \begin{cases} \frac{\pi}{100}\sin\left(\frac{\pi x}{50}\right) & 0 \le x \le 50 \\ 0 & x < 0 \text{ or } x > 50 \end{cases}$$

then the minimum score required to be awarded an 'A' is closest to

A 40 **B** 41 **C** 42 **D** 43 **E** 44

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Extended-response questions

1 The distribution of X, the life of a certain electronic component in hours, is described by the following probability density function:

$$f(x) = \begin{cases} \dfrac{a}{100}\left(1 - \dfrac{x}{100}\right) & 100 < x < 1000 \\ 0 & x \le 100 \text{ and } x \ge 1000 \end{cases}$$

a What is the value of a?

b Find the expected value of the life of the components.

c Find the median value of the life of the components.

2 The continuous random variable X has probability density function given by

$$f(x) = \begin{cases} 2 - 2x & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a Find $\Pr(X > 0.5)$.

b Find a such that $\Pr(X < a) = 0.8$.

c Find $\mathrm{E}(X)$ and $\mathrm{E}(\sqrt{X})$.

3 The probability density function of X is given by

$$f(x) = \begin{cases} \dfrac{\pi}{20}\cos\left(\dfrac{\pi(x-6)}{10}\right) & 1 \le x \le 11 \\ 0 & x < 1 \text{ or } x > 11 \end{cases}$$

a Find the median and the interquartile range of X.

b Find the mean and the variance of X.

4 A hardware shop sells a certain size nail either in a small packet at \$1 per packet, or loose at \$4 per kilogram. On any shopping day, the number, X, of packets sold is a binomial random variable with number of trials $n = 8$ and probability of success $p = 0.6$, and the weight, Y kg, of nails sold loose is a continuous random variable with probability density function f given by

$$f(y) = \begin{cases} \dfrac{2(y-1)}{25} & 1 \le y \le 6 \\ 0 & y < 1 \text{ or } y > 6 \end{cases}$$

a Find the probability that the weight of nails sold loose on any shopping day will be between 4 kg and 5 kg.

b Calculate the expected money received on any shopping day from the sale of this size nail in the shop.

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5 The continuous random variable X has the probability density function f, where

$$f(x) = \begin{cases} \dfrac{x-2}{2} & 2 \le x \le 4 \\ 0 & x < 2 \text{ or } x > 4 \end{cases}$$

By first expanding $(X - c)^2$, or otherwise, find two values of c such that

$$\mathrm{E}[(X - c)^2] = \frac{2}{3}$$

6 A wholesaler sells material offcuts in 5 m lengths. The amounts, in metres, of unuseable material are the values of a random variable, X, with probability density function:

$$f(x) = \begin{cases} k(x-1)(3-x) & 1 \le x \le 3 \\ 0 & x < 1 \text{ or } x > 3 \end{cases}$$

a Show that $k = 0.75$.

b Find the mean and variance of X.

c Find the probability that X is greater than 2.5 m.

7 The continuous random variable X has probability density function f, where

$$f(x) = \begin{cases} \dfrac{k}{12(x+1)^3} & 0 \le x \le 4 \\ 0 & x < 0 \text{ or } x > 4 \end{cases}$$

a Find k.

b Evaluate $\mathrm{E}(X + 1)$. Hence, find the mean of X.

c Use your calculator to verify your answer to part **b**.

d Find the value of $c > 0$ for which $\Pr(X \le c) = c$.

8 The yield of a variety of corn has probability density function:

$$f(x) = \begin{cases} kx & 0 \le x < 2 \\ k(4-x) & 2 \le x \le 4 \\ 0 & x < 0 \text{ or } x > 4 \end{cases}$$

a Find k.

b Find the expected value, μ, and the variance of the yield of corn.

c Find the probability $\Pr(\mu - 1 < X < \mu + 1)$.

d Find the value of a such that $\Pr(X > a) = 0.6$, giving your answer correct to one decimal place.

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Chapter 16

16

The normal distribution

Objectives

- To introduce the **standard normal distribution**.
- To introduce the family of normal distributions as transformations of the standard normal distribution.
- To investigate the effect that changing the values of the parameters defining the normal distribution has on the graph of the probability density function.
- To recognise the **mean**, **median**, **variance** and **standard deviation** of a normal distribution.
- To use technology to determine **probabilities for intervals** in the solution of problems where the normal distribution is appropriate.

The most useful continuous distribution, and one that occurs frequently, is the normal distribution. The probability density functions of normal random variables are symmetric, single-peaked, bell-shaped curves.

Data sets occurring in nature will often have such a bell-shaped distribution, as measurements on many random variables are closely approximated by a normal probability distribution.

Variables such as height, weight, IQ and the volume of milk in a milk carton are all examples of normally distributed random variables.

As well as helping us to understand better the behaviour of many real-world variables, the normal distribution also underpins the development of statistical estimation, which is the topic of Chapter 17.

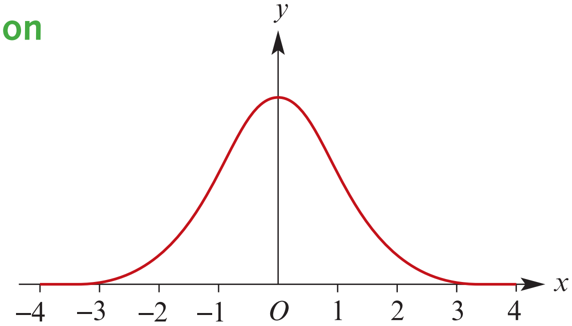
16A The normal distribution

The standard normal distribution

The simplest form of the normal distribution is a random variable with probability density function f given by

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

The domain of f is $\mathbb{R}$.



Because it is the simplest form of the normal distribution, it is given a special name: the **standard normal distribution**. The graph of the standard normal distribution is as shown.

The graph of the standard normal probability density function f is symmetric about $x = 0$, since $f(-x) = f(x)$. That is, the function f is even.

The line $y = 0$ is an asymptote: as $x \to \pm\infty$, $y \to 0$. Almost all of the area under the probability density function lies between $x = -3$ and $x = 3$.

The mean and standard deviation of the standard normal distribution

It can be seen from the graph that the mean and median of this distribution are the same, and are equal to 0. While the probability density function for the standard normal distribution cannot be integrated exactly, the value of the mean can be verified by observing the symmetry of the two integrals formed below. One is just the negative of the other.

$$\begin{aligned}
\mathrm{E}(X) &= \int_{-\infty}^{\infty} x f(x)\,dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{1}{2}x^2}\,dx \\
&= \frac{1}{\sqrt{2\pi}} \left(\int_{0}^{\infty} x e^{-\frac{1}{2}x^2}\,dx + \int_{-\infty}^{0} x e^{-\frac{1}{2}x^2}\,dx \right)
\end{aligned}$$

Thus the mean, $\mathrm{E}(X)$, of the standard normal distribution is 0.

What can be said about the standard deviation of this distribution? It can be shown that

$$\mathrm{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-\frac{1}{2}x^2}\,dx = 1$$

Therefore

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 = 1 - 0 = 1 \quad \text{and} \quad \mathrm{sd}(X) = \sqrt{\mathrm{Var}(X)} = 1$$

Standard normal distribution

A random variable with the standard normal distribution has probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

The standard normal distribution has mean $\mu = 0$ and standard deviation $\sigma = 1$.

Henceforth, we will denote the random variable of the standard normal distribution by Z.

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The general normal distribution

The normal distribution does not apply just to the special circumstances where the mean is 0 and the standard deviation is 1.

Transformations of the standard normal distribution

The graph of the probability density function for a normal distribution with mean μ and standard deviation σ may be obtained from the graph of the probability density function for the standard normal distribution by the transformation with rule:

$$(x, y) \to \left(\sigma x + \mu, \frac{y}{\sigma}\right)$$

This is a dilation of factor σ from the y-axis and a dilation of factor $\frac{1}{\sigma}$ from the x-axis, followed by a translation of μ units in the positive direction of the x-axis, for $\mu > 0$. (In Section 15D, this was discussed for probability density functions in general.)

Conversely, the transformation which maps the graph of a normal distribution with mean μ and standard deviation σ to the graph of the standard normal distribution is given by

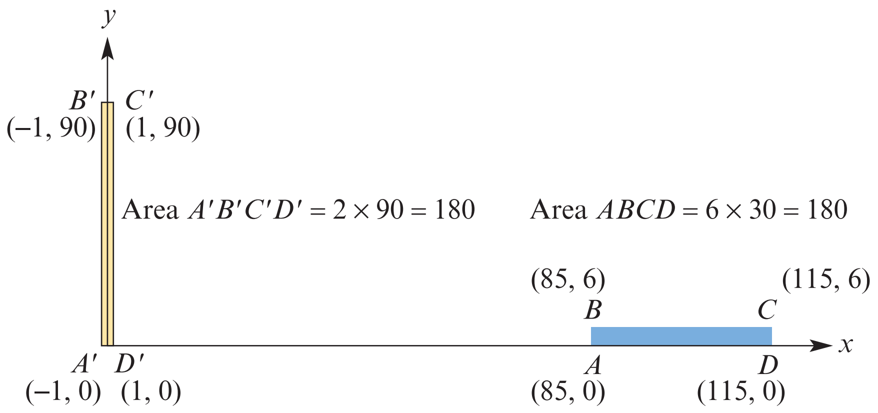
$$(x, y) \to \left(\frac{x - \mu}{\sigma}, \sigma y\right)$$

This is a translation of μ units in the negative direction of the x-axis, followed by a dilation of factor $\frac{1}{\sigma}$ from the y-axis and a dilation of factor σ from the x-axis.

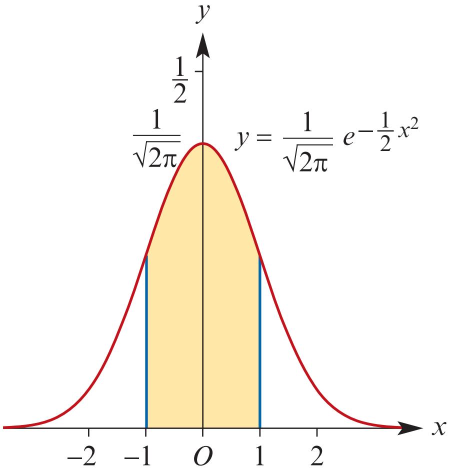
For example, if $\mu = 100$ and $\sigma = 15$, then this transformation is

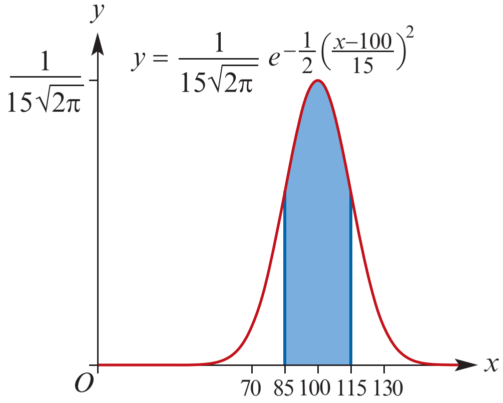
$$(x, y) \to \left(\frac{x - 100}{15}, 15y\right)$$

This transformation is area-preserving. In the following diagram, the rectangle $ABCD$ is mapped to $A'B'C'D'$. Both rectangles have an area of 180 square units.



This property enables the probabilities of any normal distribution to be determined from the probabilities of the standard normal distribution.





The shaded regions are of equal area.

This leads to the general rule for the family of normal probability distributions.

The rule for the general normal distribution

If X is a **normally distributed random variable** with mean μ and standard deviation σ, then the probability density function of X is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

and

$$\Pr(X \leq a) = \Pr\left(Z \leq \frac{a-\mu}{\sigma}\right)$$

where Z is the random variable of the standard normal distribution.

The general form of the normal density function involves two parameters, μ and σ, which are the mean (μ) and the standard deviation (σ) of that particular distribution.

When a random variable has a distribution described by a normal density function, the random variable is said to have a **normal distribution**.

As with all probability density functions, the normal density function has the fundamental properties that:

- probability corresponds to an area under the curve
- the total area under the curve is 1.

However, it has some additional special properties.

The graph of a normal density function is symmetric and bell-shaped:

- its **centre** is determined by the **mean** of the distribution
- its **width** is determined by the **standard deviation** of the distribution.

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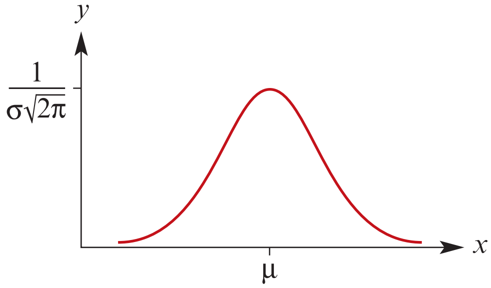
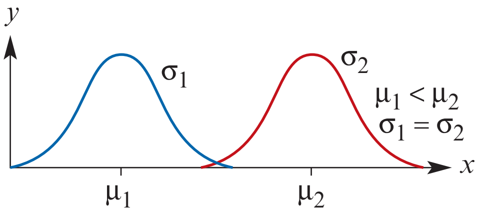
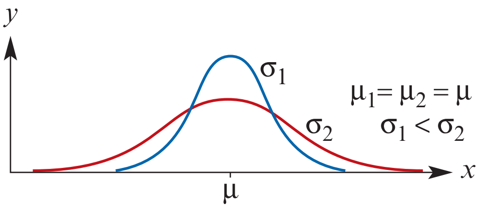
The graph of $y = f(x)$ is shown on the right.

The graph is symmetric about the line $x = \mu$, and has a maximum value of $\dfrac{1}{\sigma\sqrt{2\pi}}$, which occurs when $x = \mu$.



Thus the *location* of the curve is determined by the value of μ, and the *steepness* of the curve by the value of σ.





Irrespective of the values of the mean and standard deviation of a particular normal density function, the area under the curve within a given number of standard deviations from the mean is always the same.

Example 1

On the same set of axes, sketch the graphs of the probability density functions of the standard normal distribution and the normal distribution with:

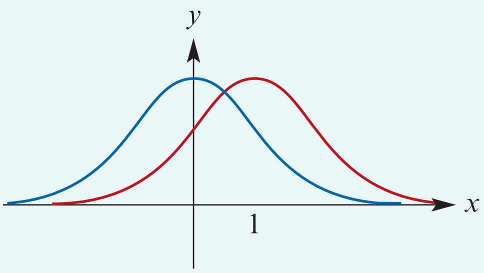
a mean 1 and standard deviation 1

b mean 1 and standard deviation 2.

(A calculator can be used to help.)

Solution

a The graph has been translated 1 unit in the positive direction of the x-axis.



The rules of the two density functions are $y = \dfrac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}x^2}$ and $y = \dfrac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}(x-1)^2}$.

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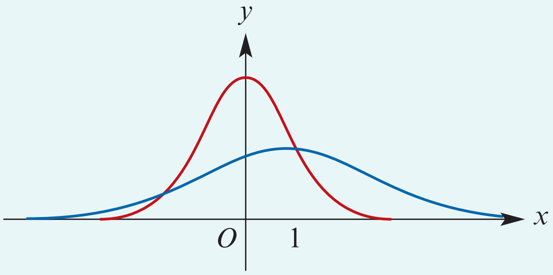
b The graph has been dilated from the y-axis by factor 2 and from the x-axis by factor $\frac{1}{2}$, and then translated 1 unit in the positive direction of the x-axis.



The rules of the two density functions are $y = \dfrac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}$ and $y = \dfrac{1}{2\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-1}{2}\right)^2}$.

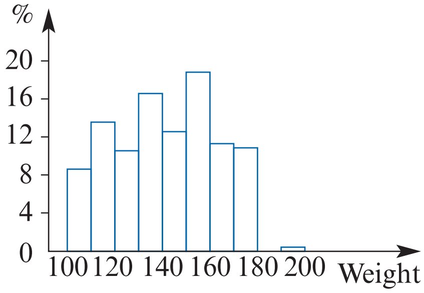
Exercise 16A

Example 1

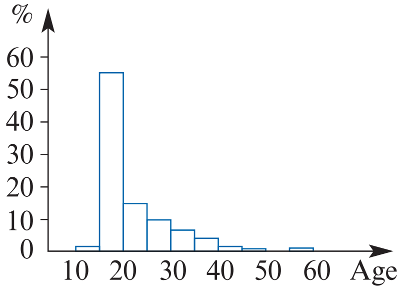
1 Both the random variables X_1 and X_2 are normally distributed, with means μ_1 and μ_2 and standard deviations σ_1 and σ_2, respectively. If $\mu_1 < \mu_2$ and $\sigma_1 < \sigma_2$, sketch both distributions on the same diagram.

2 Which of the following data distributions are approximately normally distributed?

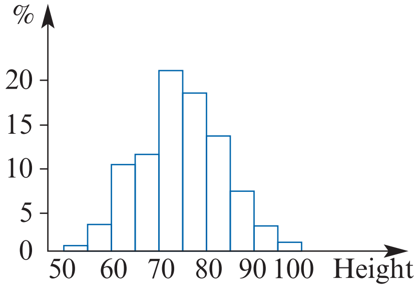
a



b



c



3 Consider the normal probability density function:

$$f(x) = \frac{1}{3\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-2}{3}\right)^2}, \quad x \in \mathbb{R}$$

a Use your calculator to find $\int_{-\infty}^{\infty} f(x)\, dx$.

b **i** Express $\text{E}(X)$ as an integral.

ii Use your calculator to evaluate the integral found in **i**.

c **i** Write down an expression for $\text{E}(X^2)$. **ii** What is the value of $\text{E}(X^2)$?

iii What is the value of σ?

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4 Consider the normal probability density function:

$$f(x) = \frac{1}{5\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x+4}{5}\right)^2}, \quad x \in \mathbb{R}$$

a Use your calculator to find $\int_{-\infty}^{\infty} f(x)\, dx$.

b **i** Express $\mathrm{E}(X)$ as an integral.

ii Use your calculator to evaluate the integral found in **i**.

c **i** Write down an expression for $\mathrm{E}(X^2)$.

ii What is the value of $\mathrm{E}(X^2)$?

iii What is the value of σ?

5 The probability density function of a normal random variable X is given by

$$f(x) = \frac{1}{10\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-3}{10}\right)^2}$$

a Write down the mean and the standard deviation of X.

b Sketch the graph of $y = f(x)$.

6 The probability density function of a normal random variable X is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x+3)^2}$$

a Write down the mean and the standard deviation of X.

b Sketch the graph of $y = f(x)$.

7 The probability density function of a normal random variable X is given by

$$f(x) = \frac{1}{\sqrt{18\pi}} e^{-\frac{1}{2}\left(\frac{x}{3}\right)^2}$$

a Write down the mean and the standard deviation of X.

b Sketch the graph of $y = f(x)$.

8 Describe the sequence of transformations which takes the graph of the probability density function of the standard normal distribution to the graph of the probability density function of the normal distribution with:

a $\mu = 3$ and $\sigma = 2$ **b** $\mu = 3$ and $\sigma = \frac{1}{2}$ **c** $\mu = -3$ and $\sigma = 2$

9 Describe the sequence of transformations which takes the graph of the probability density function of the normal distribution with the given mean and standard deviation to the graph of the probability density function of the standard normal distribution:

a $\mu = 3$ and $\sigma = 2$ **b** $\mu = 3$ and $\sigma = \frac{1}{2}$ **c** $\mu = -3$ and $\sigma = 2$

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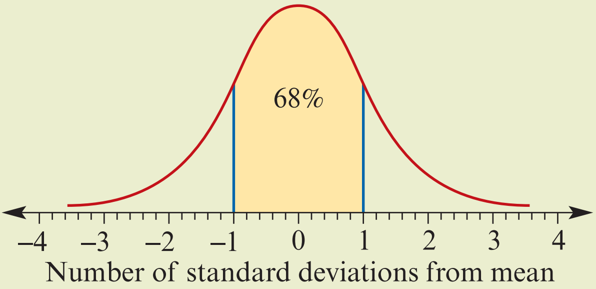
16B Standardisation and the 68–95–99.7% rule

For a set of data values that are normally distributed, approximately 68% of the values will lie within one standard deviation of the mean, approximately 95% of the values will lie within two standard deviations of the mean, and almost all (99.7%) within three standard deviations. This gives rise to what is known as the **68–95–99.7% rule**.

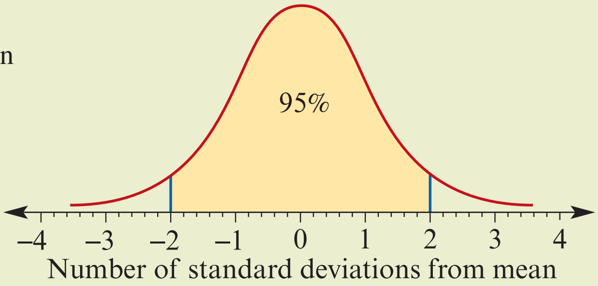
The 68–95–99.7% rule

For a normally distributed random variable, approximately:

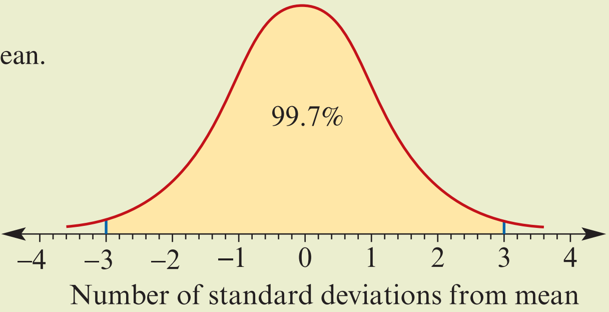
- 68% of the values lie within one standard deviation of the mean



- 95% of the values lie within two standard deviations of the mean



- 99.7% of the values lie within three standard deviations of the mean.



If we know that a random variable is approximately normally distributed, and we know its mean and standard deviation, then we can use the 68–95–99.7% rule to quickly make some important statements about the way in which the data values are distributed.

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Example 2

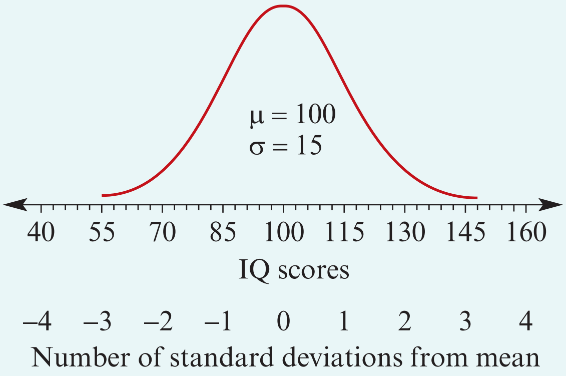
Experience has shown that the scores obtained on a commonly used IQ test can be assumed to be normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 15$.

Approximately what percentage of the distribution lies within one, two or three standard deviations of the mean?

Solution

Since the scores are normally distributed with $\mu = 100$ and $\sigma = 15$, the 68–95–99.7% rule means that approximately:

- 68% of the scores will lie between 85 and 115
- 95% of the scores will lie between 70 and 130
- 99.7% of the scores will lie between 55 and 145.



μ = 100
σ = 15
40 55 70 85 100 115 130 145 160
IQ scores
–4 –3 –2 –1 0 1 2 3 4
Number of standard deviations from mean

Note: In this example, we are using a continuous distribution to model a discrete situation.

Statements can also be made about the percentage of scores that lie in the tails of the distribution, by using the symmetry of the distribution and noting that the total area under the curve is 100%.

Example 3

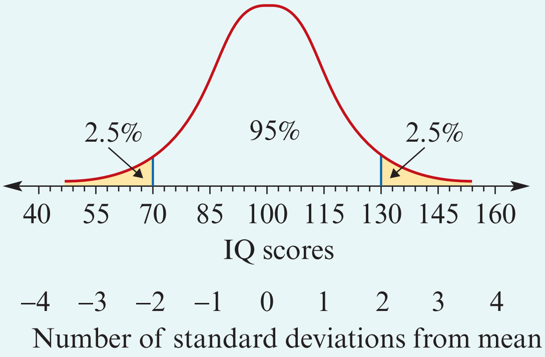
From Example 2, we know that 95% of the scores in the IQ distribution lie between 70 and 130 (that is, within two standard deviations of the mean). What percentage of the scores are *more* than two standard deviations above or below the mean (in this instance, less than 70 or greater than 130)?

Solution

If we focus our attention on the tails of the distribution, we see that 5% of the IQ scores lie outside this region.

Using the symmetry of the distribution, we can say that 2.5% of the scores are below 70, and 2.5% are above 130.

That is, if you obtained a score greater than 130 on this test, you would be in the top 2.5% of the group.



2.5%
95%
2.5%
40 55 70 85 100 115 130 145 160
IQ scores
–4 –3 –2 –1 0 1 2 3 4
Number of standard deviations from mean

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Standardised values

Clearly, the standard deviation is a natural measuring stick for normally distributed data. For example, a person who obtained a score of 112 on an IQ test with a mean of $\mu = 100$ and a standard deviation of $\sigma = 15$ is less than one standard deviation from the mean. Their score is typical of the group as a whole, as it lies well within the middle 68% of scores. In contrast, a person who scored 133 has done exceptionally well; their score is more than two standard deviations from the mean and this puts them in the top 2.5%.

Because of the additional insight provided, it is usual to convert normally distributed data to a new set of units which shows the number of standard deviations each data value lies from the mean of the distribution. These new values are called **standardised values** or ***z*-values**. To standardise a data value x, we first subtract the mean μ of the normal random variable from the value and then divide the result by the standard deviation σ. That is,

$$\text{standardised value} = \frac{\text{data value} - \text{mean of the normal curve}}{\text{standard deviation of the normal curve}}$$

or symbolically,

$$z = \frac{x-\mu}{\sigma}$$

Standardised values can be positive or negative:

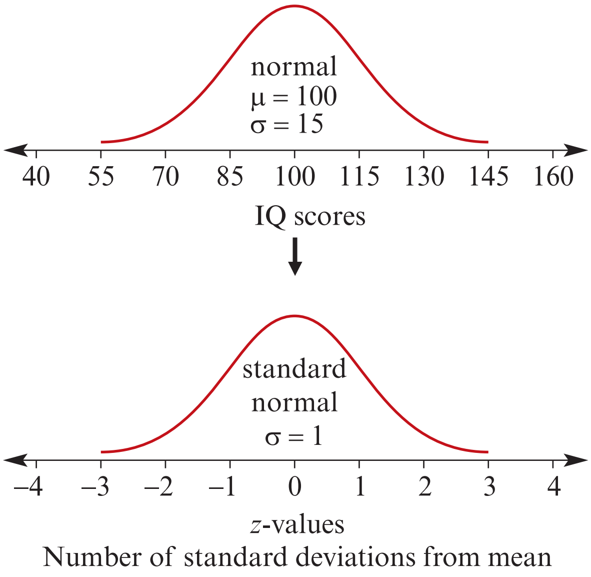
- A **positive *z*-value** indicates that the data value it represents lies **above** the mean.
- A **negative *z*-value** indicates that the data value lies **below** the mean.

For example, an IQ score of 90 lies *below* the mean and has a standardised value of

$$z = \frac{x-\mu}{\sigma} = \frac{90-100}{15} = \frac{-10}{15} \approx -0.67$$

There are as many different normal curves as there are values of μ and σ. But if the measurement scale is changed to 'standard deviations from the mean' or z-values, all normal curves reduce to the same normal curve with mean $\mu = 0$ and standard deviation $\sigma = 1$.

The figures on the right show how standardising IQ scores transforms a normal distribution with mean $\mu = 100$ and standard deviation $\sigma = 15$ into the standard normal distribution with mean $\mu = 0$ and standard deviation $\sigma = 1$.



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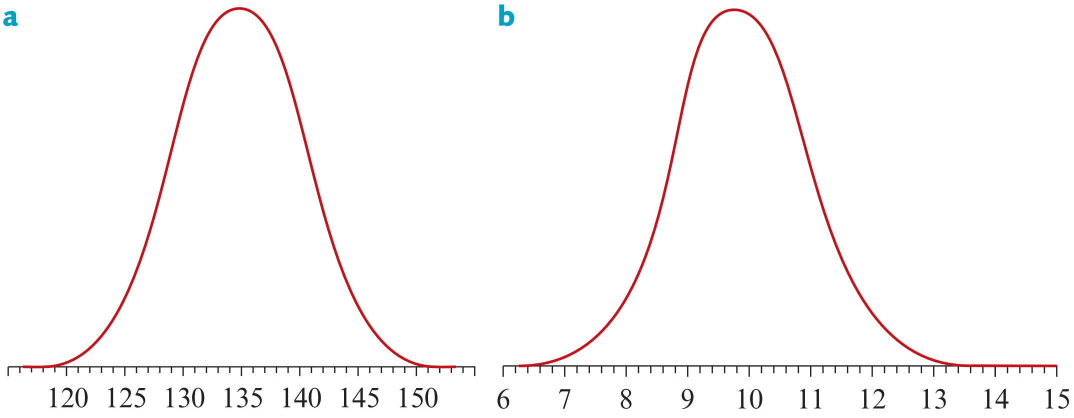
Exercise 16B

Example 2 **1** The scores obtained on an IQ test can be assumed to be normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 15$. What percentage of scores lie:

a above 115 **b** below 85

c above 130 **d** below 70?

2 State the values of the mean and standard deviation of the normal distributions shown:



Example 3 **3** The heights of young women are normally distributed with mean $\mu = 160$ cm and standard deviation $\sigma = 8$ cm. What percentage of the women would you expect to have heights:

a between 152 cm and 168 cm

b greater than 168 cm

c less than 136 cm?

4 Fill in the blanks in the following paragraph.

The age at marriage of males in the US in the 1980s was approximately normally distributed with a mean of $\mu = 27.3$ years and a standard deviation of $\sigma = 3.1$ years. From this data, we can conclude that in the 1980s about 95% of males married between the ages of ______ and ______.

5 Fill in the blanks in the following statement of the 68–95–99.7% rule.

For any normal distribution, about:

- 68% of the values lie within ______ standard deviation of the mean
- ______% of the values lie within two standard deviations of the mean
- ______% of the values lie within ______ standard deviations of the mean.

6 If you are told that in Australian adults, nostril width is approximately normally distributed with a mean of $\mu = 2.3$ cm and a standard deviation of $\sigma = 0.3$ cm, find the percentage of people with nostril widths less than 1.7 cm.

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7 The distribution of IQ scores for the inmates of a certain prison is approximately normal with mean $\mu = 85$ and standard deviation $\sigma = 15$.

a What percentage of the prison population have an IQ of 100 or higher?

b If someone with an IQ of 70 or less can be classified as having special needs, what percentage of the prison population could be classified as having special needs?

8 The distribution of the heights of navy officers was found to be normal with a mean of $\mu = 175$ cm and a standard deviation of $\sigma = 5$ cm. Determine:

a the percentage of navy officers with heights between 170 cm and 180 cm

b the percentage of navy officers with heights greater than 180 cm

c the approximate percentage of navy officers with heights greater than 185 cm.

9 The distribution of blood pressures (systolic) among women of similar ages is normal with a mean of 120 (mm of mercury) and a standard deviation of 10 (mm of mercury). Determine the percentage of women with a systolic blood pressure:

a between 100 and 140 **b** greater than 130

c greater than 120 **d** between 90 and 150.

10 The heights of women are normally distributed with mean $\mu = 160$ cm and standard deviation $\sigma = 8$ cm. What is the standardised value for the height of a woman who is:

a 160 cm tall **b** 150 cm tall **c** 172 cm tall?

11 The length of pregnancy for a human is approximately normally distributed with a mean of $\mu = 270$ days and a standard deviation of $\sigma = 10$ days. How many standard deviations away from the mean is a pregnancy of length:

a 256 days **b** 281 days **c** 305 days?

12 Michael scores 85 on the mathematics section of a scholastic aptitude test, the results of which are known to be normally distributed with a mean of 78 and a standard deviation of 5. Cheryl sits for a different mathematics ability test and scores 27. The scores from this test are normally distributed with a mean of 18 and a standard deviation of 6. Assuming that both tests measure the same kind of ability, who has the better score?

13 The following table gives a student's results in Biology and History. For each subject, the table gives the student's mark (x) and also the mean (μ) and standard deviation (σ) for the class.

	Mark (x)	Mean (μ)	Standard deviation (σ)	Standardised mark (z)
Biology	77	68.5	4.9	
History	79	75.3	4.1	

Complete the table by calculating the student's standardised mark for each subject, and use this to determine in which subject the student did best *relative* to her peers.

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14 Three students took different tests in French, English and Mathematics:

Student	Subject	Mark (x)	Mean (μ)	Standard deviation (σ)	Standardised mark (z)
Mary	French	19	15	4	
	English	42	35	8	
	Mathematics	20	20	5	
Steve	French	21	23	4	
	English	39	42	3	
	Mathematics	23	18	4	
Sue	French	15	15	5	
	English	42	35	10	
	Mathematics	19	20	5	

a Determine the standardised mark for each student on each test.

b Who is the best student in:

i French **ii** English **iii** Mathematics?

c Who is the best student overall? Give reasons for your answer.

16C Determining normal probabilities

A CAS calculator can be used to determine areas under normal curves, allowing us to find probabilities for ranges of values other than one, two or three standard deviations from the mean. The following example is for the standard normal distribution, but the same procedures can be used for any normal distribution by entering the appropriate values for μ and σ.

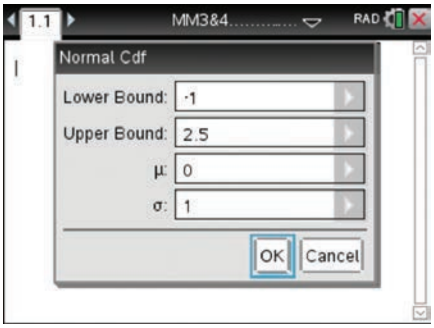
Example 4

Suppose that Z is a standard normal random variable (that is, it has mean $\mu = 0$ and standard deviation $\sigma = 1$). Find:

a $\Pr(-1 < Z < 2.5)$ **b** $\Pr(Z > 1)$

Using the TI-Nspire

a Use (menu) > **Probability** > **Distributions** > **Normal Cdf** and complete as shown.
(Use (tab) or ▼ to move between cells.)



Normal Cdf

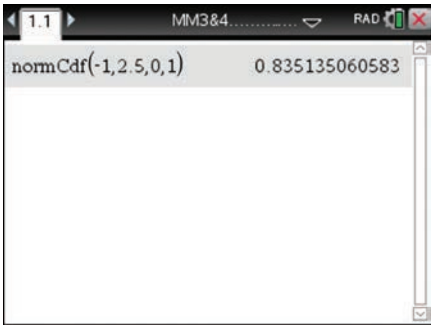
Lower Bound: -1

Upper Bound: 2.5

μ: 0

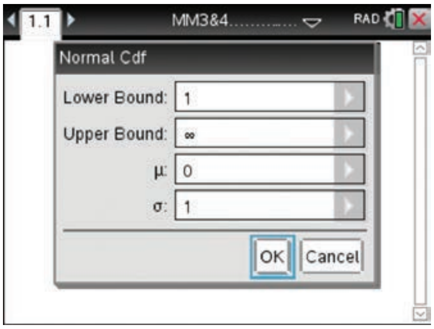
σ: 1

The answer is:
$\Pr(-1 < Z < 2.5) = 0.8351$



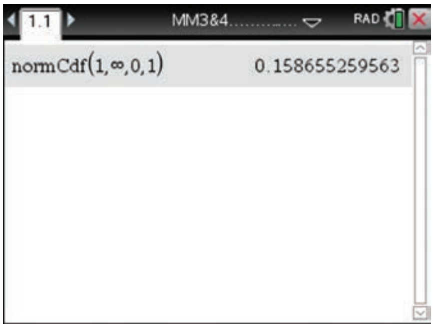
normCdf(-1,2.5,0,1) 0.835135060583

b Use (menu) > **Probability** > **Distributions** > **Normal Cdf** and complete as shown.
(The symbol ∞ can be found using (π▸) or (ctrl)(📖).)


1.1
MM3&4
RAD
Normal Cdf
OK
Cancel

Lower Bound: 1
Upper Bound: ∞
μ: 0
σ: 1

The answer is:
$\Pr(Z > 1) = 0.1587$



normCdf(1,∞,0,1) 0.158655259563

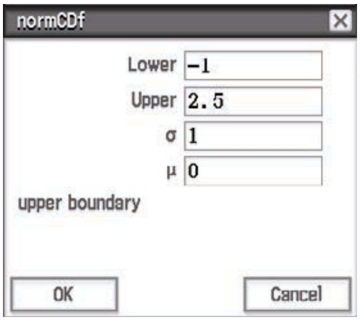
Note: You can enter the commands and parameters directly if preferred. The commands are not case sensitive.

Using the Casio ClassPad

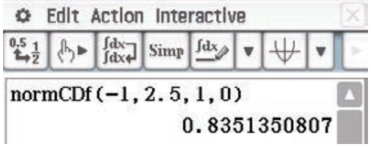
Method 1

a ■ In Main $\sqrt{\alpha}$, go to **Interactive** > **Distribution** > **Continuous** > **normCDf**.

■ Enter the lower and upper bounds and tap OK.


normCDf
OK
Cancel

Lower -1
Upper 2.5
σ 1
μ 0
upper boundary


Edit Action Interactive
Simp

normCDf(-1,2.5,1,0)
0.8351350807

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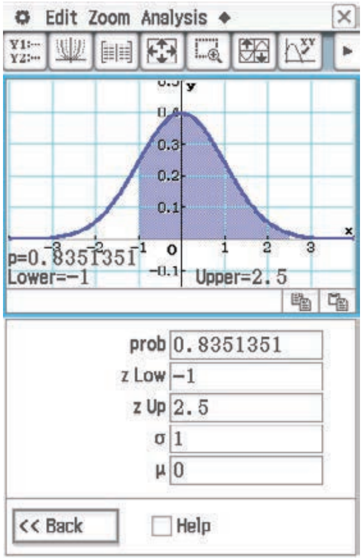
Method 2

- In **Statistics**, go to **Calc** > **Distribution** and select **Normal CD**. Tap Next.
- Enter values for the lower and upper bounds. Tap Next to view the answer.
- Select to view the graph with the answer.

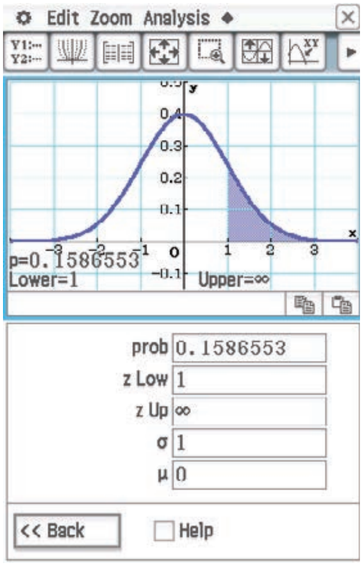
a


Edit Zoom Analysis
y
x
0.3
0.2
0.1
−1 0 1 2 3
−0.1
<< Back
☐ Help

p=0.8351351
Lower=−1
Upper=2.5

prob 0.8351351
z Low −1
z Up 2.5
σ 1
μ 0

b


Edit Zoom Analysis
y
x
0.4
0.3
0.2
0.1
−1 0 1 2 3
−0.1
<< Back
☐ Help

p=0.1586553
Lower=1
Upper=∞

prob 0.1586553
z Low 1
z Up ∞
σ 1
μ 0

The calculator can also be used to determine **percentiles** of any normal distribution.

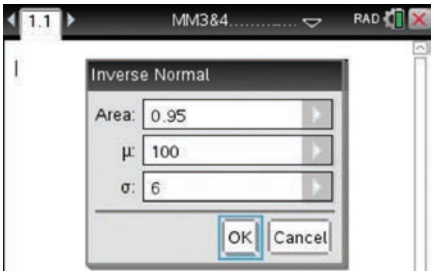
Example 5

Suppose X is normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 6$.

Find k such that $\Pr(X \leq k) = 0.95$.

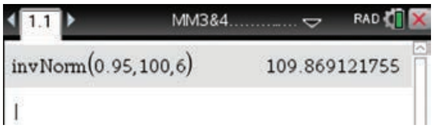
Using the TI-Nspire

Use (menu) > **Probability** > **Distributions** > **Inverse Normal** and complete as shown.


1.1
MM3&4
RAD
OK
Cancel

Inverse Normal
Area: 0.95
μ: 100
σ: 6

The value of k is 109.869.



invNorm(0.95,100,6) 109.869121755

Note: You can enter the command and parameters directly if preferred. The command is not case sensitive.

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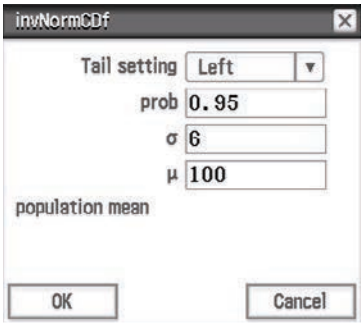
Using the Casio ClassPad

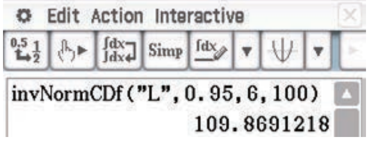
Method 1

- In Main $\sqrt{\alpha}$, go to **Interactive** > **Distribution** > **Inverse** > **InvNormCDf**.
- Set the 'Tail setting' as 'Left'.
- Enter the probability, 0.95, to the left of the required value k.
- Enter the standard deviation σ and the mean μ.
- Tap OK.

Note: The tail setting is 'Left' to indicate that we seek the value k such that 95% of the area lies to the left of k for this normal distribution.



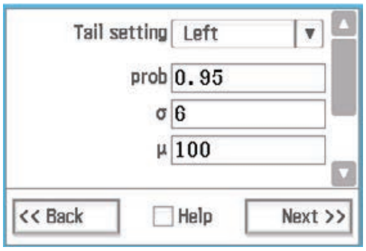
Tail setting Left
prob 0.95
σ 6
μ 100
population mean



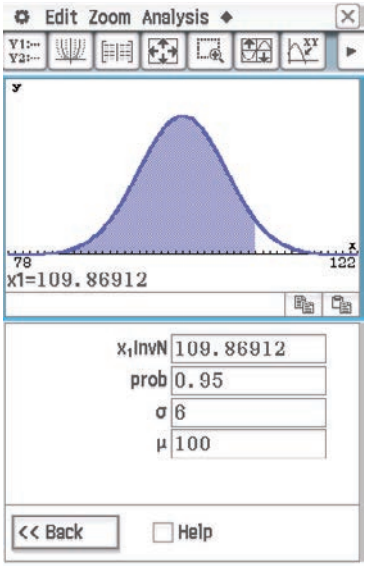
invNormCDf("L",0.95,6,100)
109.8691218

Method 2

- In the **Statistics** application Statistics, go to **Calc** > **Inv. Distribution**.
- Select **Inverse Normal CD** and tap Next.
- Set the 'Tail setting' as 'Left'.
- Enter the probability, 0.95, to the left.
- Enter the standard deviation σ and the mean μ.
- Tap Next to view the answer.
- Select ⊞ to view the graph with the answer.



Tail setting Left
prob 0.95
σ 6
μ 100



x1=109.86912

x₁InvN 109.86912
prob 0.95
σ 6
μ 100

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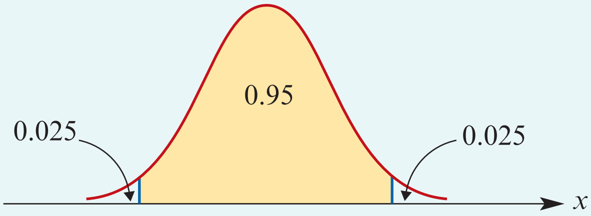
Example 6

Suppose X is normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 6$.

Find c_1 and c_2 such that $\Pr(c_1 < X < c_2) = 0.95$.

Solution

Examining the normal curve, we see that there are (infinitely) many intervals which enclose an area of 0.95. By convention, we choose the interval which leaves equal areas in each tail.



To find c_1 using the inverse-normal facility of your calculator, enter 0.025 as the area. To find c_2, enter 0.975.

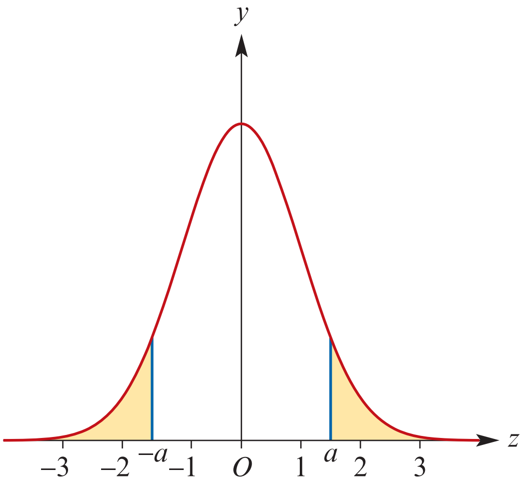
This will give the answer $c_1 = 88.240$ and $c_2 = 111.760$.

► Symmetry properties

Probabilities associated with a normal distribution can often be determined by using its symmetry properties.

Here we work with the standard normal distribution, as it is easiest to use the symmetry properties in this situation:

- $\Pr(Z > a) = 1 - \Pr(Z \leq a)$
- $\Pr(Z < -a) = \Pr(Z > a)$
- $\Pr(-a < Z < a) = 1 - 2\Pr(Z \geq a)$
  $= 1 - 2\Pr(Z \leq -a)$



Exercise 16C

Example 4 **1** Suppose Z is a standard normal random variable (that is, it has mean $\mu = 0$ and standard deviation $\sigma = 1$). Find the following probabilities, drawing an appropriate diagram in each case:

a $\Pr(Z < 2)$ **b** $\Pr(Z < 2.5)$ **c** $\Pr(Z \leq 2.5)$ **d** $\Pr(Z < 2.53)$
e $\Pr(Z \geq 2)$ **f** $\Pr(Z > 1.5)$ **g** $\Pr(Z \geq 0.34)$ **h** $\Pr(Z > 1.01)$

2 Suppose Z is a standard normal random variable. Find the following probabilities, drawing an appropriate diagram in each case:

a $\Pr(Z > -2)$ **b** $\Pr(Z > -0.5)$ **c** $\Pr(Z > -2.5)$ **d** $\Pr(Z \geq -1.283)$
e $\Pr(Z < -2)$ **f** $\Pr(Z < -2.33)$ **g** $\Pr(Z \leq -1.8)$ **h** $\Pr(Z \leq -0.95)$

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3 Suppose Z is a standard normal random variable. Find the following probabilities, drawing an appropriate diagram in each case:

a $\Pr(-1 < Z < 1)$ **b** $\Pr(-2 < Z < 2)$ **c** $\Pr(-3 < Z < 3)$

How do these results compare with the 68–95–99.7% rule discussed in Section 16B?

4 Suppose Z is a standard normal random variable. Find the following probabilities, drawing an appropriate diagram in each case:

a $\Pr(2 < Z < 3)$ **b** $\Pr(-1.5 < Z < 2.5)$

c $\Pr(-2 < Z < -1.5)$ **d** $\Pr(-1.4 < Z < -0.8)$

Example 5

5 Suppose Z is a standard normal random variable. Draw an appropriate diagram and then find the value c such that $\Pr(Z \leq c) = 0.9$.

6 Suppose Z is a standard normal random variable. Draw an appropriate diagram and then find the value c such that $\Pr(Z \leq c) = 0.75$.

7 Suppose Z is a standard normal random variable. Draw an appropriate diagram and then find the value c such that $\Pr(Z \leq c) = 0.975$.

8 Suppose Z is a standard normal random variable. Draw an appropriate diagram and then find the value c such that $\Pr(Z \geq c) = 0.95$.

9 Suppose Z is a standard normal random variable. Draw an appropriate diagram and then find the value c such that $\Pr(Z \geq c) = 0.8$.

10 Suppose Z is a standard normal random variable. Draw an appropriate diagram and then find the value c such that $\Pr(Z \leq c) = 0.10$.

11 Suppose Z is a standard normal random variable. Draw an appropriate diagram and then find the value c such that $\Pr(Z \leq c) = 0.025$.

12 Let X be a normal random variable with mean $\mu = 100$ and standard deviation $\sigma = 6$. Find:

a $\Pr(X < 110)$ **b** $\Pr(X < 105)$ **c** $\Pr(X > 110)$ **d** $\Pr(105 < X < 110)$

13 Let X be a normal random variable with mean $\mu = 40$ and standard deviation $\sigma = 5$. Find:

a $\Pr(X < 48)$ **b** $\Pr(X < 36)$ **c** $\Pr(X > 32)$ **d** $\Pr(32 < X < 36)$

14 Let X be a normal random variable with mean $\mu = 6$ and standard deviation $\sigma = 2$.

a Find c such that $\Pr(X < c) = 0.95$.

b Find k such that $\Pr(X < k) = 0.90$.

15 Let X be a normal random variable with mean $\mu = 10$ and standard deviation $\sigma = 3$.

a Find c such that $\Pr(X < c) = 0.50$.

b Find k such that $\Pr(X < k) = 0.975$.

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16 The 68–95–99.7% rule tells us approximately the percentage of a normal distribution which lies within one, two or three standard deviations of the mean. If Z is the standard normal random variable find, correct to two decimal places:

a a such that $\Pr(-a < Z < a) = 0.68$
b b such that $\Pr(-b < Z < b) = 0.95$
c c such that $\Pr(-c < Z < c) = 0.997$

17 Given that X is a normally distributed random variable with a mean of 22 and a standard deviation of 7, find:

a $\Pr(X < 26)$
b $\Pr(25 < X < 27)$
c $\Pr(X < 26 \mid 25 < X < 27)$
d c such that $\Pr(X < c) = 0.95$
e k such that $\Pr(X > k) = 0.9$
f c_1 and c_2 such that $\Pr(c_1 < X < c_2) = 0.95$

18 Let X be a normal random variable with mean $\mu = 10$ and standard deviation $\sigma = 0.5$. Find:

a $\Pr(X < 11)$
b $\Pr(X < 11 \mid X < 13)$
c c such that $\Pr(X < c) = 0.95$
d k such that $\Pr(X < k) = 0.2$
e c_1 and c_2 such that $\Pr(c_1 < X < c_2) = 0.95$

16D Solving problems using the normal distribution

The normal distribution can be used to solve many practical problems.

Example 7

The time taken to complete a logical reasoning task is normally distributed with a mean of 55 seconds and a standard deviation of 8 seconds.

a Find the probability, correct to four decimal places, that a randomly chosen person will take less than 50 seconds to complete the task.

b Find the probability, correct to four decimal places, that a randomly chosen person will take less than 50 seconds to complete the task, if it is known that this person took less than 60 seconds to complete the task.

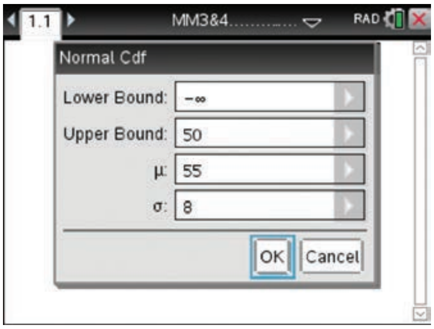
Using the TI-Nspire

a Method 1

Use (menu) > **Probability** > **Distributions** > **Normal Cdf** and complete as shown.

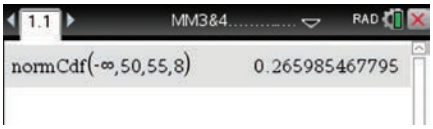
The answer is:

$\Pr(X < 50) = 0.2660$



Normal Cdf

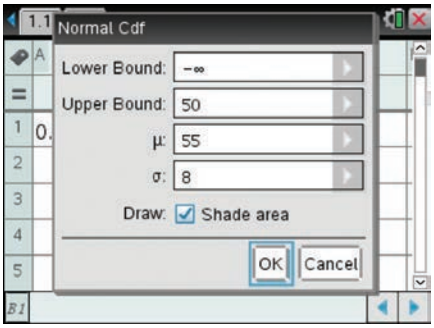
Lower Bound: -∞
Upper Bound: 50
μ: 55
σ: 8



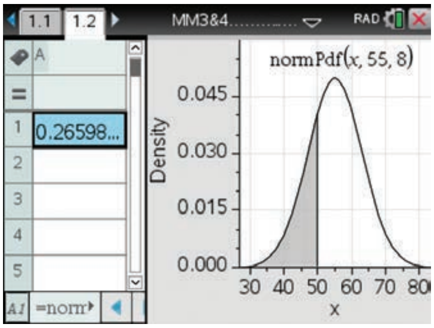
```
normCdf(-∞,50,55,8)        0.265985467795
```

Method 2

You can also solve this problem in a **Lists & Spreadsheet** page and plot the graph. Use (menu) > **Statistics** > **Distributions** > **Normal Cdf** and complete as shown below.


1.1
Normal Cdf
A
1
2
3
4
5
B1
OK
Cancel

Lower Bound: −∞
Upper Bound: 50
μ: 55
σ: 8
Draw: ☑ Shade area


1.1
1.2
MM3&4
RAD
A
1
2
3
4
5
A1 =norm
Density
0.045
0.030
0.015
0.000
30
40
50
60
70
80
x

0.26598...

normPdf(x, 55, 8)

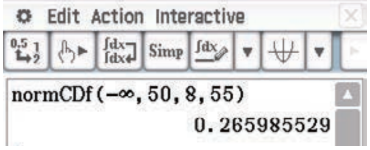
b $$\Pr(X<50 \mid X<60) = \frac{\Pr(X<50 \cap X<60)}{\Pr(X<60)}$$

$$= \frac{\Pr(X<50)}{\Pr(X<60)} = \frac{0.2660}{0.7340} = 0.3624$$

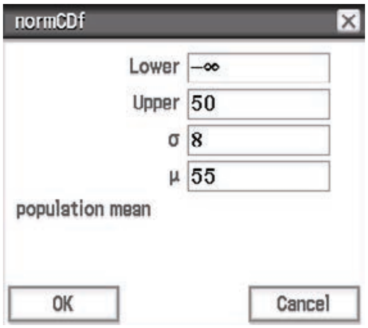
Using the Casio ClassPad

a Method 1

- In Main $\sqrt{\alpha}$, go to **Interactive** > **Distribution** > **Continuous** > **normCDf**.
- Enter values for the lower and upper bounds, the standard deviation and the mean. Tap OK.


Edit Action Interactive
Simp

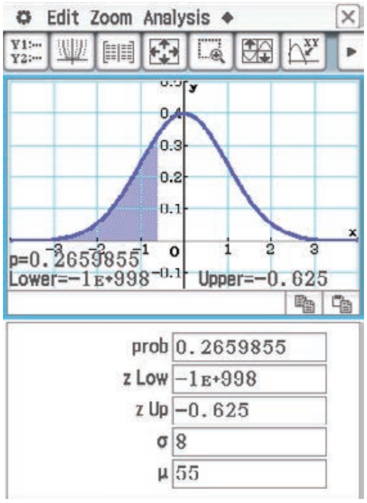
normCDf(−∞, 50, 8, 55)
0.265985529


normCDf
population mean
OK
Cancel

Lower −∞
Upper 50
σ 8
μ 55

Method 2

- In Statistics, go to **Calc** > **Distribution** and select **Normal CD**. Tap Next.
- Enter the lower and upper bounds, the standard deviation and the mean.
- Tap Next to view the answer.
- Select ⊕ to view the graph with the answer.


Edit Zoom Analysis
y
0.4
0.3
0.2
0.1
−0.1
x
−3
−2
−1
0
1
2
3

p=0.2659855
Lower=−1E+998
Upper=−0.625

prob 0.2659855
z Low −1E+998
z Up −0.625
σ 8
μ 55

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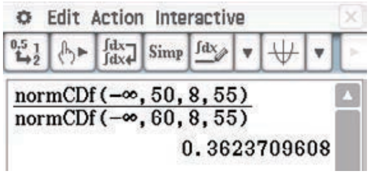
b
- In Main $\sqrt{\alpha}$, select the fraction template.
- Enter as shown and tap EXE.



```
normCDf(-∞,50,8,55)
normCDf(-∞,60,8,55)
                 0.3623709608
```

When the mean and standard deviation of a normal distribution are unknown, it is sometimes necessary to transform to the standard normal distribution. This is demonstrated in the following example.

Example 8

Limits of acceptability imposed on the lengths of a certain batch of metal rods are 1.925 cm and 2.075 cm. It is observed that, on average, 5% are rejected as undersized and 5% are rejected as oversized.

Assuming that the lengths are normally distributed, find the mean and standard deviation of the distribution.

Solution

It is given that $\Pr(X > 2.075) = 0.05$ and $\Pr(X < 1.925) = 0.05$.

Symmetry tells us that the mean is equal to

$$\mu = \frac{2.075 + 1.925}{2} = 2$$

Transforming to the standard normal gives

$$\Pr\left(Z > \frac{2.075 - \mu}{\sigma}\right) = 0.05 \quad \text{and} \quad \Pr\left(Z < \frac{1.925 - \mu}{\sigma}\right) = 0.05$$

The first equality can be rewritten as

$$\Pr\left(Z < \frac{2.075 - \mu}{\sigma}\right) = 0.95$$

Use the inverse-normal facility of your calculator to obtain

$$\frac{2.075 - \mu}{\sigma} = 1.6448\ldots \qquad \text{and} \qquad \frac{1.925 - \mu}{\sigma} = -1.6448\ldots$$

These equations confirm that $\mu = 2$.

Substitute $\mu = 2$ into the first equation and solve for σ:

$$\frac{2.075 - 2}{\sigma} = 1.6448\ldots$$

$$\therefore \qquad \sigma = 0.045596\ldots$$

Thus $\sigma = 0.0456$, correct to four decimal places.

Exercise 16D

Skillsheet

Example 7

1 Suppose that IQ scores are normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 15$.

a What is the probability that a person chosen at random has an IQ:

- **i** greater than 110
- **ii** less than 75
- **iii** greater than 130, given that they have an IQ greater than 110?

b To be allowed to join an elite club, a potential member must have an IQ in the top 5% of the population. What IQ score would be necessary to join this club?

2 The heights of women are normally distributed with a mean of $\mu = 160$ cm and a standard deviation of $\sigma = 8$ cm.

a What is the probability that a woman chosen at random would be:

- **i** taller than 155 cm
- **ii** shorter than 170 cm
- **iii** taller than 170 cm, given that her height is between 168 cm and 174 cm?

b What height would put a woman among the tallest 10% of the population?

c What height would put a woman among the shortest 20% of the population?

3 The results of a mathematics exam are normally distributed with mean $\mu = 50$ and standard deviation $\sigma = 7$.

a What is the probability that a student chosen at random has an exam mark:

- **i** greater than 60
- **ii** less than 75
- **iii** greater than 60, given that they passed? (Assume a pass mark of 50.)

b The top 15% of the class are to be awarded a distinction. What mark would be required to gain a distinction in this exam?

4 The lengths of a species of fish are normally distributed with a mean length of 40 cm and a standard deviation of 4 cm. Find the percentage of these fish having lengths:

a greater than 45 cm **b** between 35.5 cm and 45.5 cm.

Example 8

5 The weights of cats are normally distributed. It is known that 10% of cats weigh more than 1.8 kg, and 15% of cats weigh less than 1.35 kg. Find the mean and the standard deviation of this distribution.

6 The marks of a large number of students in a statistics examination are normally distributed with a mean of 48 marks and a standard deviation of 15 marks.

a If the pass mark is 53, find the percentage of students who passed the examination.

b If 8% of students gained an A on the examination by scoring a mark of at least c, find the value of c.

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7 The height of a certain population of adult males is normally distributed with mean 176 cm and standard deviation 7 cm.

a Find the probability that the height of a randomly selected male will exceed 190 cm.

b If two males are selected at random, find the probability that both of their heights will exceed 190 cm.

c Suppose 10 males are selected at random. Find the probability that at least two will have heights that exceed 190 cm.

8 **a** Machine A is packaging bags of mints with a mean weight of 300 grams. The bags are considered underweight if they weigh less than 295 grams. It is observed that, on average, 5% of bags are rejected as underweight. Assuming that the weights of the bags are normally distributed, find the standard deviation of the distribution.

b In the same factory, machine B is packaging bags of liquorice. The bags from this machine are considered underweight if they weigh less than 340 grams. It is observed that, on average, 2% of bags from machine B are rejected as underweight. Assuming that the weights are normally distributed with a standard deviation of 5 grams, find the mean of the distribution.

9 The volume of soft drink in a 1-litre bottle is normally distributed. The soft drink company needs to calibrate its filling machine. They don't want to put too much soft drink into each bottle, as it adds to their expense. However, they know they will be fined if more than 2% of bottles are more than 2 millilitres under volume. The standard deviation of the volume dispensed by the filling machine is 2.5 millilitres. What should they choose as the target volume (i.e. the mean of the distribution)? Give your answer to the nearest millilitre.

10 The weights of pumpkins sold to a greengrocer are normally distributed with a mean of 1.2 kg and a standard deviation of 0.4 kg. The pumpkins are sold in three sizes:

Small: under 0.8 kg Medium: from 0.8 kg to 1.8 kg Large: over 1.8 kg

a Find the proportions of pumpkins in each of the three sizes.

b The prices of the pumpkins are \$2.80 for a small, \$3.50 for a medium, and \$5.00 for a large. Find the expected cost for 100 pumpkins chosen at random from the greengrocer's supply.

11 Potatoes are delivered to a chip factory in semitrailer loads. A sample of 1 kg of the potatoes is chosen from each load and tested for starch content. From past experience it is known that the starch content is normally distributed with a standard deviation of 2.1.

a For a semitrailer load of potatoes with a mean starch content of 22.0:

- **i** What is the probability that the test reading is 19.5 or less?
- **ii** What reading will be exceeded with a probability of 0.98?

b If the starch content is greater than 22.0, the potatoes cannot be used for chips, and so the semitrailer load is rejected. What is the probability that a load with a mean starch content of 18.0 will be rejected?

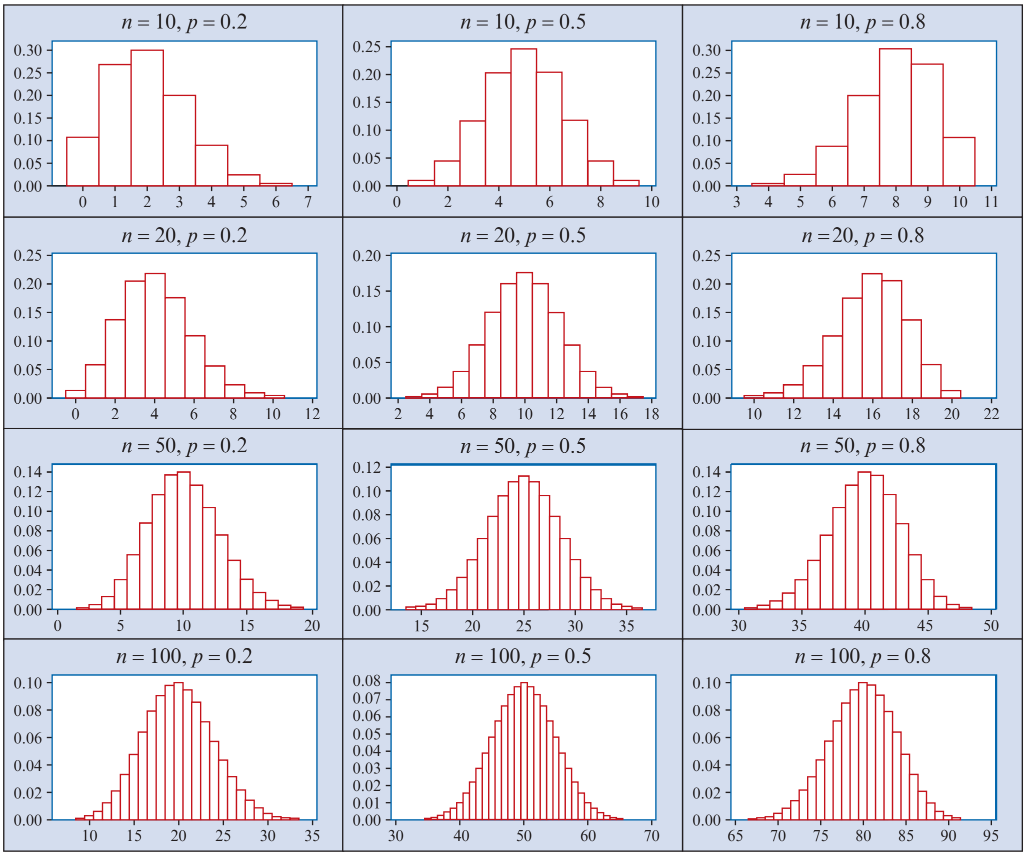
12 The amount of a certain chemical in a type A cell is normally distributed with a mean of 10 and a standard deviation of 1. The amount in a type B cell is normally distributed with a mean of 14 and a standard deviation of 2. To determine whether a cell is type A or type B, the amount of chemical in the cell is measured. The cell is classified as type A if the amount is less than a specified value c, and as type B otherwise.

a If $c = 12$, calculate the probability that a type A cell will be misclassified, and the probability that a type B cell will be misclassified.

b Find the value of c for which the two probabilities of misclassification are equal.

16E The normal approximation to the binomial distribution

We saw in Chapter 14 that the shape of the binomial distribution depends on n and p. The following plots show the binomial distribution for $n = 10, 20, 50, 100$ and $p = 0.2, 0.5, 0.8$.



We can see that, if n is small and p is close to 0 or 1, these distributions are skewed. Otherwise, they look remarkably symmetric. In fact, if n is large enough and p is not too close to 0 or 1, the binomial distribution is approximately normal. Moreover, the mean and standard deviation of this normal distribution agree with those of the binomial distribution.

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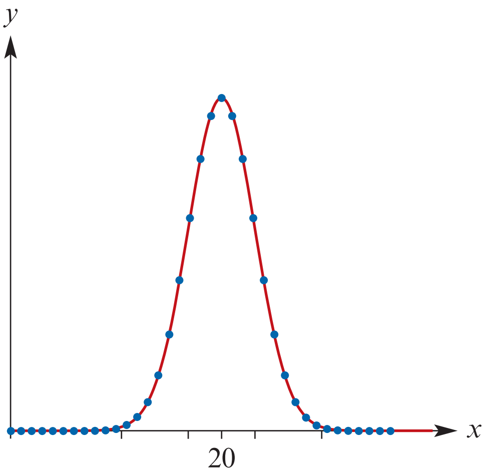
In the figure opposite, the binomial distribution with $n = 40$ and $p = 0.5$ is plotted (the blue points). This distribution has mean $\mu = 20$ and standard deviation $\sigma = \sqrt{10}$.

On the same axes, the probability density function of the normal distribution with mean $\mu = 20$ and standard deviation $\sigma = \sqrt{10}$ is drawn (the red curve).

We will see that this approximation has important uses in statistics.



When is it appropriate to use the normal approximation?

If n is large enough, the skew of the binomial distribution is not too great. In this case, the normal distribution can be used as a reasonable approximation to the binomial distribution. The approximation is generally better for larger n and when p is not too close to 0 or 1.

> If n is sufficiently large, the binomial random variable X will be approximately normally distributed, with a mean of $\mu = np$ and a standard deviation of $\sigma = \sqrt{np(1-p)}$.
>
> One rule of thumb is that:
>
> Both np and $n(1-p)$ must be greater than 5 for a satisfactory approximation.

In the example shown in the figure above, we have $np = 20$ and $n(1-p) = 20$. There are ways of improving this approximation but we will not go into that here.

Example 9

A sample of 1000 people from a certain city were asked to indicate whether or not they were in favour of the construction of a new freeway. It is known that 30% of people in this city are in favour of the new freeway. Find the approximate probability that between 270 and 330 people in the sample were in favour of the new freeway.

Solution

Let X be the number of people in the sample who are in favour of the freeway. Then we can assume that X is a binomial random variable with $n = 1000$ and $p = 0.3$.

Therefore

$$\begin{aligned} \mu &= np \\ &= 1000 \times 0.3 \\ &= 300 \end{aligned} \qquad \text{and} \qquad \begin{aligned} \sigma &= \sqrt{np(1-p)} \\ &= \sqrt{1000 \times 0.3 \times 0.7} \\ &= \sqrt{210} \end{aligned}$$

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Thus

$$\Pr(270 < X < 330) \approx \Pr\left(\frac{270 - 300}{\sqrt{210}} < Z < \frac{330 - 300}{\sqrt{210}}\right)$$

$$\approx \Pr(-2.070 < Z < 2.070)$$

$$\approx 0.9616$$

Note: When we calculate this probability directly using the binomial distribution, we find that $\Pr(270 \le X \le 330) = 0.9648$ and $\Pr(270 < X < 330) = 0.9583$.

Exercise 16E

In each of the following questions, use the normal approximation to the binomial distribution.

Example 9

1 A die is rolled 100 times. What is the probability that more than 10 sixes will be observed?

2 If 50% of the voting population in a particular state favour candidate A, what is the approximate probability that more than 156 in a sample of 300 will favour that candidate.

3 A sample of 100 people is drawn from a city in which it is known that 10% of the population is over 65 years of age. Find the approximate probability that the sample contains:

a at least 15 people who are over 65 years of age

b no more than 8 people over 65 years of age.

4 A manufacturing process produces on average 40 defective items per 1000. What is the approximate probability that a random sample of size 400 contains:

a at least 10 and no more than 20 defective items

b 25 or more defective items?

5 A survey of the entire population in a particular city found that 40% of people regularly participate in sport. What is the approximate probability that fewer than 38% of a random sample of 200 people regularly participate in sport?

6 An examination consists of 25 multiple-choice questions. Each question has four possible answers. At least 10 correct answers are required to pass the examination. Suppose that a student guesses the answer to each question.

a What is the approximate probability that the student will pass the examination?

b What is the approximate probability that the student guesses from 12 to 14 answers correctly?

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Chapter summary

Spreadsheet AS Nrich

- A special continuous random variable X, called a **normal random variable**, has a probability density function given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

  where μ and σ are the mean and standard deviation of X.
- In the special case that $\mu = 0$ and $\sigma = 1$, this probability density function defines the **standard normal distribution**. A random variable with this distribution is usually denoted by Z.
- The graph of a normal density function is a symmetric, bell-shaped curve; its centre is determined by the mean, μ, and its width by the standard deviation, σ.
- The **68–95–99.7% rule** states that, for any normal distribution:
  - approximately 68% of the values lie within one standard deviation of the mean
  - approximately 95% of the values lie within two standard deviations of the mean
  - approximately 99.7% of the values lie within three standard deviations of the mean.
- If X is a normally distributed random variable with mean μ and standard deviation σ, then to **standardise** a value x of X we subtract the mean and divide by the standard deviation:

$$z = \frac{x-\mu}{\sigma}$$

  The standardised value z indicates the number of standard deviations that the value x lies above or below the mean.
- A calculator can be used to evaluate the cumulative distribution function of a normal random variable – that is, to find the area under the normal curve up to a specified value.
- The inverse-normal facility of a calculator can be used to find the value of a normal random variable corresponding to a specified area under the normal curve.

Technology-free questions

1 Given that $\Pr(Z \le a) = p$ for the standard normal random variable Z, find in terms of p:

a $\Pr(Z > a)$ **b** $\Pr(Z < -a)$ **c** $\Pr(-a \le Z \le a)$

2 Let X be a normal random variable with mean 4 and standard deviation 1. Let Z be the standard normal random variable.

a If $\Pr(X < 3) = \Pr(Z < a)$, then $a =$ ☐.

b If $\Pr(X > 5) = \Pr(Z > b)$, then $b =$ ☐.

c $\Pr(X > 4) =$ ☐

3 A normal random variable X has mean 8 and standard deviation 3. Give the rule for a transformation that maps the graph of the density function of X to the graph of the density function for the standard normal distribution.

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4 Let X be a normal random variable with mean μ and standard deviation σ. If $\mu < a < b$ with $\Pr(X < b) = p$ and $\Pr(X < a) = q$, find:

a $\Pr(X < a \mid X < b)$ **b** $\Pr(X < 2\mu - a)$ **c** $\Pr(X > b \mid X > a)$

5 Let X be a normal random variable with mean 4 and standard deviation 2. Write each of the following probabilities in terms of Z:

a $\Pr(X < 5)$ **b** $\Pr(X < 3)$ **c** $\Pr(X > 5)$

d $\Pr(3 < X < 5)$ **e** $\Pr(3 < X < 6)$

In Questions 6 to 8, you will use the following:

$$\Pr(Z < 1) = 0.84$$

$$\Pr(Z < 2) = 0.98$$

$$\Pr(Z < 0.5) = 0.69$$

6 A machine produces metal rods with mean diameter 2.5 mm and standard deviation 0.05 mm. Let X be the random variable of the normal distribution. Find:

a $\Pr(X < 2.55)$ **b** $\Pr(X < 2.5)$

c $\Pr(X < 2.45)$ **d** $\Pr(2.45 < X < 2.55)$

7 Nuts are packed in tins such that the mean weight of the tins is 500 g and the standard deviation is 5 g. The weights are normally distributed with random variable W. Find:

a $\Pr(W > 505)$ **b** $\Pr(500 < W < 505)$

c $\Pr(W > 505 \mid W > 500)$ **d** $\Pr(W > 510)$

8 A random variable X has a normal distribution with mean 6 and standard deviation 1. Find:

a $\Pr(X < 6.5)$ **b** $\Pr(6 < X < 6.5)$

c $\Pr(6.5 < X < 7)$ **d** $\Pr(5 < X < 7)$

9 Suppose that three tests were given in your mathematics course. The class means and standard deviations, together with your scores, are listed in the table.

	μ	σ	Your score
Test A	50	11	62
Test B	47	17	64
Test C	63	8	73

On which test did you do best and on which did you do worst?

10 Let X be a normally distributed random variable with mean 10 and variance 4, and let Z be a random variable with the standard normal distribution.

a Find $\Pr(X > 10)$.

b Find b such that $\Pr(X > 13) = \Pr(Z < b)$.

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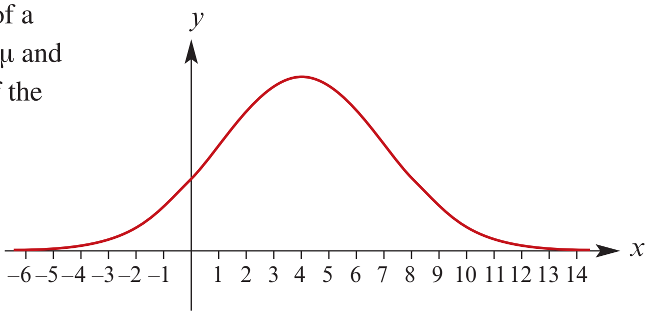
Multiple-choice questions

1 The diagram shows the graph of a normal distribution with mean μ and standard deviation σ. Which of the following statements is true?



A $\mu = 4$ and $\sigma = 3$
B $\mu = 3$ and $\sigma = 4$
C $\mu = 4$ and $\sigma = 2$
D $\mu = 3$ and $\sigma = 2$
E $\mu = 4$ and $\sigma = 4$

2 If Z is a standard normal random variable, then $\Pr(Z > 1.45) =$

A 0.1394 **B** 0.8606 **C** 0.0735 **D** 0.9625 **E** 0.0925

3 If Z is a standard normal random variable and $\Pr(Z < c) = 0.25$, then the value of c is closest to

A 0.6745 **B** −0.6745 **C** 0.3867 **D** 0.5987 **E** −0.5987

4 The random variable X has a normal distribution with mean 12 and variance 9. If Z is a standard normal random variable, then the probability that X is more than 15 is equal to

A $\Pr(Z < 1)$ **B** $\Pr(Z > 1)$ **C** $\Pr\left(Z > \frac{1}{3}\right)$
D $1 - \Pr\left(Z > \frac{1}{3}\right)$ **E** $1 - \Pr(Z > 1)$

5 If the actual length of an AFL game is normally distributed with a mean of 102 minutes and a standard deviation of 3 minutes, then the percentage of games that last more than 110 minutes is approximately

A 96.2% **B** 81.3% **C** 2.7% **D** 18.7% **E** 0.38%

6 If the number of goals that Collingwood scores in a match is a normally distributed random variable with a mean of 16 and a standard deviation of 2, then in what percentage of their matches (approximately) do they score from 10 to 22 goals?

A 5% **B** 16% **C** 68% **D** 95% **E** 99.7%

7 If X is a normally distributed random variable with mean $\mu = 6$ and standard deviation $\sigma = 3$, then the transformation which maps the graph of the density function f of X to the graph of the standard normal distribution is

A $(x, y) \to \left(\frac{x-3}{6}, 6y\right)$ **B** $(x, y) \to \left(\frac{x-6}{3}, \frac{y}{3}\right)$ **C** $(x, y) \to \left(\frac{x-6}{3}, 3y\right)$
D $(x, y) \to (3(x+6), 3y)$ **E** $(x, y) \to \left(3(x+6), \frac{y}{3}\right)$

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8 The amount of water that Steve uses to water the garden is normally distributed with a mean of 100 litres and a standard deviation of 14 litres. On 20% of occasions it takes him more than k litres to water the garden. What is the value of k?

A 88.2 **B** 110.7 **C** 120.0 **D** 111.8 **E** 114.0

9 The marks achieved by Angie in Mathematics, Indonesian and Politics, together with the mean and standard deviation for each subject, are given in the following table:

Subject	Mark	Mean (μ)	Standard deviation (σ)
Mathematics	72	72	5
Indonesian	57	59	2
Politics	68	64	4

Which of the following statements is correct?

A Angie's best subject was Politics, followed by Mathematics and then Indonesian.
B Angie's best subject was Mathematics, followed by Politics and then Indonesian.
C Angie's best subject was Politics, followed by Indonesian and then Mathematics.
D Angie's best subject was Mathematics, followed by Indonesian and then Politics.
E Angie's best subject was Indonesian, followed by Mathematics and then Politics.

10 Suppose that X is normally distributed with mean 11.3 and standard deviation 2.9. Values of c_1 and c_2 such that $\Pr(c_1 < X < c_2) = 0.90$ are closest to

A 5.5, 17.1 **B** 6.08, 16.52 **C** 15.02, 7.58 **D** 6.53, 16.07 **E** 5.62, 16.98

11 The volume of liquid in a 1-litre bottle of soft drink is a normally distributed random variable with a mean of μ litres and a standard deviation of 0.005 litres. To ensure that 99.9% of the bottles contain at least 1 litre of soft drink, the value of μ should be closest to

A 0.995 litres **B** 1.0 litres **C** 1.005 litres **D** 1.015 litres **E** 1.026 litres

12 The gestation period for human pregnancies in a certain country is normally distributed with a mean of 272 days and a standard deviation of σ days. If from a population of 1000 births there were 91 pregnancies of length less than 260 days, then σ is closest to

A 3 **B** 5 **C** 9 **D** 12 **E** 16

Extended-response questions

1 A test devised to measure mathematical aptitude gives scores that are normally distributed with a mean of 50 and a standard deviation of 10. If we wish to categorise the results so that the highest 10% of scores are designated as high aptitude, the next 20% as moderate aptitude, the middle 40% as average, the next 20% as little aptitude and the lowest 10% as no aptitude, then what ranges of scores will be covered by each of these five categories?

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2 If X is normally distributed with $\mu = 10$ and $\sigma = 2$, find the value of k such that

$$\Pr(\mu - k \le X \le \mu + k) = 0.95$$

3 Records kept by a manufacturer of car tyres suggest that the distribution of the mileage from their tyres is normal, with mean 60 000 km and standard deviation 5000 km.

a What proportion of the company's tyres last:

i less that 55 000 km

ii more than 50 000 km but less than 74 000 km

iii more than 72 000 km, given that they have already lasted more than 60 000 km?

b The company's advertising manager wishes to claim that '90% of our tyres last longer than c km'. What should c be?

c What is the probability that a customer buys five tyres at the same time and finds that they all last longer than 72 000 km?

4 The owner of a new van complained to the dealer that he was using, on average, 18 litres of petrol to drive 100 km. The dealer pointed out that the 15 litres per 100 km referred to in an advertisement was 'just a guide and actual consumption will vary'. Suppose that the distribution of fuel consumption for this make of van is normal, with a mean of 15 litres per 100 km and a standard deviation of 0.75 litres per 100 km.

a How probable is it that such a van uses at least 18 litres per 100 km?

b What does your answer to **a** suggest about the manufacturer's claim?

c Find c_1 and c_2 such that the van's fuel consumption is more than c_1 but less than c_2 with a probability of 0.95.

5 Suppose that L, the useful life (in hours) of a fluorescent tube designed for indoor gardening, is normally distributed with a mean of 600 and a standard deviation of 4. The fluorescent tubes are sold in boxes of 10. Find the probability that at least three of the tubes in a randomly selected box last longer than 605 hours.

6 The amount of anaesthetic required to cause surgical anaesthesia in patients is normally distributed, with a mean of 50 mg and a standard deviation of 10 mg. The lethal dose is also normally distributed, with a mean of 110 mg and a standard deviation of 20 mg. If a dosage that brings 90% of patients to surgical anaesthesia were used, what percentage of patients would be killed by this dose?

7 In a given manufacturing process, components are rejected if they have a particular dimension greater than 60.4 mm or less than 59.7 mm. It is found that 3% are rejected as being too large and 5% are rejected as being too small. Assume that the dimension is normally distributed.

a Find the mean and standard deviation of the distribution of the dimension, correct to one decimal place.

b Use the result of **a** to find the percentage of rejects if the limits for acceptance are changed to 60.3 mm and 59.6 mm.

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8 The hardness of a metal may be determined by impressing a hardened point into the surface of the metal and then measuring the depth of penetration of the point. Suppose that the hardness of a particular alloy is normally distributed with mean 70 and standard deviation 3.

a If a specimen is acceptable only if its hardness is between 65 and 75, what is the probability that a randomly chosen specimen has an acceptable hardness?

b If the acceptable range of hardness was $(70 - c, 70 + c)$, for what value of c would 95% of all specimens have acceptable hardness?

c If the acceptable range is the same as in **a**, and the hardness of each of 10 randomly selected specimens is independently determined, what is the expected number of acceptable specimens among the 10?

d What is the probability that at most eight out of 10 randomly selected specimens have a hardness less than 73.84?

e The profit on an acceptable specimen is \$20, while unacceptable specimens result in a loss of \$5. If \$$P$ is the profit on a randomly selected specimen, find the mean and variance of P.

9 The weekly error (in seconds) of a brand of watch is known to be normally distributed. Only those watches with an error of less than 5 seconds are acceptable.

a Find the mean and standard deviation of the distribution of error if 3% of watches are rejected for losing time and 3% are rejected for gaining time.

b Determine the probability that fewer than two watches are rejected in a batch of 10 such watches.

10 A brand of detergent is sold in bottles of two sizes: standard and large. For each size, the content (in litres) of a randomly chosen bottle is normally distributed with mean and standard deviation as given in the table:

	Mean	Standard deviation
Standard bottle	0.760	0.008
Large bottle	1.010	0.009

a Find the probability that a randomly chosen standard bottle contains less than 0.75 litres.

b Find the probability that a box of 10 randomly chosen standard bottles contains at least three bottles whose contents are each less than 0.75 litres.

c Using the results

$$\mathrm{E}(aX - bY) = a\mathrm{E}(X) - b\mathrm{E}(Y)$$
$$\mathrm{Var}(aX - bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y)$$

find the probability that there is more detergent in four randomly chosen standard bottles than in three randomly chosen large bottles. (Assume that $aX - bY$ is normally distributed.)

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Chapter 17

17

Sampling and estimation

Objectives

- To understand **random samples** and how they may be obtained.
- To define the **population proportion** and the **sample proportion**.
- To introduce the concept of the sample proportion as a random variable.
- To investigate the **sampling distribution** of the sample proportion both exactly (for small samples) and through simulation.
- To use a **normal distribution** to approximate the sampling distribution of the sample proportion.
- To use the sample proportion as a **point estimate** of the population proportion.
- To find **confidence intervals** for the population proportion.
- To introduce the concept of **margin of error**, and illustrate how this varies both with level of confidence and with sample size.

There is more to a complete statistical investigation than data analysis. First, we should concern ourselves with the methods used to collect the data. In practice, the purpose of selecting a sample and analysing the information collected from the sample is to make some sort of conclusion, or inference, about the population from which the sample was drawn. Therefore we want the sample we select to be representative of this population.

For example, consider the following questions:

- What proportion of Year 12 students intend to take a gap year?
- What proportion of people aged 18–25 regularly attend church?
- What proportion of secondary students take public transport to school?

While we can answer each of these questions for a sample of people from the group, we really want to know something about the whole group. How can we generalise information gained from a sample to the population, and how confident can we be in that generalisation?

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17A Populations and samples

The set of all eligible members of a group which we intend to study is called a **population**. For example, if we are interested in the IQ scores of the Year 12 students at ABC Secondary College, then this group of students could be considered a population; we could collect and analyse all the IQ scores for these students. However, if we are interested in the IQ scores of all Year 12 students across Australia, then this becomes the population.

Often, dealing with an entire population is not practical:

- The population may be too large – for example, all Year 12 students in Australia.
- The population may be hard to access – for example, all blue whales in the Pacific Ocean.
- The data collection process may be destructive – for example, testing every battery to see how long it lasts would mean that there were no batteries left to sell.

Nevertheless, we often wish to make statements about a property of a population when data about the entire population is unavailable.

The solution is to select a subset of the population – called a **sample** – in the hope that what we find out about the sample is also true about the population it comes from. Dealing with a sample is generally quicker and cheaper than dealing with the whole population, and a well-chosen sample will give much useful information about this population. How to select the sample then becomes a very important issue.

► Random samples

Suppose we are interested in investigating the effect of sustained computer use on the eyesight of a group of university students. To do this we go into a lecture theatre containing the students and select all the students sitting in the front two rows as our sample. This sample may be quite inappropriate, as students who already have problems with their eyesight are more likely to be sitting at the front, and so the sample may not be typical of the population. To make valid conclusions about the population from the sample, we would like the sample to have a similar nature to the population.

While there are many sophisticated methods of selecting samples, the general principle of sample selection is that the method of choosing the sample should not favour or disfavour any subgroup of the population. Since it is not always obvious if the method of selection will favour a subgroup or not, we try to choose the sample so that every member of the population has an equal chance of being in the sample. In this way, all subgroups have a chance of being represented. The way we do this is to choose the sample at random.

> A sample of size n is called a **simple random sample** if it is selected from the population in such a way that every subset of size n has an equal chance of being chosen as the sample. In particular, every member of the population must have an equal chance of being included in the sample.

To choose a sample from the group of university students, we could put the name of every student in a hat and then draw out, one at a time, the names of the students who will be in the sample.

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Choosing the sample in an appropriate manner is critical in order to obtain usable results.

Example 1

A researcher wishes to evaluate how well the local library is catering to the needs of a town's residents. To do this she hands out a questionnaire to each person entering the library over the course of a week. Will this method result in a random sample?

Solution

Since the members of the sample are already using the library, they are possibly satisfied with the service available. Additional valuable information might well be obtained by finding out the opinion of those who do not use the library.

A better sample would be obtained by selecting at random from the town's entire population, so the sample contains both people who use the library and people who do not.

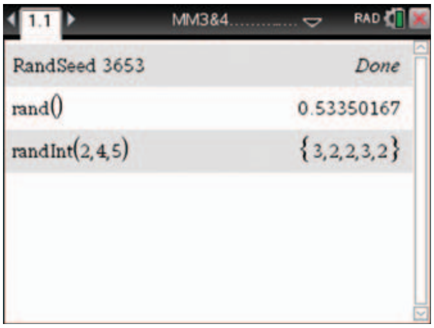
Thus, we have a very important consideration when sampling if we wish to generalise from the results of the sample.

In order to make valid conclusions about a population from a sample, we would like the sample chosen to be representative of the population as a whole. This means that all the different subgroups present in the population appear in the sample in similar proportions as they do in the population.

One very useful method for drawing random samples is to generate random numbers using a calculator or a computer.

Using the TI-Nspire

- In a **Calculator** page, go to Menu > **Probability** > **Random** > **Seed** and enter the last 4 digits of your phone number. This ensures that your random-number starting point differs from the calculator default.
- For a random number between 0 and 1, use Menu > **Probability** > **Random** > **Number**.
- For a random integer, use Menu > **Probability** > **Random** > **Integer**. To obtain five random integers between 2 and 4 inclusive, use the command randInt(2, 4, 5) as shown.



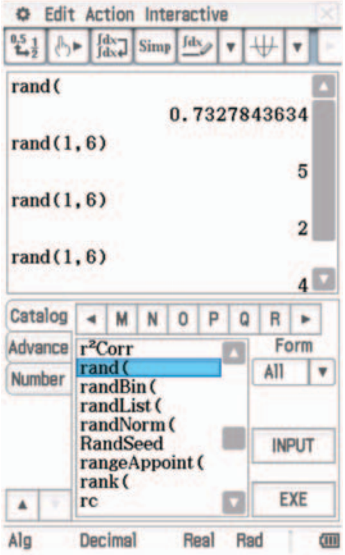
```
RandSeed 3653                 Done
rand()                  0.53350167
randInt(2,4,5)         {3,2,2,3,2}
```

Using the Casio ClassPad

- In $\overset{\text{Main}}{\sqrt{\alpha}}$, press the Keyboard button.
- Find and then select Catalog by first tapping ▼ at the bottom of the left sidebar.
- Scroll across the alphabet to the letter R.

- To generate a random number between 0 and 1:
  - In Catalog, select **rand(**.
  - Tap EXE.

- To generate three random integers between 1 and 6 inclusive:
  - In Catalog, select **rand(**.
  - Type: 1, 6)
  - Tap EXE three times.

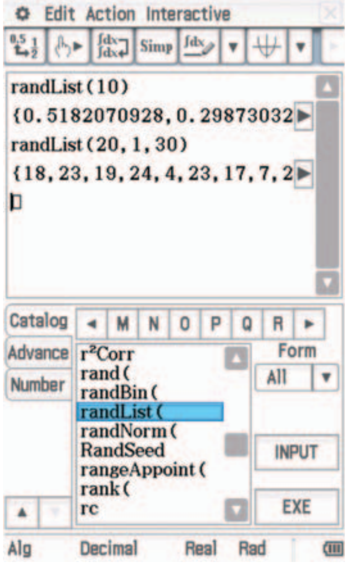


```
rand(
                0.7327843634
rand(1,6)
                           5
rand(1,6)
                           2
rand(1,6)
                           4
```

Edit Action Interactive
Simp
Catalog
M N O P Q R
Advance
Number
r²Corr
rand(
randBin(
randList(
randNorm(
RandSeed
rangeAppoint(
rank(
rc
Form
All
INPUT
EXE
Alg Decimal Real Rad

- To generate a list of 10 random numbers between 0 and 1:
  - In Catalog, select **randList(**.
  - Type: 10)
  - Tap EXE.
  - Tap ► to view all the numbers.

- To generate a list of 20 random integers between 1 and 30 inclusive:
  - In Catalog, select **randList(**.
  - Type: 20, 1, 30)
  - Tap EXE.
  - Tap ► to view all the integers.



```
randList(10)
{0.5182070928,0.29873032▶
randList(20,1,30)
{18,23,19,24,4,23,17,7,2▶
```

Edit Action Interactive
Simp
Catalog
M N O P Q R
Advance
Number
r²Corr
rand(
randBin(
randList(
randNorm(
RandSeed
rangeAppoint(
rank(
rc
Form
All
INPUT
EXE
Alg Decimal Real Rad

Example 2

Use a random number generator to select a group of six students from the following class:

- Denise
- Matt
- Teresa
- Sue
- Sharyn
- Mark
- Peter
- Nick
- Miller
- William
- Anne
- Darren
- Tom
- David
- Sally
- Janelle
- Steven
- Jane
- Georgia
- Jaimie

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Solution

First assign a number to each member of the class:

- Denise (1)
- Matt (2)
- Teresa (3)
- Sue (4)
- Sharyn (5)
- Mark (6)
- Peter (7)
- Nick (8)
- Miller (9)
- William (10)
- Anne (11)
- Darren (12)
- Tom (13)
- David (14)
- Sally (15)
- Janelle (16)
- Steven (17)
- Jane (18)
- Georgia (19)
- Jaimie (20)

Generating six random integers from 1 to 20 gives on this occasion: 4, 19, 9, 2, 13, 14. The sample chosen is thus:

Sue, Georgia, Miller, Matt, Tom, David

Note: In this example, we want a list of six random integers without repeats. We do not add a randomly generated integer to our list if it is already in the list.

The sample proportion as a random variable

Suppose that our population of interest is the class of students from Example 2, and suppose further that we are particularly interested in the proportion of female students in the class. This is called the **population proportion** and is generally denoted by p. The population proportion p is constant for a particular population.

$$\text{Population proportion } p = \frac{\text{number in population with attribute}}{\text{population size}}$$

Since there are 10 males and 10 females, the proportion of female students in the class is

$$p = \frac{10}{20} = \frac{1}{2}$$

Now consider the proportion of female students in the sample chosen:

Sue, Georgia, Miller, Matt, Tom, David

The proportion of females in the sample may be calculated by dividing the number of females in the sample by the sample size. In this case, the proportion of female students in the sample is $\frac{2}{6} = \frac{1}{3}$. This value is called the **sample proportion** and is denoted by $\hat{p}$. (We say ‘p hat’.)

$$\text{Sample proportion } \hat{p} = \frac{\text{number in sample with attribute}}{\text{sample size}}$$

Note that different symbols are used for the sample proportion and the population proportion, so that we don’t confuse them.

In this particular case, $\hat{p} = \frac{1}{3}$, which is not the same as the population proportion $p = \frac{1}{2}$. This does not mean there is a problem. In fact, each time a sample is selected the number of females in the sample will vary. Sometimes the sample proportion $\hat{p}$ will be $\frac{1}{2}$, and sometimes it will not.

- The population proportion p is a **population parameter**; its value is constant.
- The sample proportion $\hat{p}$ is a **sample statistic**; its value is not constant, but varies from sample to sample.

Example 3

Use a random number generator to select another group of six students from the same class, and determine the proportion of females in the sample.

- Denise (1)
- Matt (2)
- Teresa (3)
- Sue (4)
- Sharyn (5)
- Mark (6)
- Peter (7)
- Nick (8)
- Miller (9)
- William (10)
- Anne (11)
- Darren (12)
- Tom (13)
- David (14)
- Sally (15)
- Janelle (16)
- Steven (17)
- Jane (18)
- Georgia (19)
- Jaimie (20)

Solution

Generating another six random integers from 1 to 20 gives 19, 3, 11, 9, 15, 1.

The sample chosen is thus:

Georgia, Teresa, Anne, Miller, Sally, Denise

For this sample, we have

$$\hat{p} = \frac{5}{6}$$

Since $\hat{p}$ varies according to the contents of the random samples, we can consider the sample proportions $\hat{p}$ as being the values of a random variable, which we will denote by $\hat{P}$. We investigate this idea further in the next section.

Section summary

- A **population** is the set of all eligible members of a group which we intend to study.
- A **sample** is a subset of the population which we select in order to make inferences about the population. Generalising from the sample to the population will not be useful unless the sample is representative of the population.
- A sample of size n is called a **simple random sample** if it is selected from the population in such a way that every subset of size n has an equal chance of being chosen as the sample. In particular, every member of the population must have an equal chance of being included in the sample.
- The **population proportion** p is the proportion of individuals in the entire population possessing a particular attribute, and is constant.
- The **sample proportion** $\hat{p}$ is the proportion of individuals in a particular sample possessing the attribute, and varies from sample to sample.
- The sample proportions $\hat{p}$ are the values of a random variable $\hat{P}$.

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Exercise 17A

Example 1

1 In order to determine the sort of film in which to invest his money, a producer waits outside a theatre and asks people as they leave whether they prefer comedy, drama, horror or science fiction. Do you think this is an appropriate way of selecting a random sample of movie goers? Explain your answer.

2 A market researcher wishes to find out how people spend their leisure time. She positions herself in a shopping mall and asks shoppers as they pass to fill out a short questionnaire.

a Do you think this sample will be representative of the general population? Explain.

b How would you suggest that the sample could be chosen?

3 To investigate people's attitudes to control of gun ownership, a television station conducts a phone-in poll, where people are asked to telephone one number if they are in favour of tighter gun control, and another if they are against. Is this an appropriate method of choosing a random sample? Give reasons for your answer.

4 A researcher wishes to select five guinea pigs at random from a large cage containing 20 guinea pigs. In order to select her sample, she reaches into the cage and (gently) pulls out five guinea pigs.

a Do you think this sample will be representative of the general population? Explain.

b How would you suggest the sample could be chosen?

5 In order to estimate how much money young people spend on takeaway food, a questionnaire is sent to several schools randomly chosen from a list of all schools in the state, to be given to a random selection of students in the school. Is this an appropriate method of choosing a random sample? Give reasons for your answer.

Example 2

6 Use a random number generator to select a random sample of size 3 from the following list of people:

- Karen
- Kate
- Trinh
- Paul
- Lina
- Alexander
- Juliet
- Craig
- Conrad
- Fairlie
- Kylie
- Edward
- Kelly
- Rani
- Maree
- Janet
- Fleur
- Connie
- Aden
- Wolfgang
- Zoe
- Cara
- Noel
- Judy
- Andrew

7 In a survey to obtain adults' views on unemployment, people were stopped by interviewers as they came out of:

a a travel agency **b** a supermarket **c** an employment-services centre.

What is wrong with each of the methods of sampling listed here? Describe a better method of choosing the sample.

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8 A marine biologist wishes to estimate the total number of crabs on a rock platform which is 10 metres square. It would be impossible to count them all individually, so she places a 1-metre-square frame at five random locations on the rock platform, and counts the number of crabs in the frame. To estimate the total number, she will multiply the average number in the frame by the total area of the rock platform.

a Explain how a random number generator could be used to select the five locations for the frame.

b Will this give a good estimate of the crab population?

9 In order to survey the attitude of parents to the current uniform requirements, the principal of a school selected 100 students at random from the school roll, and then interviewed their parents. Do you think this group of parents would form a simple random sample?

10 A television station carried out a poll to find out if the public felt that mining should be allowed in a particular area. People were asked to ring one number to register a 'yes' vote and another to register a 'no' vote. The results showed that 77% of people were in favour of mining proceeding. Comment on the results.

11 A market-research company decided to collect information concerning the way people use their leisure time by phoning a randomly chosen group of 1000 people at home between 7 p.m. and 10 p.m. on weeknights. The final report was based on the responses of only the 550 people of those sampled who could be found at home. Comment on the validity of this report.

12 In a certain school, 35% of the students travel on the school bus. A group of 100 students were selected in a random sample, and 42 of them travel on the school bus. In this example:

a What is the population?

b What is the value of the population proportion p?

c What is the value of the sample proportion $\hat{p}$?

13 Of a random sample of 100 homes, 22 were found to have central heating.

a What proportion of these homes have central heating?

b Is this the value of the population proportion p or the sample proportion $\hat{p}$?

Example 3 **14** Use a random number generator to select another group of six students from the class listed below, and determine the proportion of females in the sample:

- Denise (1)
- Matt (2)
- Teresa (3)
- Sue (4)
- Sharyn (5)
- Mark (6)
- Peter (7)
- Nick (8)
- Miller (9)
- William (10)
- Anne (11)
- Darren (12)
- Tom (13)
- David (14)
- Sally (15)
- Janelle (16)
- Steven (17)
- Jane (18)
- Georgia (19)
- Jaimie (20)

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17B The exact distribution of the sample proportion

We have seen that the sample proportion varies from sample to sample. We can use our knowledge of probability to further develop our understanding of the sample proportion.

▶ Sampling from a small population

Suppose we have a bag containing six blue balls and four red balls, and from the bag we take a sample of size 4. We are interested in the proportion of blue balls in the sample. We know that the population proportion is equal to $\frac{6}{10} = \frac{3}{5}$. That is,

$$p = 0.6$$

The probabilities associated with the possible values of the sample proportion $\hat{p}$ can be calculated either by direct consideration of the sample outcomes or by using our knowledge of selections. Recall that

$$\binom{n}{x} = \frac{n!}{x!\,(n-x)!}$$

is the number of different ways to select x objects from n objects.

Example 4

A bag contains six blue balls and four red balls. If we take a random sample of size 4, what is the probability that there is one blue ball in the sample ($\hat{p} = \frac{1}{4}$)?

Solution

Method 1

Consider selecting the sample by taking one ball from the bag at a time (without replacement). The favourable outcomes are RRRB, RRBR, RBRR and BRRR, with

$$\Pr(\{\text{RRRB}, \text{RRBR}, \text{RBRR}, \text{BRRR}\})$$
$$= \left(\frac{4}{10} \times \frac{3}{9} \times \frac{2}{8} \times \frac{6}{7}\right) + \left(\frac{4}{10} \times \frac{3}{9} \times \frac{6}{8} \times \frac{2}{7}\right) + \left(\frac{4}{10} \times \frac{6}{9} \times \frac{3}{8} \times \frac{2}{7}\right) + \left(\frac{6}{10} \times \frac{4}{9} \times \frac{3}{8} \times \frac{2}{7}\right)$$
$$= \frac{4}{35}$$

Method 2

In total, there are $\binom{10}{4} = 210$ ways to select 4 balls from 10 balls.

There are $\binom{4}{3} = 4$ ways of choosing 3 red balls from 4 red balls, and there are $\binom{6}{1} = 6$ ways of choosing one blue ball from 6 blue balls.

Thus the probability of obtaining 3 red balls and one blue ball is equal to

$$\frac{\binom{4}{3} \times \binom{6}{1}}{\binom{10}{4}} = \frac{24}{210} = \frac{4}{35}$$

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The following table gives the probability of obtaining each possible sample proportion $\hat{p}$ when selecting a random sample of size 4 from the bag.

Number of blue balls in the sample	0	1	2	3	4
Proportion of blue balls in the sample, $\hat{p}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
Probability	$\frac{1}{210}$	$\frac{24}{210}$	$\frac{90}{210}$	$\frac{80}{210}$	$\frac{15}{210}$

We can see from the table that we can consider the sample proportion as a random variable, $\hat{P}$, and we can write:

- $\Pr(\hat{P} = 0) = \frac{1}{210}$
- $\Pr\left(\hat{P} = \frac{1}{4}\right) = \frac{24}{210}$
- $\Pr\left(\hat{P} = \frac{1}{2}\right) = \frac{90}{210}$
- $\Pr\left(\hat{P} = \frac{3}{4}\right) = \frac{80}{210}$
- $\Pr(\hat{P} = 1) = \frac{15}{210}$

The possible values of $\hat{p}$ and their associated probabilities together form a probability distribution for the random variable $\hat{P}$, which can summarised as follows:

$\hat{p}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$\Pr(\hat{P} = \hat{p})$	$\frac{1}{210}$	$\frac{24}{210}$	$\frac{90}{210}$	$\frac{80}{210}$	$\frac{15}{210}$

The distribution of a statistic which is calculated from a sample (such as the sample proportion) has a special name – it is called a **sampling distribution**.

Example 5

A bag contains six blue balls and four red balls. Use the sampling distribution in the previous table to determine the probability that the proportion of blue balls in a sample of size 4 is more than $\frac{1}{4}$.

Solution

$$\begin{aligned}\Pr\left(\hat{P} > \frac{1}{4}\right) &= \Pr\left(\hat{P} = \frac{1}{2}\right) + \Pr\left(\hat{P} = \frac{3}{4}\right) + \Pr(\hat{P} = 1)\\ &= \frac{90}{210} + \frac{80}{210} + \frac{15}{210}\\ &= \frac{185}{210}\\ &= \frac{37}{42}\end{aligned}$$

▶ Sampling from a large population

Generally, when we select a sample, it is from a population which is too large or too difficult to enumerate or even count – populations such as all the people in Australia, or all the cows in Texas, or all the people who will ever have asthma. When the population is so large, we assume that the probability of observing the attribute we are interested in remains constant with each selection, irrespective of prior selections for the sample.

Suppose we know that 70% of all 17-year-olds in Australia attend school. That is,

$$p = 0.7$$

We will assume that this probability remains constant for all selections for the sample.

Now consider selecting a random sample of size 4 from the population of all 17-year-olds in Australia. This time we can use our knowledge of binomial distributions to calculate the associated probability for each possible value of the sample proportion $\hat{p}$, using the probability function

$$\Pr(X = x) = \binom{4}{x} 0.7^x 0.3^{4-x} \qquad x = 0, 1, 2, 3, 4$$

The following table gives the probability of obtaining each possible sample proportion $\hat{p}$ when selecting a random sample of four 17-year-olds.

Number at school in the sample	0	1	2	3	4
Proportion at school in the sample, $\hat{p}$	0	0.25	0.5	0.75	1
Probability	0.0081	0.0756	0.2646	0.4116	0.2401

Once again, we can summarise the sampling distribution of the sample proportion as follows:

$\hat{p}$	0	0.25	0.5	0.75	1
$\Pr(\hat{P} = \hat{p})$	0.0081	0.0756	0.2646	0.4116	0.2401

The population that the sample of size $n = 4$ is being taken from is such that each item selected has a probability $p = 0.7$ of success. Thus we can define the random variable

$$\hat{P} = \frac{X}{4}$$

where X is a binomial random variable with parameters $n = 4$ and $p = 0.7$. To emphasise this we can write:

x	0	1	2	3	4
$\hat{p} = \frac{x}{4}$	0	0.25	0.5	0.75	1
$\Pr(\hat{P} = \hat{p}) = \Pr(X = x)$	0.0081	0.0756	0.2646	0.4116	0.2401

Note: The probabilities for the sample proportions, $\hat{p}$, correspond to the probabilities for the numbers of successes, x.

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Example 6

Use the sampling distribution in the previous table to determine the probability that, in a random sample of four Australian 17-year-olds, the proportion attending school is less than 50%.

Solution

$$\begin{aligned}\Pr(\hat{P} < 0.5) &= \Pr(\hat{P} = 0) + \Pr(\hat{P} = 0.25)\\ &= 0.0081 + 0.0756\\ &= 0.0837\end{aligned}$$

The mean and standard deviation of the sample proportion

Since the sample proportion $\hat{P}$ is a random variable with a probability distribution, we can determine values for the mean and standard deviation, as illustrated in the following example.

Example 7

Use the probability distribution to determine the mean and standard deviation of the sample proportion $\hat{P}$ from Example 6.

$\hat{p}$	0	0.25	0.5	0.75	1
$\Pr(\hat{P} = \hat{p})$	0.0081	0.0756	0.2646	0.4116	0.2401

Solution

By definition, the mean of $\hat{P}$ is given by

$$\begin{aligned}\mathrm{E}(\hat{P}) &= \sum \hat{p} \cdot \Pr(\hat{P} = \hat{p})\\ &= 0 \times 0.0081 + 0.25 \times 0.0756 + 0.5 \times 0.2646 + 0.75 \times 0.4116 + 1 \times 0.2401\\ &= 0.7\end{aligned}$$

Similarly, by definition,

$$\mathrm{sd}(\hat{P}) = \sqrt{\mathrm{E}(\hat{P}^2) - [\mathrm{E}(\hat{P})]^2}$$

We have

$$\begin{aligned}\mathrm{E}(\hat{P}^2) &= 0^2 \times 0.0081 + 0.25^2 \times 0.0756 + 0.5^2 \times 0.2646 + 0.75^2 \times 0.4116 + 1^2 \times 0.2401\\ &= 0.5425\end{aligned}$$

Thus

$$\mathrm{sd}(\hat{P}) = \sqrt{0.5425 - 0.7^2} = 0.2291$$

We can see from Example 7 that the mean of the sampling distribution in this case is actually the same as the value of the population proportion (0.7). Is this always true? Can we determine the mean and standard deviation of the sample proportion without needing to find the probability distribution?

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If we are selecting a random sample of size n from a large population, then we can assume that the sample proportion is of the form

$$\hat{P} = \frac{X}{n}$$

where X is a binomial random variable with parameters n and p. From Chapter 14, the mean and variance of X are given by

$$\mathrm{E}(X) = np \qquad \text{and} \qquad \mathrm{Var}(X) = np(1-p)$$

Thus we can determine

$$\begin{aligned} \mathrm{E}(\hat{P}) &= \mathrm{E}\left(\frac{X}{n}\right) \\ &= \frac{1}{n}\mathrm{E}(X) && \text{since } \mathrm{E}(aX+b) = a\mathrm{E}(X)+b \\ &= \frac{1}{n} \times np \\ &= p \end{aligned}$$

and

$$\begin{aligned} \mathrm{Var}(\hat{P}) &= \mathrm{Var}\left(\frac{X}{n}\right) \\ &= \left(\frac{1}{n}\right)^2 \mathrm{Var}(X) && \text{since } \mathrm{Var}(aX+b) = a^2\mathrm{Var}(X) \\ &= \frac{1}{n^2} \times np(1-p) \\ &= \frac{p(1-p)}{n} \end{aligned}$$

If we are selecting a random sample of size n from a large population, then the mean and standard deviation of the sample proportion $\hat{P}$ are given by

$$\mathrm{E}(\hat{P}) = p \quad \text{and} \quad \mathrm{sd}(\hat{P}) = \sqrt{\frac{p(1-p)}{n}}$$

(The standard deviation of a sample statistic is called the **standard error**.)

Example 8

Use these rules to determine the mean and standard deviation of the sample proportion $\hat{P}$ from Example 6. Are they the same as those found in Example 7?

Solution

$$\mathrm{E}(\hat{P}) = p = 0.7$$

$$\mathrm{sd}(\hat{P}) = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.7(1-0.7)}{4}} = 0.2291$$

These are the same as those obtained in Example 7.

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Example 9

Suppose that 70% of 17-year-olds in Australia attend school. If a random sample of size 20 is chosen from this population, find:

a the probability that the sample proportion is equal to the population proportion (0.7)

b the probability that the sample proportion lies within one standard deviation of the population proportion

c the probability that the sample proportion lies within two standard deviations of the population proportion.

Solution

a If the sample proportion is $\hat{p} = 0.7$ and the sample size is 20, then the number of school students in the sample is $0.7 \times 20 = 14$. Thus

$$\begin{aligned}\Pr(\hat{P} = 0.7) &= \Pr(X = 14)\\ &= \binom{20}{14} 0.7^{14} 0.3^{6} = 0.1916\end{aligned}$$

b We have

$$\begin{aligned}\operatorname{sd}(\hat{P}) &= \sqrt{\frac{p(1-p)}{n}}\\ &= \sqrt{\frac{0.7(1-0.7)}{20}} = 0.1025\end{aligned}$$

Since $0.7 - 0.1025 = 0.5975$ and $0.7 + 0.1025 = 0.8025$, we find

$$\begin{aligned}\Pr(0.5975 \le \hat{P} \le 0.8025) &= \Pr(11.95 \le X \le 16.05)\\ &= \Pr(12 \le X \le 16) \qquad \text{since } X \text{ takes integer values}\\ &= 0.7795\end{aligned}$$

c Since $0.7 - 2 \times 0.1025 = 0.495$ and $0.7 + 2 \times 0.1025 = 0.905$, we find

$$\begin{aligned}\Pr(0.495 \le \hat{P} \le 0.905) &= \Pr(9.9 \le X \le 18.1)\\ &= \Pr(10 \le X \le 18)\\ &= 0.9752\end{aligned}$$

Section summary

- The distribution of a statistic which is calculated from a sample is called a **sampling distribution**.
- The **sample proportion** $\hat{P} = \dfrac{X}{n}$ is a random variable, where X is the number of favourable outcomes in a sample of size n.
- The distribution of $\hat{P}$ is known as the **sampling distribution** of the sample proportion.
- When the population is *small*, the sampling distribution of the sample proportion $\hat{P}$ can be determined using our knowledge of selections.

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- When the population is *large*, the sampling distribution of the sample proportion $\hat{P}$ can be determined by assuming that X is a binomial random variable with parameters n and p. In this case, the mean and standard deviation of $\hat{P}$ are given by

$$\mathrm{E}(\hat{P}) = p \quad \text{and} \quad \mathrm{sd}(\hat{P}) = \sqrt{\frac{p(1-p)}{n}}$$

Exercise 17B

Skillsheet

Example 4, 5

1 Consider a bag containing five blue and five red balls.

a What is p, the proportion of blue balls in the bag?

b If samples of size 3 are taken from the bag, without replacement, then a sample could contain 0, 1, 2 or 3 blue balls. What are the possible values of the sample proportion $\hat{p}$ of blue balls associated with each of these samples?

c Construct a probability distribution table which summarises the sampling distribution of the sample proportion of blue balls when samples of size 3 are taken from the bag, without replacement.

d Use the sampling distribution from **c** to determine the probability that the proportion of blue balls in the sample is more than 0.5. That is, find $\Pr(\hat{P} > 0.5)$.

2 A company employs a sales team of 20 people, consisting of 12 men and 8 women.

a What is p, the proportion of men in the sales team?

b Five salespeople are to be selected at random to attend an important conference. What are the possible values of the sample proportion $\hat{p}$ of men in the sample?

c Construct a probability distribution table which summarises the sampling distribution of the sample proportion of men when samples of size 5 are selected from the sales team.

d Use the sampling distribution from **c** to determine the probability that the proportion of men in the sample is more than 0.7.

e Find $\Pr(0 < \hat{P} < 0.7)$ and hence find $\Pr(\hat{P} < 0.7 \mid \hat{P} > 0)$.

3 A pond contains eight gold and eight black fish.

a What is p, the proportion of gold fish in the pond?

b Three fish are to be selected at random. What are the possible values of the sample proportion $\hat{p}$ of gold fish in the sample?

c Construct a probability distribution table which summarises the sampling distribution of the sample proportion of gold fish when samples of size 3 are selected from the pond.

d Use the sampling distribution from **c** to determine the probability that the proportion of gold fish in the sample is more than 0.25.

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4 A random sample of three items is selected from a batch of 10 items which contains four defectives.

a What is p, the proportion of defectives in the batch?

b What are the possible values of the sample proportion $\hat{p}$ of defectives in the sample?

c Construct a probability distribution table which summarises the sampling distribution of the sample proportion of defectives in the sample.

d Use the sampling distribution from **c** to determine the probability that the proportion of defectives in the sample is more than 0.5.

e Find $\Pr(0 < \hat{P} < 0.5)$ and hence find $\Pr(\hat{P} < 0.5 \mid \hat{P} > 0)$.

Example 6

5 Suppose that a fair coin is tossed four times, and the number of heads observed.

a What is p, the probability that a head is observed when a fair coin is tossed?

b What are the possible values of the sample proportion $\hat{p}$ of heads in the sample?

c Construct a probability distribution table which summarises the sampling distribution of the sample proportion of heads in the sample.

d Use the sampling distribution from **c** to determine the probability that the proportion of heads in the sample is more than 0.7.

6 Suppose that the probability of a male child is 0.5, and that a family has five children.

a What are the possible values of the sample proportion $\hat{p}$ of male children in the family?

b Construct a probability distribution table which summarises the sampling distribution of the sample proportion of male children in the family.

c Use the sampling distribution from **b** to determine the probability that the proportion of male children in the family is less than 0.4.

d Find $\Pr(\hat{P} > 0 \mid \hat{P} < 0.8)$.

7 Suppose that, in a certain country, the probability that a person is left-handed is 0.2. If four people are selected at random from that country:

a What are the possible values of the sample proportion $\hat{p}$ of left-handed people in the sample?

b Construct a probability distribution table which summarises the sampling distribution of the sample proportion of left-handed people in the sample.

c Find $\Pr(\hat{P} > 0.5 \mid \hat{P} > 0)$.

Example 7

8 Use the sampling distribution from Question 5 to determine the mean and standard deviation of the sample proportion $\hat{P}$ of heads observed when a fair coin is tossed four times.

9 Use the sampling distribution from Question 6 to determine the mean and standard deviation of the sample proportion $\hat{P}$ of male children in a family of five children.

10 Use the sampling distribution from Question 7 to determine the mean and standard deviation of the sample proportion $\hat{P}$ of left-handed people when a sample of four people are selected.

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Example 8 **11** Suppose that the probability of rain on any day is 0.3. Find the mean and standard deviation of the sample proportion of rainy days which might be observed in the month of June.

12 In a certain country, it is known that 40% of people speak more than one language. If a sample of 100 people is selected, find the mean and standard deviation of the sample proportion of people who speak more than one language.

13 An examination consists of 100 multiple-choice questions, each with five possible answers. Find the mean and standard deviation of the sample proportion of correct answers that will be achieved if a student guesses every answer.

Example 9 **14** Suppose that 65% of people in Australia support an AFL team. If a random sample of size 20 is chosen from this population, find:

- **a** the probability that the sample proportion is equal to the population proportion
- **b** the probability that the sample proportion lies within one standard deviation of the population proportion
- **c** the probability that the sample proportion lies within two standard deviations of the population proportion.

17C Approximating the distribution of the sample proportion

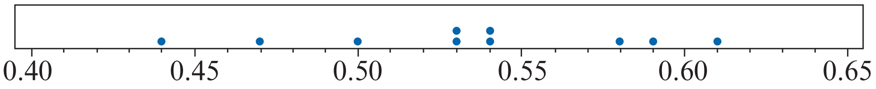
In the previous section, we used our knowledge of probability to determine the exact distribution of the sample proportion. Working out the exact probabilities associated with a sample proportion is really only practical when the sample size is quite small (say less than 10). In practice, we are rarely working with such small samples. But we can overcome this problem by approximating the distribution of the sample proportion.

Suppose, for example, we know that 55% of people in Australia have blue eyes ($p = 0.55$) and that we are interested in the values of the sample proportion $\hat{p}$ which might be observed when samples of size 100 are drawn at random from the population.

If we select one sample of 100 people and find that 50 people have blue eyes, then the value of the sample proportion is $\hat{p} = \dfrac{50}{100} = 0.5$.

If a second sample of 100 people is selected and this time 58 people have blue eyes, then the value of the sample proportion for this second sample is $\hat{p} = \dfrac{58}{100} = 0.58$.

Continuing in this way, after selecting 10 samples, the values of $\hat{p}$ that are observed might look like those in the following dotplot:



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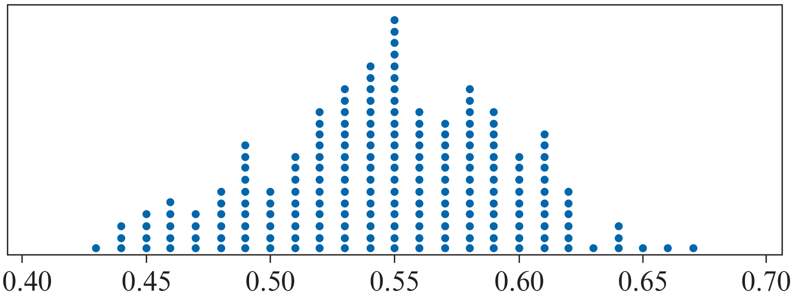
It is clear that the proportion of people with blue eyes in the sample, $\hat{p}$, is varying from sample to sample: from as low as 0.44 to as high as 0.61 for these particular 10 samples.

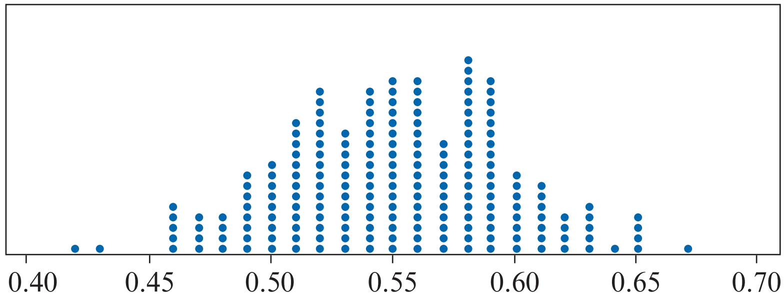
What does the distribution of the sample proportions look like if we continue with this sampling process?

The following dotplot summarises the values of $\hat{p}$ observed when 200 samples (each of size 100) were selected from a population in which the probability of having blue eyes is 0.55. We can see from the dotplot that the distribution is reasonably symmetric, centred at 0.55, and has values ranging from 0.43 to 0.67.



What does the distribution look like when another 200 samples (each of size 100) are selected at random from the same population?

The following dotplot shows the distribution obtained when this experiment was repeated. Again, the distribution is reasonably symmetric, centred at 0.55, and has values ranging from 0.42 to 0.67.



It seems reasonable to infer from these examples that, while there will be variation in the details of the distribution each time we take a collection of samples, the distribution of the values of $\hat{p}$ observed tends to conform to a predictable shape, centre and spread.

Actually, we already know from Chapter 16 that, when the sample size is large enough, the distribution of a binomial random variable is well approximated by the normal distribution. We have also seen that the rule of thumb for the normal approximation to the binomial distribution to apply is that both np and $n(1 - p)$ should be greater than 5.

The dotplots confirm the reasonableness of the normality assumption with regard to the sample proportion $\hat{P}$, which can be considered to be a linear function of a binomial random variable.

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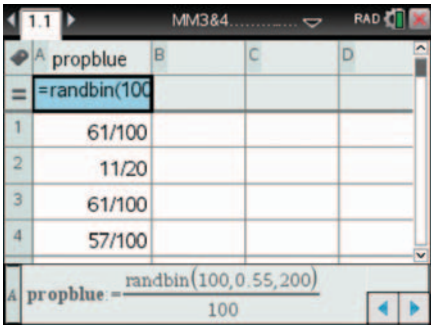
Repeated sampling can be investigated using a calculator.

Example 10

Assume that 55% of people in Australia have blue eyes. Use your calculator to illustrate a possible distribution of sample proportions $\hat{p}$ that may be obtained when 200 different samples (each of size 100) are selected from the population.

Using the TI-Nspire

- To generate the sample proportions:
  - Start from a **Lists & Spreadsheet** page.
  - Name the list 'propblue' in Column A.
  - In the formula cell of Column A, enter the formula using Menu > **Data** > **Random** > **Binomial** and complete as: $= \text{randbin}(100, 0.55, 200)/100$



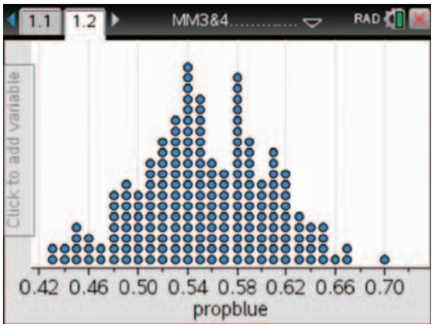
	A propblue	B	C	D
=	=randbin(10			
1	61/100			
2	11/20			
3	61/100			
4	57/100			

A propblue: $=\dfrac{\text{randbin}(100,0.55,200)}{100}$

Note: The syntax is: randbin(*sample size*, *population proportion*, *number of samples*) To calculate as a proportion, divide by the sample size.

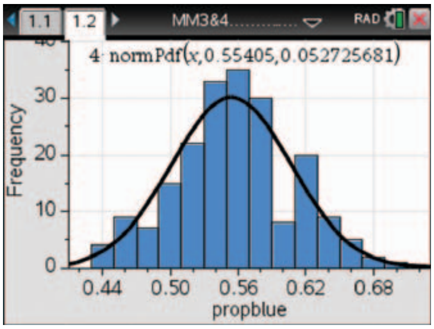
- To display the distribution of sample proportions:
  - Insert a **Data & Statistics** page (ctrl I or ctrl doc ▾).
  - Click on 'Click to add variable' on the x-axis and select 'propblue'. A dotplot is displayed.

Note: You can recalculate the random sample proportions by using ctrl R while in the **Lists & Spreadsheet** page.



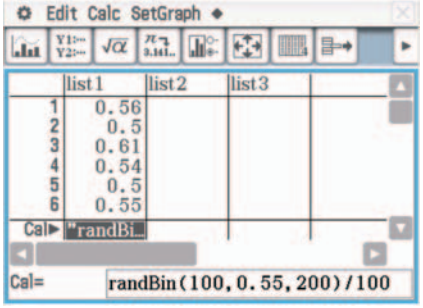
- To fit a normal curve to the distribution:
  - Menu > **Plot Type** > **Histogram**
  - Menu > **Analyze** > **Show Normal PDF**

Note: The calculated Normal PDF, based on the data set, is superimposed on the plot, showing the mean and standard deviation of the sample proportion.



Using the Casio ClassPad

- To generate the sample proportions:
  - Open the **Statistics** application.
  - Tap the ‘Calculation’ cell at the bottom of list1.
  - Type: randBin(100, 0.55, 200)/100
  - Tap `Set`.

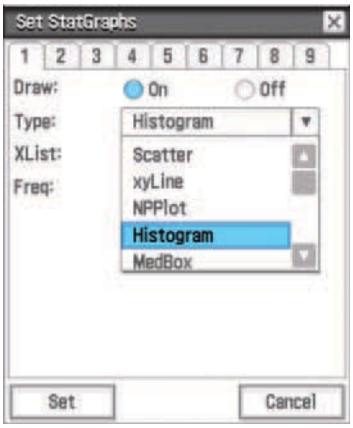

Edit Calc SetGraph

	list1	list2	list3
1	0.56		
2	0.5		
3	0.61		
4	0.54		
5	0.5		
6	0.55		
Cal▶	"randBi		

Cal= randBin(100, 0.55, 200)/100

Note: The syntax is: randBin(*sample size*, *population proportion*, *number of samples*) To calculate as a proportion, divide by the sample size.

- To display the distribution of sample proportions:
  - Tap on the **Set StatGraphs** icon, select the type ‘Histogram’ and tap `Set`.
  - Tap on the graph icon in the toolbar.
  - In the **Set Interval** window, enter the values shown below and tap `OK`.

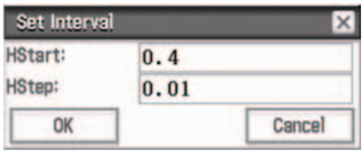

Set StatGraphs
1 2 3 4 5 6 7 8 9
Scatter
xyLine
NPPlot
Histogram
MedBox
Set
Cancel

Draw: ● On ○ Off

Type: Histogram

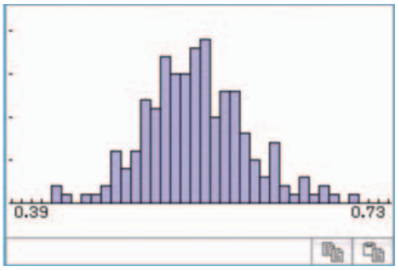
XList:

Freq:

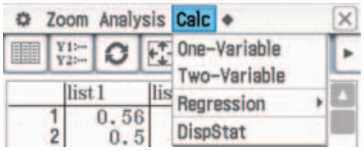

Set Interval
OK
Cancel

HStart: 0.4

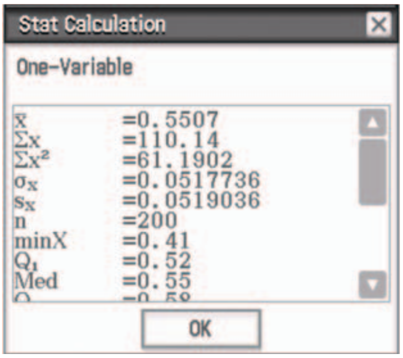
HStep: 0.01



- To obtain statistics from the distribution, select **Calc** > **One–Variable**. Tap `OK`.


Zoom Analysis Calc
One-Variable
Two-Variable
Regression
DispStat
list1
1 0.56
2 0.5

Note: The mean of the sample proportions, $\bar{x}$, estimates the population proportion.


Stat Calculation
OK

One-Variable

$\bar{x}$ =0.5507
Σx =110.14
Σx^2 =61.1902
σx =0.0517736
sx =0.0519036
n =200
minX =0.41
Q_1 =0.52
Med =0.55

When the sample size n is large, the sample proportion $\hat{P}$ has an approximately normal distribution, with mean $\mu = p$ and standard deviation $\sigma = \sqrt{\dfrac{p(1-p)}{n}}$.

Thus, when samples of size $n = 100$ are selected from a population in which the proportion of people with blue eyes is $p = 0.55$, the distribution of the sample proportion $\hat{P}$ is approximately normal, with mean and standard deviation given by

$$\mu = \mathrm{E}(\hat{P}) = 0.55 \quad \text{and} \quad \sigma = \mathrm{sd}(\hat{P}) = \sqrt{\frac{0.55 \times 0.45}{100}} = 0.0497$$

Example 11

Assume that 60% of people have a driver's licence. Using the normal approximation, find the approximate probability that, in a randomly selected sample of size 200, more than 65% of people have a driver's licence.

Solution

Here $n = 200$ and $p = 0.6$. Since n is large, the distribution of $\hat{P}$ is approximately normal, with mean $\mu = p = 0.6$ and standard deviation

$$\sigma = \sqrt{\frac{0.6(1-0.6)}{200}} = 0.0346$$

Thus the probability that more than 65% of people in the sample have a driver's licence is

$$\Pr(\hat{P} > 0.65) = 0.0745 \quad \text{(correct to four decimal places)}$$

Section summary

When the sample size n is large, the sample proportion $\hat{P}$ has an approximately normal distribution, with mean $\mu = p$ and standard deviation $\sigma = \sqrt{\dfrac{p(1-p)}{n}}$.

Exercise 17C

In each of the following questions, use the normal approximation to the binomial distribution.

Example 11

1 Find the approximate probability that, in the next 50 tosses of a fair coin, the proportion of heads observed will be less than or equal to 0.46.

2 In a large city, 12% of the workforce are unemployed. If 300 people from the workforce are selected at random, find the approximate probability that more than 10% of the people surveyed are unemployed.

3 It is known that on average 50% of the children born at a particular hospital are female. Find the approximate probability that more than 60% of the next 25 children born at that hospital will be female.

4 A car manufacturer expects 10% of cars produced to require minor adjustments before they are certified as ready for sale. What is the approximate probability that more than 15% of the next 200 cars inspected will require minor adjustments?

5 Past records show that on average 30% of the workers at a particular company have had one or more accidents in the workplace. What is the approximate probability that less than 20% of a random sample of 50 workers have had one or more accidents?

6 Sacha is shooting at a target which she has a probability of 0.6 of hitting. What is the approximate probability that:

a the proportion of times she hits the target in her next 100 attempts is less than 0.8

b the proportion of times she hits the target in her next 100 attempts is between 0.6 and 0.8

c the proportion of times she hits the target in her next 100 attempts is between 0.7 and 0.8, given that it is more than 0.6?

7 Find the approximate probability that, in the next 100 tosses of a fair coin, the proportion of heads will be between 0.4 and 0.6.

8 A machine has a probability of 0.1 of producing a defective item.

a What is the approximate probability that, in the next batch of 1000 items produced, the proportion of defective items will be between 0.08 and 0.12?

b What is the approximate probability that, in the next batch of 1000 items produced, the proportion of defective items will be between 0.08 and 0.12, given that we know that it is greater than 0.10?

9 The proportion of voters in the population who favour Candidate A is 52%. Of a random sample of 400 voters, 230 indicated that they would vote for Candidate A at the next election.

a What is the value of the sample proportion, $\hat{p}$?

b Find the approximate probability that, in a random sample of 400 voters, the proportion who favour Candidate A is greater than or equal to the value of $\hat{p}$ observed in this particular sample.

10 A manufacturer claims that 90% of their batteries will last more than 100 hours. Of a random sample of 250 batteries, 212 lasted more than 100 hours.

a What is the value of the sample proportion, $\hat{p}$?

b Find the approximate probability that, in a random sample of 250 batteries, the proportion lasting more than 100 hours is less than or equal to the value of $\hat{p}$ observed in this particular sample.

c Does your answer to **b** cause you to doubt the manufacturer's claim?

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17D Confidence intervals for the population proportion

In practice, the reason we analyse samples is to further our understanding of the population from which they are drawn. That is, we know what is in the sample, and from that knowledge we would like to infer something about the population.

▶ Point estimates

Suppose, for example, we wish to know the proportion of primary school children in Australia who regularly use social media. The value of the population proportion p is unknown. As already mentioned, collecting information about the whole population is generally not feasible, and so a random sample must suffice. What information can be obtained from a single sample? Certainly, the sample proportion $\hat{p}$ gives some indication of the value of the population proportion p, and can be used when we have no other information.

> The value of the sample proportion $\hat{p}$ can be used to estimate the population proportion p. Since this is a single-valued estimate, it is called a **point estimate** of p.

Thus, if we select a random sample of 20 Australian primary school children and find that the proportion who use social media is 0.7, then the value $\hat{p} = 0.7$ serves as an estimate of the unknown population proportion p.

▶ Interval estimates

The value of the sample proportion $\hat{p}$ obtained from a single sample is going to change from sample to sample, and while sometimes the value will be close to the population proportion p, at other times it will not. To use a single value to estimate p can be rather risky. What is required is an interval that we are reasonably sure contains the parameter value p.

> An **interval estimate** for the population proportion p is called a **confidence interval** for p.

We have already seen that, when the sample size n is large, the sample proportion $\hat{P}$ has an approximately normal distribution with $\mu = p$ and $\sigma = \sqrt{\dfrac{p(1-p)}{n}}$.

By standardising, we can say that the distribution of the random variable

$$\frac{\hat{P} - p}{\sqrt{\frac{p(1-p)}{n}}}$$

is approximated by that of a standard normal random variable Z.

We know that $\Pr(-1.96 < Z < 1.96) = 0.95$, correct to two decimal places, and therefore

$$\Pr\left(-1.96 < \frac{\hat{P} - p}{\sqrt{\frac{p(1-p)}{n}}} < 1.96\right) \approx 0.95$$

Multiplying through gives

$$\Pr\left(-1.96\sqrt{\frac{p(1-p)}{n}} < \hat{P} - p < 1.96\sqrt{\frac{p(1-p)}{n}}\right) \approx 0.95$$

Further simplifying, we obtain

$$\Pr\left(\hat{P} - 1.96\sqrt{\frac{p(1-p)}{n}} < p < \hat{P} + 1.96\sqrt{\frac{p(1-p)}{n}}\right) \approx 0.95$$

Remember that what we want to do is to use the value of the sample proportion $\hat{p}$ obtained from a single sample to calculate an interval that we are fairly certain (say 95% certain) contains the true population proportion p (which we do not know).

In order to do this, we need to make one further approximation, and substitute $\hat{p}$ for p in our estimate of the standard deviation σ of $\hat{P}$.

An approximate **95% confidence interval** for p is given by

$$\left(\hat{p} - 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$$

where:

- p is the population proportion (unknown)
- $\hat{p}$ is a value of the sample proportion
- n is the size of the sample from which $\hat{p}$ was calculated.

Note: In order to use this rule to calculate a confidence interval, the criteria for the normal approximation to the binomial distribution must apply. Therefore, from Chapter 16, we require both np and $n(1-p)$ to be greater than 5.

Example 12

Find an approximate 95% confidence interval for the proportion p of primary school children in Australia who regularly use social media, if we select a random sample of 20 children and find the sample proportion $\hat{p}$ to be 0.7.

Solution

Since $\hat{p} = 0.7$ and $n = 20$, we have

$$\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{0.7 \times 0.3}{20}} = 0.1025$$

and so a 95% confidence interval for p is

$$(0.7 - 1.96 \times 0.1025,\ 0.7 + 1.96 \times 0.1025) = (0.499, 0.901)$$

Thus, based on a sample of size 20 and a sample estimate of 0.7, an approximate 95% confidence interval for the population proportion p is $(0.499, 0.901)$.

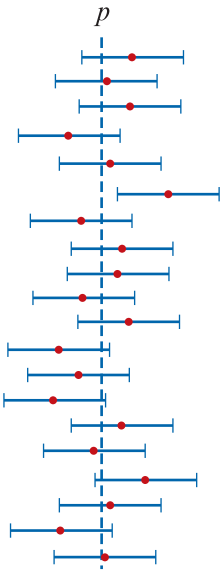
Interpretation of confidence intervals

The confidence interval found in Example 12 should not be interpreted as meaning that $\Pr(0.499 < p < 0.901) = 0.95$. In fact, such a statement is meaningless, as p is a constant and either does or does not lie in the stated interval.

The particular confidence interval found is just one of any number of confidence intervals which could be found for the population proportion p, each one depending on the particular value of the sample proportion $\hat{p}$.

The correct interpretation of the confidence interval is that we expect approximately 95% of such intervals to contain the population proportion p. Whether or not the particular confidence interval obtained contains the population proportion p is generally not known.

If we were to repeat the process of taking a sample and calculating a confidence interval many times, the result would be something like that indicated in the diagram.



The diagram shows the confidence intervals obtained when 20 different samples were drawn from the same population. The round dot indicates the value of the sample estimate in each case. The intervals vary, because the samples themselves vary. The value of the population proportion p is indicated by the vertical line, and it is of course constant.

It is quite easy to see from the diagram that none of the values of the sample estimate is exactly the same as the population proportion, but that all the intervals except one (19 out of 20, or 95%) have captured the value of the population proportion, as would be expected in the case of a 95% confidence interval.

Using a calculator to determine confidence intervals

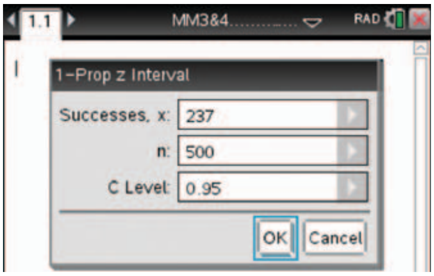
Example 13

A survey found that 237 out of 500 undergraduate university students questioned intended to take a postgraduate course in the future. Find a 95% confidence interval for the proportion of undergraduates intending to take a postgraduate course.

Using the TI-Nspire

In a **Calculator** page:

- Use (Menu) > **Statistics** > **Confidence Intervals** > **1–Prop z Interval**.
- Enter the values $x = 237$ and $n = 500$ as shown.



1–Prop z Interval

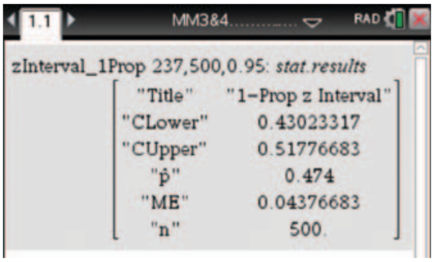
Successes, x: 237

n: 500

C Level: 0.95

- The 'CLower' and 'CUpper' values give the 95% confidence interval (0.43, 0.52).

Note: 'ME' stands for margin of error, which is covered in the next subsection.



```
zInterval_1Prop 237,500,0.95: stat.results
 "Title"    "1-Prop z Interval"
 "CLower"       0.43023317
 "CUpper"       0.51776683
 "p̂"              0.474
 "ME"           0.04376683
 "n"              500.
```

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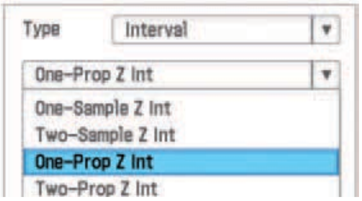
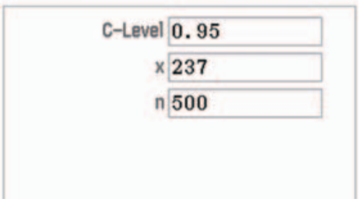
Using the Casio ClassPad

- In Statistics, go to **Calc** > **Interval**.
- Select **One–Prop Z Int** and tap `Next`.
- Enter the values C-Level = 0.95, $x = 237$ and $n = 500$ as shown below. Tap `Next`.


Type Interval

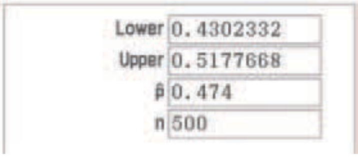
One–Prop Z Int


C-Level 0.95

x 237

n 500

- The ‘Lower’ and ‘Upper’ values give the 95% confidence interval (0.43, 0.52).


Lower 0.4302332

Upper 0.5177668

$\hat{p}$ 0.474

n 500

Precision and margin of error

In Example 12, we found an approximate 95% confidence interval (0.499, 0.901) for the proportion p of primary school children in Australia who use social media, based on a sample of size 20. Therefore we predict that the population proportion p is somewhere in the range of approximately 50% to 90%! But this interval is so wide as to be not very helpful.

Example 14

Find an approximate 95% confidence interval for the proportion p of primary school children in Australia who regularly use social media, if we select a random sample of 200 children and find the sample proportion $\hat{p}$ to be 0.7.

Solution

Since $\hat{p} = 0.7$ and $n = 200$, we have

$$\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{0.7 \times 0.3}{200}}$$

$$= 0.0324$$

and so a 95% confidence interval for p is

$$(0.7 - 1.96 \times 0.0324,\ 0.7 + 1.96 \times 0.0324) = (0.636, 0.764)$$

Thus, based on a sample of size 200 and a sample estimate of 0.7, an approximate 95% confidence interval for the population proportion p is (0.636, 0.764).

Note: This interval is much narrower than the one determined in Example 12, which was based on a sample of size 20.

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Often we discuss the confidence interval in terms of its width or, more formally, in terms of the distance between the sample estimate and the endpoints of the confidence interval.

That is, we find it useful to make statements such as 'we predict the proportion of people who will vote Labor in the next election as 52% ± 2%'. Here the sample estimate is 52%, and the distance between the sample estimate and the endpoints is 2%.

The distance between the sample estimate and the endpoints of the confidence interval is called the **margin of error** (M). For a 95% confidence interval,

$$M = 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

We can see from this rule that the margin of error is a function of the sample size n, and that one way to make the interval narrower (that is, to increase the precision of the estimate) is to increase the sample size.

Example 15

Determine the sample size required to achieve a margin of error of 2% in an approximate 95% confidence interval for the proportion p of primary school children in Australia who use social media, if the sample proportion $\hat{p}$ is found to be 0.7.

Solution

Substituting $M = 0.02$ and $\hat{p} = 0.7$ in the expression for the margin of error gives

$$0.02 = 1.96\sqrt{\frac{0.7 \times 0.3}{n}}$$

Solving for n:

$$\left(\frac{0.02}{1.96}\right)^2 = \frac{0.7 \times 0.3}{n}$$

$$\therefore \quad n = 0.7 \times 0.3 \times \left(\frac{1.96}{0.02}\right)^2 \approx 2016.84$$

Thus, to achieve a margin of error of 2%, we need a sample of size 2017.

Of course, it is highly unlikely that we will know the value of the sample proportion $\hat{p}$ before we have selected the sample. Thus it is usual to substitute an estimated value into the equation in order to determine the sample size before we select the sample. This estimate can be based on our prior knowledge of the population or on a pilot study. If we denote this estimated value for the sample proportion by p^*, we can write

$$M = 1.96\sqrt{\frac{p^*(1-p^*)}{n}}$$

Rearranging to make n the subject of the equation, we find

$$M^2 = 1.96^2\left(\frac{p^*(1-p^*)}{n}\right)$$

$$\therefore \quad n = \left(\frac{1.96}{M}\right)^2 p^*(1-p^*)$$

A 95% confidence interval for a population proportion p will have margin of error approximately equal to a specified value of M when the sample size is

$$n = \left(\frac{1.96}{M}\right)^2 p^*(1 - p^*)$$

where p^* is an estimated value for the population proportion p.

Changing the level of confidence

So far we have only considered 95% confidence intervals, but in fact we can choose any level of confidence for a confidence interval. What is the effect of changing the level of confidence?

Consider again a 95% confidence interval:

$$\left(\hat{p} - 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$$

From our knowledge of the normal distribution, we can say that a 99% confidence interval will be given by

$$\left(\hat{p} - 2.58\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + 2.58\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$$

In general, a C% confidence interval is given by

$$\left(\hat{p} - k\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + k\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$$

where k is such that

$$\Pr(-k < Z < k) = \frac{C}{100}$$

Example 16

Calculate and compare 90%, 95% and 99% confidence intervals for the proportion p of primary school children in Australia who regularly use social media, if we select a random sample of 200 children and find the sample proportion $\hat{p}$ to be 0.7.

Solution

From Example 14, we know that the 95% confidence interval is (0.636, 0.764).

The 90% confidence interval is

$$\left(0.7 - 1.65\sqrt{\frac{0.7 \times 0.3}{200}},\ 0.7 + 1.65\sqrt{\frac{0.7 \times 0.3}{200}}\right) = (0.647, 0.753)$$

The 99% confidence interval is

$$\left(0.7 - 2.58\sqrt{\frac{0.7 \times 0.3}{200}},\ 0.7 + 2.58\sqrt{\frac{0.7 \times 0.3}{200}}\right) = (0.616, 0.784)$$

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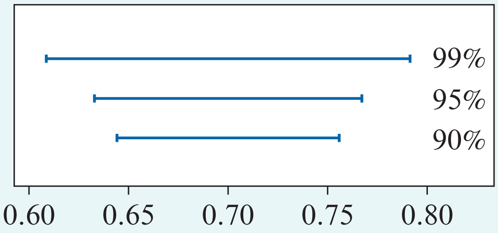
It is helpful to use a diagram to compare these confidence intervals:



From the diagram, it can be clearly seen that the effect of being more confident that the confidence interval captures the true value of the population proportion means that a wider interval is required.

Section summary

- The value of the sample proportion $\hat{p}$ can be used to estimate the population proportion p. Since this is a single-valued estimate, it is called a **point estimate** of p.
- An **interval estimate** for the population proportion p is called a **confidence interval** for p.
- An approximate **95% confidence interval** for p is given by

$$\left(\hat{p} - 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$$

  where:
  - p is the population proportion (unknown)
  - $\hat{p}$ is a value of the sample proportion
  - n is the size of the sample from which $\hat{p}$ was calculated.
- The distance between the sample estimate and the endpoints of the confidence interval is called the **margin of error** (M) and, for a 95% confidence interval,

$$M = 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

- A 95% confidence interval for a population proportion p will have margin of error approximately equal to a specified value of M when the sample size is

$$n = \left(\frac{1.96}{M}\right)^2 p^*(1-p^*)$$

  where p^* is an estimated value for the population proportion p.
- In general, a C% confidence interval is given by

$$\left(\hat{p} - k\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + k\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$$

  where k is such that $\Pr(-k < Z < k) = \dfrac{C}{100}$.

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Exercise 17D

Skillsheet

1 A quality-control engineer in a factory needs to estimate the proportion of bags of potato chips packed by a certain machine that are underweight. The engineer takes a random sample of 100 bags and finds that eight of them are underweight.

Example 12

- **a** Find a point estimate for p, the proportion of bags packed by the machine that are underweight.
- **b** Calculate a 95% confidence interval for p.

2 A newspaper wants to estimate the proportion of its subscribers who believe that the government should be allowed to tap telephones without a court order. It selects a random sample of 250 subscribers, and finds that 48 of them believe that the government should have this power.

- **a** Find a point estimate for p, the proportion of subscribers who believe that the government should be allowed to tap telephones without a court order.
- **b** Calculate a 95% confidence interval for p.

3 The lengths of stay in hospital among patients is of interest to health planners. A random sample of 100 patients was investigated, and 20 were found to have stayed longer than 7 days.

- **a** Find a point estimate for p, the proportion of patients who stay in hospital longer than 7 days.
- **b** Calculate a 95% confidence interval for p.

Example 13, 14

4 Given that 132 out of 400 randomly selected adult males are cigarette smokers, find a 95% confidence interval for the proportion of adult males in the population who smoke.

5 Of a random sample of 400 voters in a particular electorate, 210 indicated that they would vote for the Labor party at the next election.

- **a** Use this information to find a 95% confidence interval for the proportion of Labor voters in the electorate.
- **b** A random sample of 4000 voters from the same electorate was taken, and this time 2100 indicated that they would vote for Labor at the next election. Find a 95% confidence interval for the proportion of Labor voters in the electorate.
- **c** Compare your answers to parts **a** and **b**.

6 A manufacturer claims that 90% of their batteries will last more than 50 hours.

- **a** Of a random sample of 250 batteries, 212 lasted more than 50 hours. Use this information to find a 95% confidence interval for the proportion of batteries lasting more than 50 hours.
- **b** An inspector requested further information. A random sample of 2500 batteries was selected and this time 2120 lasted more than 50 hours. Use this information to find a 95% confidence interval for the proportion of batteries lasting more than 50 hours.
- **c** Compare your answers to parts **a** and **b**.

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Example 15

7 Determine the size of sample required to achieve a margin of error of 2% in an approximate 95% confidence interval when the sample proportion $\hat{p}$ is 0.8.

8 Determine the size of sample required to achieve a margin of error of 5% in an approximate 95% confidence interval when the sample proportion $\hat{p}$ is 0.2.

9 Samar is conducting a survey to estimate the proportion of people in Victoria who would support reducing the driving age to 16. He knows from previous studies that this proportion is about 30%.

- **a** Determine the size of sample required for the survey to achieve a margin of error of 3% in an approximate 95% confidence interval for this proportion.
- **b** Determine the size of sample required for the survey to achieve a margin of error of 2% in an approximate 95% confidence interval for this proportion.
- **c** Compare your answers to parts **a** and **b**.

10 Bob is thinking of expanding his pizza delivery business to include a range of desserts. He would like to know the proportion of his clients who would order dessert from him, and so he intends to ask a number of his clients what they think.

- **a** Bob thinks that the proportion of his clients who would order dessert is around 0.3. Determine the size of sample required for Bob to achieve a margin of error of 2% in an approximate 95% confidence interval for this proportion.
- **b** Bob's business partner Phil thinks that the proportion of clients who would order dessert is around 0.5. Determine the size of sample required to achieve a margin of error of 2% in an approximate 95% confidence interval for this proportion.
- **c** What is the effect on the margin of error if:
  - **i** Bob is correct, but they use the sample size from Phil's estimate
  - **ii** Phil is correct, but they use the sample size from Bob's estimate?
- **d** What sample size would you recommend that Bob and Phil use?

Example 16

11 When a coin thought to be biased was tossed 100 times, it came up heads 60 times. Calculate and compare 90%, 95% and 99% confidence intervals for the probability of observing a head when that coin is tossed.

12 In a survey of attitudes to climate change, a total of 537 people from a random sample of 1000 people answered no to the question 'Do you think the government is doing enough to address global warming?' Calculate and compare 90%, 95% and 99% confidence intervals for the proportion of people in Australia who would answer no to that question.

Chapter summary

Spreadsheet

AS

Nrich

- A **population** is the set of all eligible members of a group which we intend to study.
- A **sample** is a subset of the population which we select in order to make inferences about the population. Generalising from the sample to the population will not be useful unless the sample is representative of the population.
- A sample of size n is called a **simple random sample** if it is selected from the population in such a way that every subset of size n has an equal chance of being chosen as the sample. In particular, every member of the population must have an equal chance of being included in the sample.
- The **population proportion** p is the proportion of individuals in the entire population possessing a particular attribute, and is constant.
- The **sample proportion** $\hat{p}$ is the proportion of individuals in a particular sample possessing the attribute, and varies from sample to sample.
- The sample proportion $\hat{P} = \dfrac{X}{n}$ is a random variable, where X is the number of favourable outcomes in a sample of size n. The distribution of the random variable $\hat{P}$ is known as the **sampling distribution** of the sample proportion.
- When the population is *small*, the sampling distribution of the sample proportion $\hat{P}$ can be determined using our knowledge of selections.
- When the population is *large*, the sampling distribution of the sample proportion $\hat{P}$ can be determined by assuming that X is a binomial random variable with parameters n and p. In this case, the mean and standard deviation of $\hat{P}$ are given by

$$\mathrm{E}(\hat{P}) = p \quad \text{and} \quad \mathrm{sd}(\hat{P}) = \sqrt{\frac{p(1-p)}{n}}$$

- When the sample size n is large, the sample proportion $\hat{P}$ has an approximately normal distribution, with mean $\mu = p$ and standard deviation $\sigma = \sqrt{\dfrac{p(1-p)}{n}}$.
- If the value of the sample proportion $\hat{p}$ is used as an estimate of the population proportion p, then it is called a **point estimate** of p.
- An approximate **95% confidence interval** for p is given by

$$\left(\hat{p} - 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$$

  where:
  - p is the population proportion (unknown)
  - $\hat{p}$ is a value of the sample proportion
  - n is the size of the sample from which $\hat{p}$ was calculated.
- The distance between the sample estimate and the endpoints of the confidence interval is called the **margin of error** (M) and, for a 95% confidence interval,

$$M = 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

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- A 95% confidence interval for a population proportion p will have margin of error approximately equal to a specified value of M when the sample size is

$$n = \left(\frac{1.96}{M}\right)^2 p^*(1 - p^*)$$

where p^* is an estimated value for the population proportion p.

Technology-free questions

1 A company has 2000 employees, 700 of whom are female. A random sample of 100 employees was selected, and 40 of them were female. In this example:

a What is the population?

b What is the value of the population proportion p?

c What is the value of the sample proportion $\hat{p}$?

2 To study the effectiveness of yoga for reducing stress levels, a researcher measured the stress levels of 50 people who had just enrolled in a 10-week introductory yoga course, and then measured their stress levels at the end the course.

a Do you think that this sample will be representative of the general population? Explain your answer.

b How would you suggest that the sample could be chosen?

3 A coin is tossed 100 times, and k heads observed.

a Give a point estimate for p, the probability of observing a head when the coin is tossed.

b Write down an expression for a 95% confidence interval for p.

4 A sample of n people were asked whether they thought that income tax in Australia was too high, and 90% said yes.

a What is the value of the sample proportion $\hat{p}$?

b Write down an expression for M, the margin of error for this estimate at the 95% confidence level, in terms of n.

c If the number of people in the sample were doubled, what would be the effect on the margin of error M?

5 Suppose that 40 independent random samples were taken from a large population, and that a 95% confidence interval for the population proportion p was computed from each of these samples.

a How many of the 95% confidence intervals would you expect to contain the population proportion p?

b Write down an expression for the probability that all 40 confidence intervals contain the population proportion p.

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6 Suppose that 50 independent random samples were taken from a large population, and that a 90% confidence interval for the population proportion p was computed from each of these samples.

a How many of the 90% confidence intervals would you expect to contain the population proportion p?

b Write down an expression for the probability that at least 49 of the 50 confidence intervals contain the population proportion p.

7 A newspaper determined that an approximate 95% confidence interval for the proportion of people in Australia who regularly read the news online was $(0.50, 0.70)$.

a What was the value of $\hat{p}$ which was used to determine this confidence interval?

b What is the margin of error?

c How could the newspaper increase the precision of their study?

Multiple-choice questions

1 In order to estimate the ratio of males to females at a school, a teacher determines the number of males and the number of females in a particular class. The ratio that he then calculates is called a

A sample **B** sample statistic **C** population parameter
D population **E** sample parameter

2 In a complete census of the population of a particular community, it is found that 59% of families have two or more children. Here '59%' represents the value of a

A sample **B** sample statistic **C** population parameter
D population **E** sample parameter

3 From a random sample, a 95% confidence interval for the population proportion p is found to be $(0.7, 0.8)$. This means that

A the population proportion $p = 0.6$

B the probability that the population proportion p lies in the interval $(0.7, 0.8)$ is 0.95

C the probability that the population proportion p lies in the interval $(0.7, 0.8)$ is 0.05

D 95% of random samples lead to confidence intervals which contain the population proportion p

E none of these

4 A survey showed that 15 out of a random sample of 50 football supporters attend at least one match per season. If this information is used to find a 95% confidence interval for the proportion of all football supporters who attend at least one match per season, then the margin of error will be

A 0.3 **B** 0.004 **C** 0.065 **D** 0.254 **E** 0.127

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5 Of a random sample of 50 golfers, four were found to play golf left-handed. A 95% confidence interval for the proportion of golfers in the population who play left-handed is given by

A $(0.053, 0.107)$ **B** $(0.026, 0.134)$ **C** $(0.005, 0.155)$
D $(0.006, 0.154)$ **E** $(0.075, 0.085)$

6 Fourteen of a random sample of 88 people said they prefer to watch the news on a particular channel. A 95% confidence interval for the proportion of people in the population who prefer to watch the news on that channel is given by

A $(0.085, 0.233)$ **B** $(0.259, 0.359)$ **C** $(0.157, 0.161)$
D $(0.120, 0.198)$ **E** $(0.083, 0.236)$

7 If the sample proportion remains unchanged, then an increase in the level of confidence will lead to a confidence interval which is

A narrower **B** wider **C** unchanged **D** asymmetric
E cannot be determined from the information given

8 Which of the following statements is true?

I The centre of a confidence interval is a population parameter.
II The bigger the margin of error, the smaller the confidence interval.
III The confidence interval is a type of point estimate.
IV A population proportion is an example of a point estimate.

A I only **B** II only **C** III only **D** IV only **E** none of these

9 If a researcher increases her sample size by a factor of 4, then the width of a 95% confidence interval would

A increase by a factor of 2 **B** increase by a factor of 4 **C** decrease by a factor of 2
D decrease by a factor of 4 **E** none of these

10 The Education Department in a certain state wishes to determine the percentage of teachers who are considering leaving the profession in the next two years. They believe it to be about 25%. How large a sample should be taken to find the answer to within ±3% at the 95% confidence level?

A 6 **B** 33 **C** 534 **D** 752 **E** 897

11 Which of the following statements is true?

A We use sample statistics to estimate population parameters.
B We use sample parameters to estimate population statistics.
C We use population parameters to estimate sample statistics.
D We use population statistics to estimate sample parameters.
E none of the above

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12 A sampling distribution can best be described as a distribution which

A gives the possible range of values of the sample statistic
B describes how a statistic's value will change from sample to sample
C describes how samples do not give reliable estimates
D gives the distribution of the values observed in particular sample
E none of the above

13 A survey is conducted to determine the percentage of students in Year 12 who intend to go straight to university after they finish secondary school. In a random sample of 100 students, 78% indicated this intention. A 95% confidence interval for the percentage of students in Year 12 who intend to go straight to university is

A 68.2% to 87.8%
B 68.5% to 87.5%
C 69.9% to 86.1%
D 71.2% to 84.8%
E 73.9% to 82.1%

14 In Question 13, what is one way to decrease the width of the confidence interval?

A increase the sample size
B use a smaller confidence level
C use a higher confidence level
D both A and B are correct
E both A and C are correct

Extended-response questions

1 A survey is being planned to estimate the proportion of people in Australia who think that university fees should be abolished. The organisers of the survey want the error in the approximate 95% confidence interval for this proportion to be no more than ±2%. They have no prior information about the value of the proportion.

a Plot that sample size, $n = \left(\dfrac{1.96}{M}\right)^2 p^*(1 - p^*)$, against p^* for $0 \le p^* \le 1$.
b For what value of p^* is the sample size the maximum?
c What value of n would you recommend be used for the survey?
d Show that the maximum sample size required for the error in an approximate 95% confidence interval to be no more than M is approximately $n = \dfrac{1}{M^2}$.

2 It is known that 60% of the voters in a particular electorate support the Liberal party. A sample of 100 voters is taken. Let $\hat{P}$ be the random variable for the sampling distribution of the sample proportion. Use the normal approximation to find:

a $\Pr(\hat{P} > 0.65)$
b $\Pr(0.5 < \hat{P} < 0.65)$

3 **a** Summer is investigating the probability that a drawing pin will land point-up when tossed. She tosses the drawing pin 100 times, and finds that it lands point-up 57 times. Determine an approximate 95% confidence interval for the probability that the drawing pin lands point-up when tossed.

b Four of Summer's friends decide to repeat her investigation, each tossing the drawing pin 100 times. They each calculate an approximate 95% confidence interval based on their own data, making five confidence intervals in all.

- **i** What is the probability that all five confidence intervals contain the true value of p, the probability that the drawing pin will land point-up when tossed?
- **ii** What is the probability that none of the confidence intervals contain p?
- **iii** What is the probability that at least one of the confidence intervals does not contain p?
- **iv** How many of these five confidence intervals would you expect to contain p?

c Summer's four friends obtained the following results, each based on tossing the drawing pin 100 times and counting the number of times that it lands point-up:

- Emma 67
- Chloe 72
- Maddie 55
- Regan 60

Summer suggests that the best estimate of p would be obtained by pooling their results. Based on all the data collected, determine an approximate 95% confidence interval for p.

4 A landscape gardener wishes to estimate how many carp live in his very large ornamental lake. He is advised that the best way to do this is through capture–recapture sampling.

a Suppose that there are N carp in the lake and he captures 500 of them, tags them and then releases them back into the lake. Write down an expression for the proportion of tagged carp in the lake.

b The next day, a sample of 400 carp is captured from the lake, and he finds that there are 60 tagged carp in this sample. What is the proportion of tagged carp in the second sample?

c If the second sample is representative of the population, we expect the proportion of tagged carp in the second sample to be the same as the proportion of tagged carp in the lake. That is,

$$\frac{60}{400} \approx \frac{500}{N}$$

Use this equation to find an estimate for the number of carp in the lake.

d Show that an expression for a 95% confidence interval for the proportion of tagged carp in the lake can be written as

$$0.15 - 1.96\sqrt{\frac{0.1275}{400}} < \frac{500}{N} < 0.15 + 1.96\sqrt{\frac{0.1275}{400}}$$

e Use this inequality to find an approximate 95% confidence interval for the number of carp in the lake.

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18

Revision of Chapters 13–17

18A Technology-free questions

1 The function

$$f(x) = \begin{cases} k\cos(\pi x) & \text{if } \dfrac{3}{2} < x < \dfrac{5}{2} \\ 0 & \text{otherwise} \end{cases}$$

is a probability density function for the continuous random variable X.

a Find the value of k.

b Find the median of X.

c Find $\Pr\left(X < \dfrac{7}{4} \,\middle|\, X < 2\right)$.

d Find $\Pr\left(X > \dfrac{9}{4} \,\middle|\, X > \dfrac{7}{4}\right)$.

2 The random variable X has the following probability distribution.

x	0	1	2	3	4
$\Pr(X = x)$	0.3	0.2	0.1	0.3	0.1

Find:

a $\Pr(X > 3 \mid X > 1)$

b $\Pr(X > 1 \mid X \le 3)$

c the mean of X

d the variance of X

3 The continuous random variable X has probability density function given by

$$f(x) = \begin{cases} kx(6 - x) & \text{if } 0 < x < 6 \\ 0 & \text{otherwise} \end{cases}$$

Find:

a the value of k

b $\Pr(X < 4)$

c the median of X

d the mean of X

e $\Pr(X < 2 \mid X < 3)$

f $\Pr(X > 2 \mid X < 4)$

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4 Eight coloured balls are placed in a box: 3 red balls, 2 black balls, 2 green balls and a yellow ball. A ball is randomly withdrawn from the box and not returned, and then a second ball is randomly withdrawn.

- **a** What is the probability of withdrawing a red ball first and a green ball second?
- **b** What is the probability of obtaining one green and one red ball?
- **c** What is the probability that the second ball withdrawn is not red, given that the first ball withdrawn is red?
- **d** What is the probability that neither of the two balls withdrawn is red?
- **e** What is the probability of obtaining two balls of the same colour?

5 Two events A and B are such that $\Pr(A) = \frac{4}{7}$ and $\Pr(B) = \frac{1}{3}$. Find $\Pr(A' \cap B)$ if:

- **a** $\Pr(A \cup B) = \frac{5}{7}$
- **b** A and B are mutually exclusive.

6 Two events A and B are such that $\Pr(A) = \frac{3}{4}$, $\Pr(B \mid A) = \frac{1}{5}$ and $\Pr(B' \mid A') = \frac{4}{7}$. Find:

- **a** $\Pr(A \cap B)$
- **b** $\Pr(B)$
- **c** $\Pr(A \mid B)$

7 Janette and four friends each have an independent probability of 0.45 of winning a prize. Find the probability that:

- **a** exactly two of the friends win a prize
- **b** Janette and only one friend win a prize.

8 The random variable X has probability distribution:

x	1	2	3	4	5
$\Pr(X = x)$	a	0.3	0.1	0.2	b

- **a** Given that $E(X) = 2.34$, find the values of a and b.
- **b** Find $\operatorname{Var}(X)$.

9 The probability that a certain football club has all its first-team players fit is 0.7. When the club has a fully fit team, the probability of it winning a home game is 0.9. When the team is not fully fit, the probability of winning a home game is 0.4.

- **a** Find the probability that the team will win its next home game.
- **b** Given that the team did not win its last home game, find the probability that the team was fully fit.

10 The random variable X has probability density function:

$$f(x) = \begin{cases} (x-a)(2a-x) & \text{if } a \le x \le 2a \\ 0 & \text{otherwise} \end{cases}$$

- **a** Show that $a^3 = 6$.
- **b** Find $E(X)$.

11 The random variable X is normally distributed with mean 40 and standard deviation 2. If $\Pr(36 < X < 44) = q$, find $\Pr(X > 44)$ in terms of q.

12 The probability density function of a random variable X is given by

$$f(x) = \begin{cases} 2(1-x) & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the value a of X such that $\Pr(X < a) = \frac{3}{4}$.

13 A biased coin is tossed three times. On each toss, the probability of a head is p.

a Find, in terms of p, the probability that all three tosses show tails.

b If the probability of three tails is equal to 8 times the probability of three heads, find p.

14 Suppose that an approximate 95% confidence interval for the population proportion is given by the interval (a, b).

a Write down an expression for the sample proportion in terms of a and b.

b Write down an expression for the margin of error for this confidence interval in terms of a and b.

18B Multiple-choice questions

1 A box contains 12 red balls and 4 green balls. A ball is selected at random from the box and not replaced, and then a second ball is drawn. The probability that the two balls are both green is equal to

A $\frac{1}{4}$ **B** $\frac{1}{16}$ **C** $\frac{3}{64}$ **D** $\frac{1}{8}$ **E** $\frac{1}{20}$

2 A test consists of six true/false questions. The probability that a student who guesses will obtain six correct answers is

A 0.9844 **B** 0.0278 **C** 0.5 **D** 0.0156 **E** 0.17

3 A random variable X has the following probability distribution.

x	1	2	3	4
$\Pr(X = x)$	$4c^2$	$5c^2$	$4c^2$	$3c^2$

The value of c is

A 0.5 **B** 0.0263 **C** 0.1622 **D** 0.25 **E** 0.0625

4 Suppose that a spinner numbered 1, 2, 3, 4, 5, 6 is spun until a '3' appears, and the number of spins is noted. The sample space for this random experiment is

A $\{1, 2, 3, 4, 5, 6\}$ **B** $\{0, 1, 2, 3, 4, 5, 6\}$ **C** $\{1, 2, 3, 4, \dots\}$

D $\{3\}$ **E** $\{1, 2, 3\}$

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Questions 5–8 refer to the following probability distribution.

x	4	6	7	9
$\Pr(X = x)$	0.3	0.2	0.1	0.4

5 For this probability distribution, the mean, $E(X)$, is equal to

A 6.7 **B** 0.275 **C** 6.5 **D** 2.75 **E** 2.59

6 For this probability distribution, the variance, $\text{Var}(X)$, is equal to

A 19.45 **B** 4.41 **C** 6.7 **D** 2.1 **E** 0.61

7 Let $Y = 2X - 1$, where X has the probability distribution given in the table. The probability distribution of Y is

A

y	4	6	7	9
$\Pr(Y = y)$	0.3	0.2	0.1	0.4

B

y	8	12	14	18
$\Pr(Y = y)$	0.3	0.2	0.1	0.4

C

y	4	6	7	9
$\Pr(Y = y)$	0.6	0.4	0.2	0.8

D

y	6	10	12	16
$\Pr(Y = y)$	0.3	0.2	0.1	0.4

E

y	7	11	13	17
$\Pr(Y = y)$	0.3	0.2	0.1	0.4

8 Let $Z = 4 - X$, where X has the probability distribution given in the table. The variance of Z is

A 11.59 **B** 2.1 **C** 0.41 **D** 4.41 **E** −0.41

9 Suppose that in Melbourne the probability of the temperature exceeding 30°C on a particular day is 0.6 if the temperature exceeded 30°C on the previous day, and 0.25 if it did not. If the temperature exceeds 30°C on Monday, then the probability that it exceeds 30°C on Wednesday is

A 0.36 **B** 0.10 **C** 0.60 **D** 0.30 **E** 0.46

10 If a random variable X is such that $E(X) = 11$ and $E(X^2) = 202$, then the standard deviation of X is equal to

A 191 **B** 13.82 **C** 9 **D** 3.72 **E** 81

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11 A set of test scores has a probability distribution with mean $\mu = 50$ and standard deviation $\sigma = 10$. Which of the following intervals contains about 95% of the test scores?

A $(40, 60)$ **B** $(30, 70)$ **C** $(20, 80)$ **D** $(46.84, 53.16)$ **E** $(43.68, 56.32)$

12 If three fair coins are tossed, what is the probability that there are at least two heads?

A $\frac{1}{3}$ **B** $\frac{6}{7}$ **C** $\frac{1}{4}$ **D** $\frac{1}{2}$ **E** $\frac{1}{8}$

13 Let X be a binomial random variable with parameters $n = 400$ and $p = 0.1$. Then $\mathrm{E}(X)$, the mean of X, is equal to

A 36 **B** 6 **C** 40 **D** 6.32 **E** 360

14 Which of the following does *not* define a binomial random variable?

A A die is rolled 10 times, and the number of sixes observed.

B A die is rolled until a six is obtained, and the number of rolls counted.

C A die is rolled five times, and the number of even numbers showing observed.

D A sample of 20 people is chosen from a large population, and the number of females counted.

E A student guesses the answer to every question on a multiple-choice test, and the number of correct answers is noted.

15 Let X be a binomial random variable with parameters $n = 900$ and $p = 0.2$. The standard deviation of X is equal to

A 18 **B** 144 **C** 180 **D** 13.42 **E** 12

16 Let X be a binomial random variable with a variance of 9.4248. If $n = 42$, then the probability of success p is equal to

A 0.45 **B** 0.22 **C** 0.34 **D** 0.68 **E** 0.34 or 0.66

17 If p is the probability of success in one trial, then $\binom{7}{5} p^5 (1 - p)^2$ is the probability of

A exactly two failures **B** exactly two successes **C** at least two failures
D exactly five failures **E** more failures than successes

18 The proportion of female students at a particular university is 0.2. A sample of 10 students is chosen at random from the entire student population. What is the probability that the sample contains exactly four female students?

A 0.0881 **B** 0.5000 **C** 0.0328 **D** 0.0016 **E** 0.9672

19 Mai decides to call five friends to invite each of them to a party. The probability of a friend not being at home when Mai calls is 0.4. What is the probability that Mai finds at least one of her friends at home?

A 0.0778 **B** 0.9222 **C** 0.0102 **D** 0.9898 **E** 0.0768

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20 If a random variable X has probability density function given by

$$f(x) = \begin{cases} kx^3 + \frac{3}{4}x & \text{if } 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

then k is equal to

A $-\frac{3}{16}$ **B** $\frac{6}{25}$ **C** $-\frac{9}{16}$ **D** $-\frac{1}{8}$ **E** $-\frac{3}{8}$

21 If a random variable X has probability density function given by

$$f(x) = \begin{cases} \frac{1}{9}(4x - x^2) & \text{if } 0 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

then $\Pr(X \le 2)$ is closest to

A 0.6667 **B** 0.4074 **C** 0.5926 **D** 0.4444 **E** 0.5556

22 A random variable X has probability density function:

$$f(x) = \begin{cases} \frac{8}{3}(1 - x) & \text{if } 0 < x < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

The median of X is closest to

A 0.222 **B** 0.667 **C** 0.250 **D** 1.791 **E** 0.209

23 A random variable X has probability density function:

$$f(x) = \begin{cases} 2\left(1 - \frac{1}{x^2}\right) & \text{if } 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

The mean of X is closest to

A 1 **B** 1.614 **C** 2 **D** 1.5 **E** 0.609

24 The probability of obtaining a z-value which falls between $z = -1.0$ and $z = 0$ for a standard normal distribution is approximately

A 0.05 **B** 0.20 **C** 0.34 **D** 0.68 **E** 0.16

25 For a normal probability distribution, which of the following is true?

A The mean is always positive.

B No value can be more than four standard deviations away from the mean.

C The area under the normal curve is approximately equal to 1.

D The standard deviation is always positive.

E The standard deviation is less than the mean.

26 If X is a normally distributed random variable with mean $\mu = 2$ and standard deviation $\sigma = 0.5$, then the probability that X is greater than 2.6 is closest to

A 0.8849 **B** 0.9918 **C** 0.1151 **D** 0.0082 **E** 0.0302

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27 If X is a normally distributed random variable with mean $\mu = 2$ and standard deviation $\sigma = 2$, then the probability that X is less than -2 is

A 0.1587 **B** 0.8413 **C** 0.9772 **D** 0.1228 **E** 0.0228

28 If X is a normally distributed random variable with mean $\mu = 3$ and variance $\sigma^2 = 0.4$, then the probability that X is greater than -2.73 is

A 1 **B** 0 **C** 0.9115 **D** 0.0885 **E** 0.5537

29 If X is a normally distributed random variable with mean $\mu = 2$ and variance $\sigma^2 = 4$, then $\Pr(1 < X < 2.5)$ is

A 0.5987 **B** 0.2902 **C** 0.6915 **D** 0.4013 **E** 0.3085

30 An automatic dispensing machine fills cups with cordial. If the amount of cordial in the cup is a normally distributed random variable with a mean of 50 mL and a standard deviation of 2 mL, then 90% of the cups contain more than

A 44.87 mL **B** 53.29 mL **C** 46.71 mL **D** 52.56 mL **E** 47.44 mL

31 Lengths of blocks of cheese are found to be normally distributed with a mean of 10 cm and a variance of 0.5. Then 95% of the blocks of cheese are shorter than

A 11.39 cm **B** 8.84 cm **C** 11.16 cm **D** 8.61 cm **E** 9.18 cm

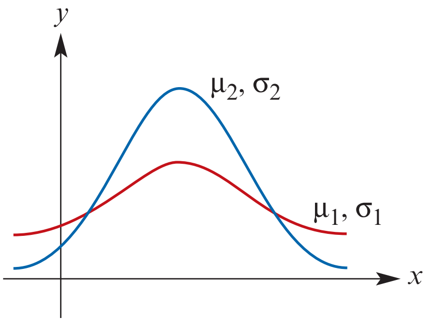
32 Assume that X is a normally distributed random variable with mean $\mu = 1$ and variance $\sigma^2 = 2.25$. If $\Pr(\mu - k < X < \mu + k) = 0.7$, then $k =$

A 1.555 **B** 1.037 **C** 0.787 **D** 0.524 **E** 2.332

33 The weight of a packet of biscuits is known to be normally distributed with a mean of 1 kg. If a packet is more than 0.05 kg underweight, it is unacceptable. If it is found that 3% of packets are unacceptable, then the standard deviation of the weight is

A 1.881 **B** 0.027 **C** 10.488 **D** 0.030 **E** 37.616

34 The diagram shows the probability density functions of two normally distributed random variables, one with mean μ_1 and standard deviation σ_1, and the other with mean μ_2 and standard deviation σ_2.
Which of the following statements is true?



A $\mu_1 = \mu_2,\ \sigma_1 < \sigma_2$

B $\mu_1 = \mu_2,\ \sigma_1 > \sigma_2$

C $\mu_1 > \mu_2,\ \sigma_1 = \sigma_2$

D $\mu_1 < \mu_2,\ \sigma_1 = \sigma_2$

E $\mu_1 = \mu_2,\ \sigma_1 = \sigma_2$

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35 If the heights of a certain population of men are normally distributed with a mean of 173 cm and a variance of 25, then about 68% of men in the population have heights in the interval (in cm)

A $(148, 198)$ **B** $(168, 178)$ **C** $(163, 183)$ **D** $(158, 188)$ **E** $(123, 223)$

36 In a random sample of 200 people, 38% said they would rather watch tennis on television than attend the match. An approximate 95% confidence interval for the proportion of people in the population who prefer to watch tennis on television is

A $(0.136, 0.244)$ **B** $(0.313, 0.447)$ **C** $(0.285, 0.475)$
D $(0.255, 0.505)$ **E** $(0.292, 0.468)$

37 For a fixed sample, an increase in the level of confidence will lead to a confidence interval which is

A narrower **B** wider **C** unchanged **D** asymmetric
E cannot be determined from the information given

38 Which of the following statements are true?

I The lower the level of confidence, the smaller the confidence interval.
II The larger the sample size, the smaller the confidence interval.
III The smaller the sample size, the smaller the confidence interval.
IV The higher the level of confidence, the smaller the confidence interval.

A I and II **B** I and III **C** II only **D** II and IV **E** none of these

39 If a researcher decreases her sample size by a factor of 2, then the width of a 95% confidence interval would

A increase by a factor of 2 **B** increase by a factor of $\sqrt{2}$
C decrease by a factor of $\sqrt{2}$ **D** decrease by a factor of 4
E none of these

18C Extended-response questions

1 In each of a sequence of trials, the probability of the occurrence of a certain event is $\frac{1}{2}$, except that this event cannot occur in two consecutive trials.

a Show that the probability of this event occurring:

i exactly twice in three trials is $\frac{1}{4}$ **ii** exactly twice in four trials is $\frac{1}{2}$.

b What is the probability of this event occurring exactly twice in five trials?

2 Katia and Mikki play a game in which a fair six-sided die is thrown five times:

- Katia will receive \$1 from Mikki if there is an odd number of sixes
- Mikki will receive \$$x$ from Katia if there is an even number of sixes.

Find the value of x so that the game is fair. (Note that the number 0 is even.)

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3 A newspaper seller buys papers for 50 cents and sells them for 75 cents, and cannot return unsold papers. Daily demand has the following distribution, and each day's demand is independent of the previous day's demand.

Number of customers	24	25	26	27	28	29	30
Probability	0.05	0.10	0.10	0.25	0.25	0.15	0.10

If the newspaper seller stocks too many papers, a loss is incurred. If too few papers are stocked, potential profit is lost because of the excess demand. Let s represent the number of newspapers stocked, and X the daily demand.

a If P is the newspaper seller's profit for a particular stock level s, find an expression for P in terms of s and X.

b Find the expected value of the profit, $\mathrm{E}(P)$, when $s = 26$.

c Hence find an expression for the expected profit when s is unknown.

d By evaluating the expression for expected profit for different values of s, determine how many papers the newspaper seller should stock.

4 Let X be a random variable with mean μ and variance σ^2. Show that, if $Z = \dfrac{X - \mu}{\sigma}$, then $\mathrm{E}(Z) = 0$ and $\mathrm{Var}(Z) = 1$.

5 Anne and Jane play a game against each other, which starts with Anne aiming to throw a bean bag into a circle marked on the ground.

a The probability that the bean bag lands entirely inside the circle is $\frac{1}{2}$, and the probability that it lands on the rim of the circle is $\frac{1}{3}$.

- **i** Show that the probability that the bean bag lands entirely outside the circle is $\frac{1}{6}$.
- **ii** What is the probability that two successive throws land outside the circle?
- **iii** What is the probability that, for two successive throws, the first lands on the rim of the circle and the second inside the circle?

b Jane then shoots at a target on which she can score 10, 5 or 0. With any one shot, Jane scores 10 with probability $\frac{2}{5}$, scores 5 with probability $\frac{1}{10}$, and scores 0 with probability $\frac{1}{2}$. With exactly two shots, what is the probability that her total score is:

i 20 **ii** 10?

c If the bean bag thrown by Anne lands outside the circle, then Jane is allowed two shots at her target; if the bean bag lands on the rim of the circle, then Jane is allowed one shot; if it lands inside the circle, then Jane is not allowed any shots. Find the probability that Jane scores a total of 10 as a result of any one throw from Anne.

6 The lifetime of a certain brand of light bulb is normally distributed with a mean of $\mu = 400$ hours and a standard deviation of $\sigma = 50$ hours.

a Find the probability that a randomly chosen light bulb will last more than 375 hours.

b The light bulbs are sold in boxes of 10. Find the probability that at least nine of the bulbs in a randomly selected box will last more than 375 hours.

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7 A large taxi company determined that the distance travelled annually by each taxi is normally distributed with a mean of 80 000 km and a standard deviation of 20 000 km.

a What is the probability that a randomly selected taxi will travel between 56 000 km and 60 000 km in a year?

b What percentage of taxis can be expected to travel either less than 48 000 km or more than 96 000 km in a year?

c How many of the 250 taxis in the fleet are expected to travel between 48 000 km and 96 000 km in a year?

d At least how many kilometres would be travelled by 85% of the taxis?

8 The weight of cereal in boxes, packed by a particular machine, is normally distributed with a mean of μ g and a standard deviation of $\sigma = 5$ g.

a A box is considered underweight if it weighs less than 500 g.

- **i** Find the proportion of boxes that will be underweight if $\mu = 505$ g.
- **ii** Find the value of μ required to ensure that only 1% of boxes are underweight.

b As a check on the setting of the machine, a random sample of five boxes is chosen and the setting changed if more than one of them is underweight. Find the probability that the setting of the machine is changed if $\mu = 505$ g.

9 A factory has two machines that produce widgets. The time taken, X seconds, to produce a widget using machine A is normally distributed, with a mean of 10 seconds and a standard deviation of 2 seconds. The time taken, Y seconds, to produce a widget using machine B has probability density function given by

$$f(y) = \begin{cases} k(y-8) & \text{if } 8 < y < 12 \\ 0 & \text{otherwise} \end{cases}$$

a **i** Find the value of k.

- **ii** Show that machine A has a greater probability of producing a widget in less than 11 seconds than machine B.
- **iii** Find which machine, on average, is quicker in producing widgets.

b Suppose that 60% of the widgets manufactured at the factory are produced by machine A, and 40% by machine B. If a widget selected at random is known to have been produced in less than 10 seconds, what is the probability that it was produced by machine A?

10 The random variable X has probability function given by

$$\Pr(X = x) = \begin{cases} \dfrac{c}{x} & \text{if } x = 1, 2, \ldots, 6 \\ 0 & \text{otherwise} \end{cases}$$

where c is a constant. Find the value of:

a c

b $\mathrm{E}(X)$

c $\mathrm{Var}(X)$

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11 The queuing time, X minutes, at the box office of a movie theatre has probability density function:

$$f(x) = \begin{cases} kx(100 - x^2) & \text{if } 0 \le x \le 10 \\ 0 & \text{otherwise} \end{cases}$$

a Find:

- **i** the value of k
- **ii** the mean of X
- **iii** the probability that a moviegoer will have to queue for more than 3 minutes
- **iv** the probability that a moviegoer will have to queue for more than 3 minutes, given that she queues for less than 7 minutes.

b If 10 moviegoers go independently to the theatre, find the probability that at least five of them will be required to queue for more than 3 minutes.

12 Electronic sensors of a certain type fail when they become too hot. The temperature at which a randomly chosen sensor fails is T°C, where T is modelled as a normal random variable with mean μ and standard deviation σ.

a In a laboratory test, 98% of a random sample of sensors continued working at a temperature of 80°C, but only 4% continued working at 104°C.

- **i** Show the given information on a sketch of the distribution of T.
- **ii** Determine estimates for the values of μ and σ.

b More extensive tests confirmed that T is normally distributed, but with $\mu = 94.5$ and $\sigma = 5.7$. Use these values in the rest of the question.

- **i** What proportion of sensors will operate in boiling water (i.e. at 100°C)?
- **ii** The manufacturers wish to quote a safe operating temperature at which 99% of the sensors will work. What temperature should they quote?

13 A flight into an airport is declared to be 'on time' if it touches down within 3 minutes either side of the advertised arrival time; otherwise, it is declared early or late. On any one occasion, the probability that a fight is on time is 0.5 and the probability that it is late is 0.3. The time of arrival of a particular flight on any one day is independent of the time of arrival on any other day.

a Calculate the probability that:

- **i** on any given day, the flight arrives early
- **ii** on any given day, the flight does not arrive late
- **iii** the flight arrives on time on three consecutive days
- **iv** in any given week, the flight arrives late on Monday, but is on time for all the remaining four weekdays.

b In a given week of five days, find the probability that:

- **i** the flight is late exactly once
- **ii** the flight is early exactly twice.

c The airline is reported to the authority if the flight is late on more than two occasions in a five-day week. Find the probability that this happens.

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14 Jam is packed in tins of nominal net weight 1 kg. The actual weight of jam delivered to a tin by the filling machine is normally distributed about the mean weight set on the machine, with a standard deviation of 12 g.

a If the machine is set to 1 kg, find the probability that a tin chosen at random contains less than 985 g.

b It is a legal requirement that no more than 1% of tins contain less than the nominal weight. Find the minimum setting of the filling machine which will meet this requirement.

15 In a factory, machines A, B and C are all producing springs of the same length. Of the total production of springs in the factory, machine A produces 35% and machines B and C produce 25% and 40% respectively. Of their production, machines A, B and C produce 3%, 6% and 5% defective springs respectively.

a Find the probability that:

- **i** a randomly selected spring is produced by machine A and is defective
- **ii** a randomly selected spring is defective.

b Given that a randomly selected spring is defective, find the probability that it was produced by machine C.

c Given that a randomly selected spring is not defective, find the probability that it was produced by either machine A or machine B.

16 An electronic game comes with five batteries. The game only needs four batteries to work. But because the batteries are sometimes faulty, the manufacturer includes five of them with the game. Suppose that X is the number of good batteries included with the game. The probability distribution of X is given in the following table.

x	0	1	2	3	4	5
$\Pr(X = x)$	0.01	0.02	0.03	0.04	0.45	0.45

a Use the information in the table to:

- **i** find μ, the expected value of X
- **ii** find σ, the standard deviation of X, correct to one decimal place
- **iii** find, exactly, the proportion of the distribution that lies within two standard deviations of the mean
- **iv** find the probability that a randomly selected game works, i.e. find $\Pr(X \geq 4)$.

b The electronic games are packed in boxes of 20. Whether or not an electronic game in a box will work is independent of any other game in the box working. Let Y be the number of working games in a box.

- **i** Name the distribution of Y.
- **ii** Find the expected number of working games in a box.
- **iii** Find the standard deviation of the number of working games in a box.
- **iv** Find the probability that a randomly chosen box will contain at least 19 working games.

17 There are n identical black balls and n identical white balls. A blue box contains 3 black balls and $n - 3$ white balls. A red box contains $n - 3$ black balls and 3 white balls. A ball is taken at random from the red box and put in the blue box. A ball is then taken at random from the blue box.

a Find the probability, in terms of n, that the ball taken from the blue box is:

i black **ii** white.

b Find the probability, in terms of n, that the first ball is black given that the second is white.

18 In a study of the prevalence of red hair in a certain country, researchers collected data from a random sample of 1800 adults.

a Of the 1000 females in the sample, they found that 10% had red hair. Calculate an approximate 95% confidence interval for the proportion with red hair in the female population.

b Of the 800 males in the sample, they found that 10% had red hair. Calculate an approximate 95% confidence interval for the proportion with red hair in the male population.

c Why is the width of the confidence interval for males different from the width of the confidence interval for females?

d How should the sample of 1800 adults be chosen to ensure that the widths of the two confidence intervals are the same when the sample proportions are the same?

e Assume that there are 1000 females and 800 males in the sample, and that the proportion of females in the sample with red hair is 10%. What sample proportion of red-headed males would result in the 95% confidence interval for the proportion with red hair in the female population and the 95% confidence interval for the proportion with red hair in the male population being of the same width?

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19

Revision of Chapters 1–18

19A Technology-free questions

1 Let $f(x) = x^2 + 6$ and $g(x) = 3x + 1$. Write down the rule of $f(g(x))$.

2 For the simultaneous linear equations

$$\begin{aligned} kx + 3y &= 0 \\ 4x + (k+2)y &= 0 \end{aligned}$$

where k is a real constant, find the value(s) of k for which there are infinitely many solutions.

3 Find the equation of the image of the graph of $y = \frac{1}{x}$ under the transformation defined by the matrix $\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$, and describe a sequence of transformations that maps the graph of $y = \frac{1}{x}$ onto its image.

4 **a** Let $f(x) = (5x^3 - 3x)^7$. Find $f'(x)$. **b** Let $f(x) = 2xe^{4x}$. Evaluate $f'(0)$.

5 **a** Differentiate $x^2 \log_e(2x)$ with respect to x. **b** For $f(x) = \frac{\sin x}{2x+1}$, find $f'\left(\frac{\pi}{2}\right)$.

6 **a** Let $f(x) = e^{\sin(2x)}$. Find $f'(x)$. **b** Let $f(x) = 3x\tan(2x)$. Evaluate $f'\left(\frac{\pi}{3}\right)$.

7 Find the general solution to the equation $\sin(2x) - \cos(2x) = 0$.

8 Let $f\colon [-\pi, \pi] \to \mathbb{R}$, $f(x) = 4\sin\left(2\left(x + \frac{\pi}{6}\right)\right)$.

a Write down the amplitude and period of the function f.

b Sketch the graph of the function f. Label the axis intercepts and the endpoints with their coordinates.

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9 Sketch the graph of $f\colon [-1, \infty) \setminus \{2\} \to \mathbb{R}$, $f(x) = 1 - \dfrac{4}{x-2}$. Label all axis intercepts, and label each asymptote with its equation.

10 For the function $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 5e^{x-1} - 3$:

a find the rule for the inverse function f^{-1}

b find the domain of the inverse function f^{-1}.

11 Solve the equation $\cos\left(\dfrac{5x}{2}\right) = \dfrac{1}{2}$ for $x \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.

12 Let $g\colon \mathbb{R} \to \mathbb{R}$, $g(x) = 5x^2$. Show that $g(u+v) + g(u-v) = 2(g(u) + g(v))$.

13 Find the average value of $y = e^x$ over the interval $[0, 4]$.

14 The graph of $y = ax^3 + bx + c$ has intercepts $(0, 6)$ and $(-2, 0)$ and has a turning point where $x = -1$.

a Find the value of c.

b Write down two simultaneous equations in a and b from the given information.

c Hence find the values of a and b.

15 Let $g\colon \mathbb{R} \to \mathbb{R}$, $g(x) = 3 - e^{2x}$.

a Find the rule and domain of the function g^{-1}.

b Sketch the graph of $y = g(g^{-1}(x))$ for its maximal domain.

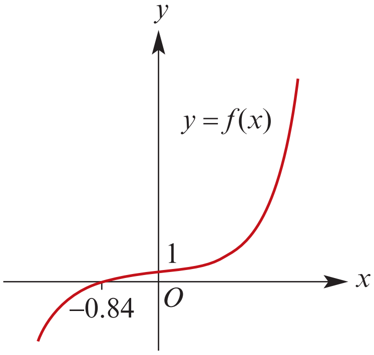
16 The graph of the piecewise-defined function

$$f(x) = \begin{cases} -2x^4 + 1 & \text{if } x \le 0 \\ 2x^4 + 1 & \text{otherwise} \end{cases}$$

is shown.

a Draw the graph of the derivative function f'.

b Write down a rule for the derivative function.



17 Find an antiderivative of $\dfrac{1}{1-3x}$ with respect to x, for $x < \dfrac{1}{3}$.

18 Let $f\colon \mathbb{R} \setminus \{\frac{1}{2}\} \to \mathbb{R}$ where $f(x) = \dfrac{3}{2x-1} + 3$. Find f^{-1}, the inverse function of f.

19 Solve the equation $\tan(2x) = -\sqrt{3}$ for $x \in \left(-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right) \cup \left(\dfrac{\pi}{4}, \dfrac{3\pi}{4}\right)$.

20 Let X be a normally distributed random variable with a mean of 84 and a standard deviation of 6. Let Z be the standard normal random variable.

a Find the probability that X is greater than 84.

b Use the result that $\Pr(Z < 1) = 0.84$ to find the probability that $78 < X < 90$.

c Find the probability that $X < 78$ given that $X < 84$.

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21 The probability density function of a random variable X is given by

$$f(x) = \begin{cases} \dfrac{x}{24} & \text{if } 1 \le x \le 7 \\ 0 & \text{otherwise} \end{cases}$$

a Find $\Pr(X < 3)$.

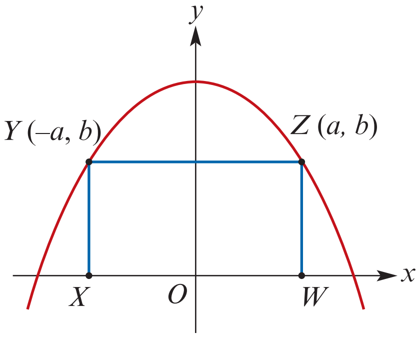
b If $b \in [1, 7]$ and $\Pr(X \ge b) = \dfrac{3}{8}$, find b.

22 A tangent to the graph of $y = x^{\frac{1}{3}}$ has equation $y = \dfrac{1}{3}x + a$. Find the value(s) of a.

23 A rectangle $XYZW$ has two vertices on the x-axis and the other two vertices on the graph of $y = 16 - 4x^2$, as shown in the diagram.

a Find the area, A, of rectangle $XYZW$ in terms of a.

b Find the maximum value of A and the value of a for which this occurs.



24 Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = -3x^2 + 2bx + 9$ with $\int_{-1}^{3} f(x)\, dx = 32$. Find the value of b.

25 Simone has either a sandwich or pasta for lunch. If she has a sandwich, the probability that she has a sandwich again the next day is 0.6. If she has pasta, the probability that she has pasta again the next day is 0.3. Suppose that Simone has a sandwich for lunch on a Monday. What is the probability that she has pasta for lunch on the following Wednesday?

26 A player in a game of chance can win \$0, \$1, \$2 or \$3. The amount won, \$$X$, is a random variable with probability distribution given by:

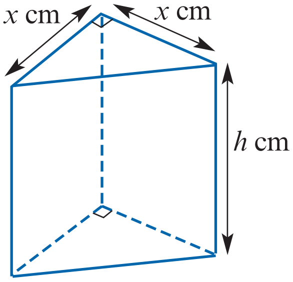
x	0	1	2	3
$\Pr(X = x)$	0.6	0.2	0.15	0.05

a Find the mean of X.

b What is the probability that a player wins the same amount from two games?

27 Every Thursday night, Chris either goes to the gym or goes for a run. If he goes to the gym one Thursday, the probability that he goes to the gym the next Thursday is 0.5. If he goes for a run one Thursday, the probability that he goes for a run the next Thursday is 0.6. If Chris goes to the gym one Thursday, what is the probability that he goes for a run on exactly two of the next three Thursdays?

28 A brick is made in the shape of a right triangular prism. The triangular end is a right-angled isosceles triangle, with the equal sides of length x cm. The height of the brick is h cm. The volume of the brick is 2000 cm^3.



a Find an expression for h in terms of x.

b Show that the total surface area, A cm^2, of the brick is given by $A = \dfrac{4000\sqrt{2} + 8000}{x} + x^2$.

c Find the value of x^3 if the brick has minimum surface area.

29 In order to measure the effect of alcohol on reaction time, an investigator selects a random sample of subjects from a group of diners in a restaurant.

a Do you think this sample will be representative of the general population? Explain your answer.

b How would you suggest that the sample could be chosen?

30 A coin is tossed 100 times, and 53 heads observed.

a Give a point estimate for p, the probability of a head when the coin is tossed.

b Write down an expression for a 95% confidence interval for p.

31 A sample of n people were asked whether they thought that Australians had access to adequate hospital care, and 37% said no.

a What is the value of the sample proportion, $\hat{p}$?

b Write down an expression for M, the margin of error for this estimate at the 95% confidence level, in terms of n.

c If the number of people in the sample were halved, what would be the effect on M?

19B Multiple-choice questions

1 The simultaneous linear equations

$$\begin{aligned} mx - 2y &= 0 \\ 6x - (m+4)y &= 0 \end{aligned}$$

where m is a real constant, have a unique solution provided

A $m \in \{-6, 2\}$
B $m \in \mathbb{R} \setminus \{-6, 2\}$
C $m \in \{-2, 6\}$
D $m \in \mathbb{R} \setminus \{-2, 6\}$
E $m \in \mathbb{R} \setminus \{0\}$

2 The general solution to the equation $\sin(2x) = 1$ is, where n is an integer,

A $x = n\pi + \dfrac{\pi}{4}$
B $x = \dfrac{n\pi}{2} + (-1)^n \dfrac{\pi}{2}$
C $x = 2n\pi + \dfrac{\pi}{4}$ or $x = 2n\pi - \dfrac{\pi}{4}$
D $x = \dfrac{n\pi}{4} + (-1)^n \dfrac{\pi}{4}$
E $x = n\pi + \dfrac{\pi}{4}$ or $x = 2n\pi + \dfrac{\pi}{4}$

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3 Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$. Which one of the following is *not* true?

A $f(xy) = f(x)f(y)$ **B** $f(x) + f(-x) = 0$ **C** $f(2x) = 8f(x)$

D $f(x - y) = f(x) - f(y)$ **E** $f(y - x) + f(x - y) = 0$

4 Define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 5x + 1 & \text{if } x \geq -\frac{4}{5} \\ -5x - 7 & \text{if } x < -\frac{4}{5} \end{cases}$$

Which of the following statements is *not* true about this function?

A The graph of f is continuous everywhere.

B The graph of f' is continuous everywhere.

C $f(x) \geq -3$, for all values of x. **D** $f'(x) = 5$, for all $x > 0$.

E $f'(x) = -5$, for all $x < -2$.

5 Let $k = \int_{-6}^{-2} \left(\frac{2}{x}\right) dx$. Then e^k is equal to

A $\log_e 3$ **B** 1 **C** $\frac{1}{9}$ **D** 9 **E** $\frac{1}{3}$

6 The average value of the function with rule $f(x) = \log_e(x + 2)$ over the interval $[0, 3]$ is

A $\frac{-1}{5}$ **B** $\log_e 6$ **C** $\frac{\log_e 5}{4}$ **D** $\frac{5\log_e 5 - 4}{4}$ **E** $\frac{5\log_e 5 - 3\log_e 3 - 4}{4}$

7 The average value of the function $y = \sin(2x)$ over the interval $\left[0, \frac{\pi}{2}\right]$ is

A $\frac{2}{\pi}$ **B** $\frac{\pi}{2}$ **C** 0.5 **D** 0 **E** π

8 The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

The equation of the image of the curve $y = x^2$ under T is

A $2y = (x - 5)^2 + 2$ **B** $3y = (x - 5)^2 + 3$ **C** $9y = (x + 5)^2 - 9$

D $9y = (x - 5)^2 + 9$ **E** $y = \left(\frac{x}{5} - 5\right)^2 + 2$

9 If $f(x) = e^{3x}$, for all real x, and $[f(x)]^3 = f(y)$, then y is equal to

A e^{9x} **B** $9x$ **C** $3x$ **D** $9x^3$ **E** $(3x)^3$

10 The continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} \sin(2x) & \text{if } 0 \leq x \leq \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

The value of a such that $\Pr(X > a) = 0.25$ is closest to

A 0.25 **B** 0.75 **C** 1.04 **D** 1.05 **E** 1.09

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11 The radius of a sphere is increasing at a rate of 4 cm/min. When the radius is 2 cm, the rate of increase of the volume of the sphere (in cm^3/min) is

A 32π **B** 48π **C** 64π **D** 96π **E** 120π

12 The function f is a probability density function, with rule

$$f(x) = \begin{cases} 1 + 2e^{\frac{x}{k}} & \text{if } 0 \le x \le 2k \\ 0 & \text{otherwise} \end{cases}$$

Hence k is equal to

A $\frac{1}{2}e^{-2}$ **B** $1 + e^2$ **C** e^{-2} **D** $1 - e^{-2}$ **E** 1

13 The random variable X has a normal distribution with a mean of 8 and a standard deviation of 0.25. If Z has the standard normal distribution, then the probability that X is less than 7.5 is equal to

A $\Pr(Z > 2)$ **B** $\Pr(Z < -1.5)$ **C** $\Pr(Z < 1)$ **D** $\Pr(Z \ge 1.5)$ **E** $\Pr(Z < -4)$

14 The graph of $y = 2kx - 2$ intersects the graph of $y = x^2 + 12x$ at two points for

A $k = 12$ **B** $k > 6 + \sqrt{2}$ or $k < 6 - \sqrt{2}$

C $4 < k < 7$ **D** $5 < k < 7$ **E** $6 - \sqrt{2} < k < 6 + \sqrt{2}$

15 The set of solutions to the equation $e^{4x} - 7e^{2x} + 12 = 0$ is

A $\{3, 4\}$ **B** $\{-4, -3\}$ **C** $\{-2, -\sqrt{3}, \sqrt{3}, 2\}$

D $\{\log_e \sqrt{3}, \log_e 2\}$ **E** $\{-\log_e \sqrt{3}, \log_e \sqrt{3}, \log_e 2\}$

16 The graph of the function $f\colon [0, \infty) \to \mathbb{R}$, where $f(x) = 7x^{\frac{3}{2}}$, is reflected in the x-axis and then translated 3 units to the right and 4 units down. The equation of the new graph is

A $y = 7(x - 3)^{\frac{3}{2}} + 4$ **B** $y = -7(x - 3)^{\frac{3}{2}} - 4$ **C** $y = -7(x + 3)^{\frac{3}{2}} - 1$

D $y = -7(x - 4)^{\frac{3}{2}} + 3$ **E** $y = 7(x - 4)^{\frac{3}{2}} + 3$

17 If a random variable X has probability density function given by

$$f(x) = \begin{cases} \frac{1}{8}x & \text{if } 0 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

then $\mathrm{E}(X)$ is equal to

A $\frac{1}{2}$ **B** 1 **C** $\frac{8}{3}$ **D** $\frac{16}{3}$ **E** 2

18 The function $f\colon [a, \infty) \to \mathbb{R}$ with rule $f(x) = \log_e((x - 2)^4)$ will have an inverse function if and only if

A $a \ge 3$ **B** $a \le -2$ **C** $a < 2$ **D** $a \ge 0$ **E** $a \ge -1$

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19 The transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ which maps the curve with equation $y = e^x$ to the curve with equation $y = e^{(2x+4)} - 3$ could have rule $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) =$

A $\begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ **B** $\begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -4 \\ 3 \end{bmatrix}$

D $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ **E** $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

20 Assume that $f'(x) = g'(x)$ with $f(1) = 2$ and $g(x) = -xf(x)$. Then $f(x) =$

A $g(x) + 4x + 4$ **B** $g'(x) + 4$ **C** $g(x) + 4x$ **D** $\dfrac{4 - 4x}{x + 1}$ **E** $g(x) + 4$

21 The function f satisfies the equation $f\left(\dfrac{xy}{2}\right) = 2f(x)f(y)$, for x and y any non-zero real numbers. A possible rule for the function is

A $f(x) = \dfrac{1}{x}$ **B** $f(x) = \dfrac{x}{2}$ **C** $f(x) = 2x$ **D** $f(x) = 2^x$ **E** $f(x) = \cos x$

22 The random variable X has the following probability distribution.

x	0	1	2
$\Pr(X = x)$	a	b	0.6

If the mean of X is 1.6, then

A $a = 0.3$ and $b = 0.7$ **B** $a = 0.2$ and $b = 0.2$ **C** $a = 0.4$ and $b = 0.4$

D $a = 0.1$ and $b = 0.3$ **E** $a = 0$ and $b = 0.4$

23 The simultaneous equations $(m - 4)x + 6y = 6$ and $2x + (m - 3)y = 2m - 10$ have no solution for

A $m \in \mathbb{R} \setminus \{0, 7\}$ **B** $m \in \mathbb{R} \setminus \{0\}$ **C** $m \in \mathbb{R} \setminus \{7\}$ **D** $m = 7$ **E** $m = 0$

Questions 24 and 25 are based on the following information:

An exit poll of 1000 randomly selected voters found that 520 favoured candidate A.

24 An approximate 95% confidence interval for the proportion of voters in favour of candidate A is

A (0.484, 0.546) **B** (0.422, 0.618) **C** (0.494, 0.546)

D (0.489, 0.551) **E** (0.479, 0.561)

25 On the basis of this confidence interval, what would be your prediction for the result of the election?

A predict a win for candidate A

B predict a loss for candidate A

C too close to make any prediction

D cannot tell as we do not know the number of candidates

E none of the above

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19C Extended-response questions

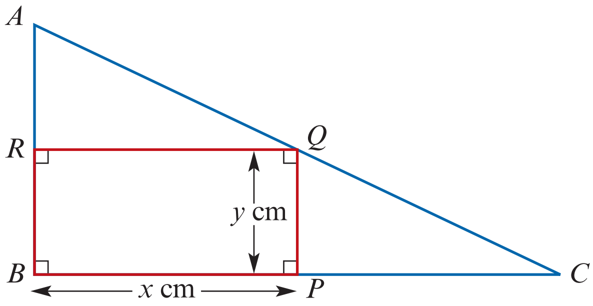
1 **a** **i** Find the coordinates of the stationary point for the curve with equation

$$y = \frac{16x^3 + 4x^2 + 1}{2x^2}$$

ii Determine the nature of this stationary point.

b The right-angled triangle ABC shown in the diagram has side lengths $AB = 5$ cm and $AC = 13$ cm.

The rectangle $BPQR$ is such that its vertices P, Q and R lie on the line segments BC, CA and AB respectively.



i Given that $BP = x$ cm and $PQ = y$ cm, show that $y = \dfrac{60 - 5x}{12}$.

ii Find the area of the rectangle, A cm^2, in terms of x.

iii Find the maximum value of this area as x varies.

2 A theoretical model of the relationship between two variables, x and y, predicts the values given in the table.

x	0	1	3
y	6	0	0

a An equation of the form $y = k(x - p)(x - q)$ is suggested, where p, q and k are constants and $p < q$.
Use the information in the table to find p, q and k.

b A series of experiments is carried out to test this model. The values of y when $x = 0, 1, 3$ are found to be as predicted. But when $x = 2$, the value of y is found to be 2. After further discussion, a new model is proposed with an equation of the form

$$y = m(x - p)^2(x - q)$$

where p and q have the values already calculated and m is a constant.

i Find the value of m.

ii Obtain the equation of this new model in the form $y = ax^3 + bx^2 + cx + d$.

iii Sketch the graph of y against x. State the coordinates of the stationary points and the nature of each of these points.

3 A curve C has equation $y = ax - x^2$, where a is a positive constant.

a Sketch C, showing clearly the coordinates of the axis intercepts.

b Calculate the area of the finite region bounded by C and the x-axis, giving your answer in terms of a.

c The lines $x = \dfrac{1}{3}a$ and $x = \dfrac{2}{3}a$ intersect C at the points A and B respectively.

i Find, in terms of a, the y-coordinates of A and B.

ii Calculate the area of the finite region bounded by C and the straight line AB, giving your answer in terms of a.

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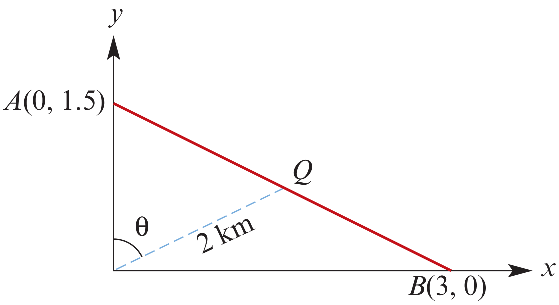


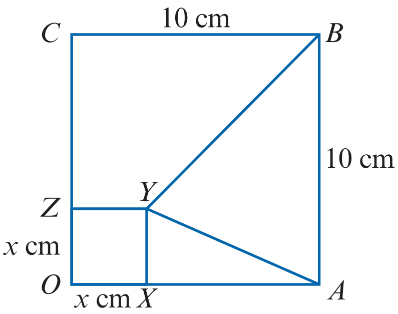
4 **a** Find the equation of the straight line joining the points $A(0, 1.5)$ and $B(3, 0)$.

b Let $y = \sin\theta + 2\cos\theta$.

- **i** Find $\dfrac{dy}{d\theta}$.
- **ii** Solve the equation $\dfrac{dy}{d\theta} = 0$ for θ, where $0° \le \theta \le 90°$.
- **iii** State the coordinates of the stationary point of $y = \sin\theta + 2\cos\theta$, where $0° \le \theta \le 90°$.
- **iv** It can be shown that $\sin\theta + 2\cos\theta$ can be written in the form $r\sin(\theta + \alpha)$. Use the result of **iii** and the fact that $y = 2$ when $\theta = 0$ to find the values of r and α.
- **v** Use addition of ordinates and the result of **iv** to sketch the graph of y against θ for $0° \le \theta \le 90°$.

c The figure shows a map of a region of wetland. The units of the coordinates are kilometres, and the y-axis points due north. A walker leaves her car somewhere on the straight road between A and B. She walks in a straight line for a distance of 2 km to a monument at the origin O. While she is looking at the monument, a fog comes down, and so she cannot see her way back to her car. She needs to work out the bearing on which she should walk.

- **i** Write down the coordinates of a point Q which is 2 km from O on a bearing of θ.
- **ii** Show that, for Q to be on the road between A and B, the angle θ must satisfy the equation $2\sin\theta + 4\cos\theta = 3$.
- **iii** Use the result of **b iv** to solve this equation for θ, where $0° \le \theta \le 90°$.



5 A square piece of card $OABC$, of side length 10 cm, is cut into four pieces by removing a square $OXYZ$ of side length x cm as shown, and then cutting out the triangle ABY.

a
- **i** Find A cm^2, the sum of the areas of $OXYZ$ and ABY, in terms of x.
- **ii** Find the domain of the function which determines this area.
- **iii** Sketch the graph of the function, with domain determined in **ii**.
- **iv** State the minimum value of this area.

b
- **i** Find the rule for the function of x which represents the area of triangle AXY.
- **ii** Sketch the graph of this function for a suitable domain.

c Find the ratio of the areas of the four pieces when the area of triangle AXY is a maximum.

6 The number of people unemployed in a particular population can be modelled by the function

$$f(t) = 1000(t^2 - 10t + 44)e^{\frac{-t}{10}}$$

where t is the number of months after January 2012 and $0 \le t \le 35$.

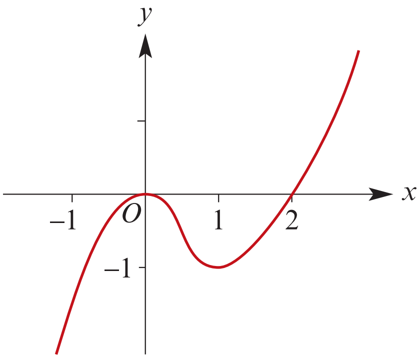
a Use this function to find an expression for:

- **i** the rate of increase of the number unemployed
- **ii** the rate of increase of this rate of increase.

b Find the values of t for which:

- **i** the number unemployed was increasing
- **ii** the rate of increase of the number unemployed was going down
- **iii** the number unemployed was increasing and the rate of increase of the number unemployed was going down.

7 The graph of $y = f(x)$ is shown.



a Sketch the graph of:

- **i** $y = 2f(x)$
- **ii** $y = f(2x)$
- **iii** $y = f(-x)$
- **iv** $y = -f(x)$
- **v** $y = f(x + 2)$

b Explain why f does not have an inverse function.

c **i** Sketch the graph of the function $g\colon (2, \infty) \to \mathbb{R}$, $g(x) = f(x)$.

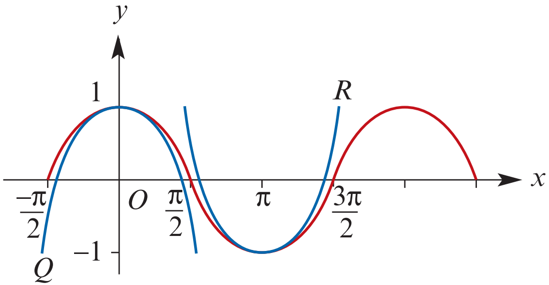
ii Sketch the graph of g^{-1}.

d **i** Given that $g\colon (2, \infty) \to \mathbb{R}$ where $g(x) = x^2(x - 2)$, calculate the gradient of the graph of $y = g(x)$ at the point $(3, 9)$.

ii Hence find the gradient of $y = g^{-1}(x)$ at the point $(9, 3)$.

8 The diagram shows part of the graph of $y = \cos x$ and the graphs of two quadratic functions, denoted by Q and R, which approximate to the cosine function around $x = 0$ and $x = \pi$ respectively.

The equation of Q is $y = 1 - \frac{1}{2}x^2$.



a **i** Find an estimate of $\cos 0.1$ by using the approximation $y = 1 - \frac{1}{2}x^2$.

ii Find an approximation for the solution to the equation $\cos x = 0.98$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, by solving the quadratic equation $1 - \frac{1}{2}x^2 = 0.98$.

b **i** The graph Q can be transformed into R by a reflection in the x-axis, followed by a translation. Use this fact to find an equation for the graph R.

ii Estimate the value of $\cos 3$ using this approximation.

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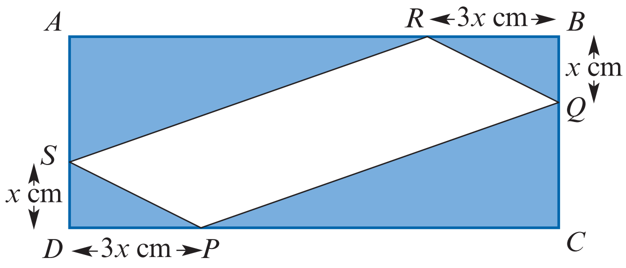
9 In the figure, $ABCD$ is a rectangle with $AB = 30$ cm and $AD = 10$ cm. The shaded portions are cut away, leaving the parallelogram $PQRS$, where $BQ = SD = x$ cm and $RB = DP = 3x$ cm.



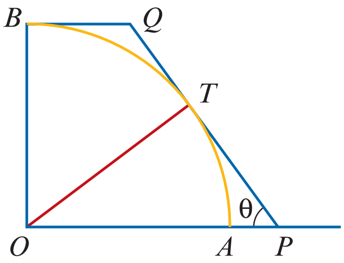
a Find the area, S cm^2, of the parallelogram in terms of x.

b Find the allowable values of x.

c Find the value of x for which S is a maximum.

d Sketch the graph of S against x for a suitable domain.

10 In the figure, OAB is a quadrant of a circle of radius 1 unit. The line segment OA is extended to a point P. From P, a tangent to the quadrant is drawn, touching it at T and meeting another tangent, BQ, at Q. Let $\angle OPQ = \theta$.



a **i** Find the length OP as a function of θ.

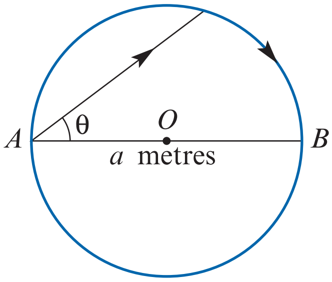
ii Find the length BQ as a function of θ.

b Show that the area, S, of trapezium $OPQB$ is given by $\dfrac{2-\cos\theta}{2\sin\theta}$.

c Show that $\dfrac{dS}{d\theta} = \dfrac{2-4\cos\theta}{4\sin^2\theta}$.

d Find the minimum value of S and the distance AP when S is a minimum.

11 A dog is at point A on the edge of a circular lake of diameter a metres, and she wishes to reach her owner who is at the diametrically opposite point B. The dog can swim at $\frac{1}{2}$ m/s and run at 1 m/s.



a If she swims in a direction making an angle of θ with AB and then runs round the edge of the lake to B, show that the time taken, T s, is given by

$$T = a(\theta + 2\cos\theta)$$

b On one set of axes, sketch the graphs of $y = 200\theta$ and $y = 400\cos\theta$ for $0 \le \theta \le \frac{\pi}{2}$. Using addition of ordinates, sketch the graph of

$$y = 200(\theta + 2\cos\theta)$$

(Find the maximum value of y for $0 \le \theta \le \frac{\pi}{2}$ by finding $\dfrac{dy}{d\theta}$ and then solving the equation $\dfrac{dy}{d\theta} = 0$.)

c Sketch the graph of $T = a(\theta + 2\cos\theta)$ for $0 \le \theta \le \frac{\pi}{2}$ and state the minimum value of T.

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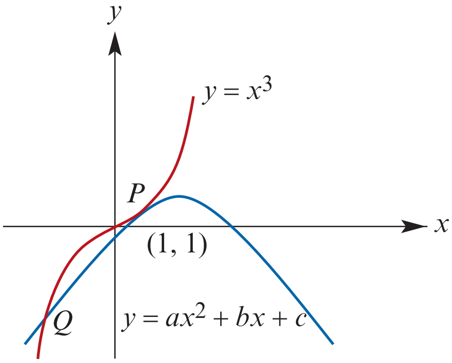
12 **a** **i** Show that, if $f(x) = (x - 1)g(x)$ and $f'(x) = (x - 1)h(x)$, where $g(x)$ and $h(x)$ are polynomials, then $(x - 1)$ must be a factor of $g(x)$.

ii Let $F(x) = x^3 - kx^2 - (3 - 2k)x - (k - 2)$. Show that $F(1) = F'(1) = 0$.

iii Using the results of **i** and **ii**, solve the equation $F(x) = 0$.

b The parabola $y = ax^2 + bx + c$ and the cubic $y = x^3$ touch at $P(1, 1)$ (and have the same gradient at this point). The curves also meet at Q.



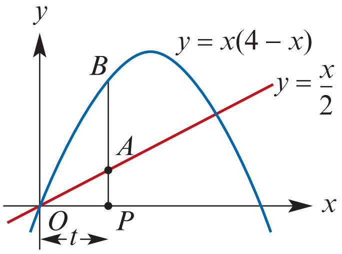
i Find b and c in terms of a.

ii If the coordinates of Q are (h, k), find h in terms of a. (Use the result of **a**.)

iii If Q has coordinates $(-2, -8)$, find the values of a, b and c.

iv If Q has coordinates $(-3, -27)$, find the values of a, b and c.

13 The point P has coordinates $(t, 0)$, where $0 < t < \frac{7}{2}$. The line PAB is parallel to the y-axis.



a Let Z be the length of AB. Find Z in terms of t.

b Sketch the graph of Z against t.

c State the maximum value of Z and the value of t for which it occurs.

14 A study is being conducted of the numbers of male and female children in families in a certain population.

a A simple model is that each child in any family is equally likely to be male or female, and that the sex of each child is independent of the sex of any previous children in the family. Using this model, calculate the probability that in a randomly chosen family of four children:

i there will be two males and two females

ii there will be exactly one female, given that there is at least one female.

b An alternative model is that the first child in any family is equally likely to be male or female, but that, for any subsequent children, the probability that they will be of the same sex as the previous child is $\frac{3}{5}$. Using this model, calculate the probability that in a randomly chosen family of four children:

i all four will be of the same sex

ii no two consecutive children will be of the same sex

iii there will be two males and two females.

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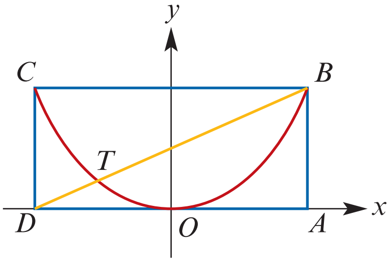
15 In the figure, $ABCD$ is a rectangle with $OA = OD = a$ and $AB = b$. The equation of the parabola BOC is $y = kx^2$.



a Express k in terms of a and b.

b If BD cuts the parabola at T, find:

  i the equation of the straight line BD

  ii the coordinates of T.

c Show that the area bounded by the parabola and the line BC is $\frac{4}{3}ab$ square units.

d Let S_1 be the area of the region bounded by the line segment BT and the curve BOT. Let S_2 be the area of the region bounded by the curve CT and the line segments BC and BT. Find the ratio $S_1 : S_2$.

16 A certain type of brass washer is manufactured as follows. A length of brass rod is cut cross-sectionally into pieces of mean thickness 0.25 cm, with a standard deviation of 0.002 cm. These brass slices are then put through a machine that punches out a circular hole of mean diameter 0.5 cm through the middle of the slice, with a standard deviation of 0.05 cm. The thickness of the washers and the diameters of the holes are known to be normally distributed, and do not depend on each other.

a Find the probability that a randomly selected washer will:

  i have a thickness of less than 0.253 cm

  ii have a thickness of less than 0.247 cm

  iii have a hole punched with a diameter greater than 0.56 cm

  iv have a hole punched with a diameter less than 0.44 cm.

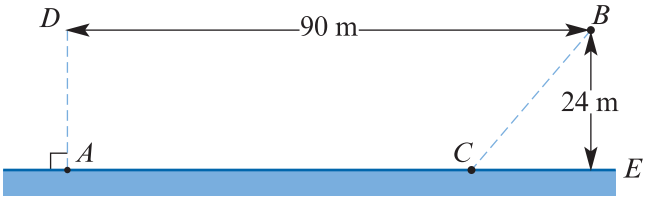
b The brass washers are acceptable only if they are between 0.247 cm and 0.253 cm in thickness with a hole of diameter between 0.44 cm and 0.56 cm. Find:

  i the percentage of washers that are rejected

  ii the expected number of washers of acceptable thickness in a batch of 1000 washers

  iii the expected number of washers of acceptable thickness that will be rejected in a batch of 1000 washers.

17 A ditch is to be dug to connect the points A and B in the figure. The earth on the same side of AE as B is hard, and the earth on the other side is soft.



The cost of digging hard earth is \$200 per metre and soft earth is \$100 per metre. Find the position of point C, where the turn is made, that will minimise the cost.

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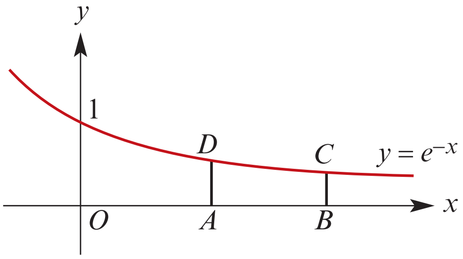
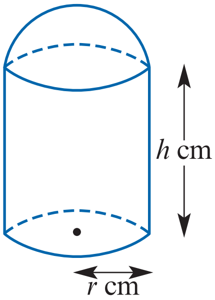
18 The diagram shows the graph of $y = e^{-x}$. The points A and B have coordinates $(n, 0)$ and $(n + 1, 0)$ respectively, and the points C and D on the curve are such that AD and BC are parallel to the y-axis.



- **a** **i** Find the equation of the tangent to $y = e^{-x}$ at the point D.
  - **ii** Find the intercept of the tangent with the x-axis.
- **b** **i** Find the area of the region $ABCD$.
  - **ii** The line BD divides the region into two parts. Find the ratio of the areas of these two parts.

19 A closed capsule is to be constructed as shown in the diagram. It consists of a circular cylinder of height h cm with a flat base of radius r cm. It is surmounted by a hemispherical cap.



- **a** **i** Show that the volume of the capsule, V cm^3, is given by $V = \dfrac{\pi r^2}{3}(3h + 2r)$.
  - **ii** Show that the surface area of the capsule, S cm^2, is given by $S = \pi r(2h + 3r)$.
- **b** **i** If $V = \pi a^3$, where a is a positive constant, find h in terms of a and r.
  - **ii** Hence find S in terms of a and r.
- **c** **i** By using addition of ordinates, sketch the graph of S against r for a suitable domain.
  - **ii** Find the coordinates of the turning point by first finding $\dfrac{dS}{dr}$, and then solving the equation $\dfrac{dS}{dr} = 0$ for r and determining the corresponding value of S.

20 A manufacturer sells cylinders whose diameters are normally distributed with mean 3 cm and standard deviation 0.002 cm. The selling price is \$$s$ per cylinder and the cost of manufacture is \$1 per cylinder. A cylinder is returned and the purchase money is refunded if the diameter of the cylinder is found to differ from 3 cm by more than d cm. A returned cylinder is regarded as a total loss to the manufacturer. The probability that a cylinder is returned is 0.25.

- **a** Find d.
- **b** The profit, \$$Q$, per cylinder is a random variable. Give the possible values of Q in terms of s, and the probabilities of these values.
- **c** Express the mean and standard deviation of Q in terms of s.

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21 The length of a certain species of worm has a normal distribution with mean 20 cm and standard deviation 1.5 cm.

a Find the probability that a randomly selected worm has a length greater than 22 cm.

b If the lengths of the worms are measured to the nearest centimetre, find the probability that a randomly selected worm has its length measured as 20 cm.

c If five worms are randomly selected, find the probability that exactly two will have their lengths measured as 20 cm (to the nearest centimetre).

22 The amount of coal, P tonnes, produced by x miners in one shift is given by the rule:

$$P = \frac{x^2}{90}(56 - x) \quad \text{where } 1 \le x \le 40$$

a Find $\dfrac{dP}{dx}$.

b **i** Sketch the graph of P against x for $1 \le x \le 40$.

ii State the maximum value of P.

c Write down an expression in terms of x for the average production per miner in the shift. Denote the average production per miner by A (in tonnes).

i Sketch the graph of A against x for $1 \le x \le 40$.

ii State the maximum value of A and the value of x for which it occurs.

23 Consider the family of quadratic functions with rules of the form

$$f(x) = (k + 2)x^2 + (6k - 4)x + 2$$

where k is an arbitrary constant.

a Sketch the graph of f when:

i $k = 0$ **ii** $k = -2$ **iii** $k = -4$

b Find the coordinates of the turning point of the graph of $y = f(x)$ in terms of k. If the coordinates of the turning point are (a, b), find:

i $\{k : a > 0\}$ **ii** $\{k : a = 0\}$ **iii** $\{k : b > 0\}$ **iv** $\{k : b < 0\}$

c For what values of k is the turning point a local maximum?

d By using the discriminant, state the values of k for which:

i $f(x)$ is a perfect square

ii there are no solutions to the equation $f(x) = 0$.

24 **a** Find the solution to the equation $e^{2-2x} = 2e^{-x}$.

b Let $y = e^{2-2x} - 2e^{-x}$.

i Find $\dfrac{dy}{dx}$. **ii** Solve the equation $\dfrac{dy}{dx} = 0$.

iii State the coordinates of the turning points of $y = e^{2-2x} - 2e^{-x}$.

iv Sketch the graph of $y = e^{2-2x} - 2e^{-x}$ for $x \ge 0$.

c State the set of values of k for which the equation $e^{2-2x} - 2e^{-x} = k$ has two distinct positive solutions.

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25 **a** Sketch, on a single clear diagram, the graphs of:

i $y = x^2$ **ii** $y = (x+a)^2$ **iii** $y = b(x+a)^2$ **iv** $y = b(x+a)^2 + c$

where a, b and c are positive constants with $b > 1$.

b Show that $\dfrac{2x^2+4x+5}{x^2+2x+1} = \dfrac{3}{(x+1)^2} + 2$, for all values except $x = -1$.

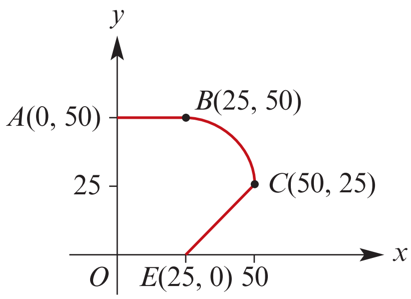
c Hence state precisely a sequence of transformations by which the graph of $y = \dfrac{2x^2+4x+5}{x^2+2x+1}$ may be obtained from the graph of $y = \dfrac{1}{x^2}$.

d Evaluate $\displaystyle\int_0^1 \frac{2x^2+4x+5}{x^2+2x+1}\,dx$.

e Sketch the graphs of $y = \dfrac{1}{x^2}$ and $y = \dfrac{3}{(x+1)^2} + 2$ on the one set of axes, and indicate the region for which the area has been determined in **d**.

26 A real-estate agent has a block of land to sell. An x–y coordinate grid is placed with the origin at O, as shown in the diagram.

The block of land is $OABCE$, where OA, AB, CE and EO are straight line segments and the curve through points B and C is part of a parabola with equation of the form $y = ax^2 + 4x + c$.



a Find the equation of line segments:

i AB **ii** EC

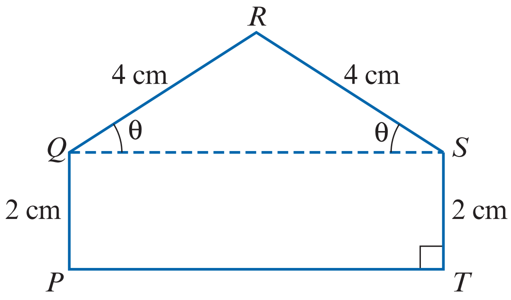
b Find the values of a and c and hence find the equation of the parabola through points B and C.

c Find the area of:

i the rectangle $OEBA$ **ii** the region EBC (with boundaries as defined above)

iii the block of land.

27 In the diagram, $PQRST$ is a thin metal plate, where $PQST$ is a rectangle with $PQ = 2$ cm and QRS is an isosceles triangle with $QR = RS = 4$ cm.



a Show that the area of the metal plate, A cm^2, is given by

$$A = 16(\cos\theta + \cos\theta\sin\theta)$$

for $0 < \theta < \dfrac{\pi}{2}$.

b Show that $\dfrac{dA}{d\theta} = 16(1 - \sin\theta - 2\sin^2\theta)$.

c Solve the equation $\dfrac{dA}{d\theta} = 0$ for $0 < \theta < \dfrac{\pi}{2}$ by first solving $16(1 - a - 2a^2) = 0$ for a.

d Hence sketch the graph of A against θ for $0 < \theta < \dfrac{\pi}{2}$ and state the maximum value of A.

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28 The length of an engine part must be between 4.81 cm and 5.20 cm. In mass production, it is found that 0.8% are too short and 3% are too long. Assume that the lengths are normally distributed.

a Find the mean and standard deviation of this distribution.

b Each part costs \$4 to produce; those that turn out to be too long are shortened at an extra cost of \$2, and those that turn out to be too short are rejected. Find the expected total cost of producing 100 parts that meet the specifications.

29 The temperature, T°C, of water in a kettle at time t minutes is given by the formula

$$T = \theta + Ae^{-kt}$$

where θ°C is the temperature of the room in which the kettle sits.

a Assume that the room is of constant temperature 21°C. At 2:23 p.m., the water in the kettle boils at 100°C. After 10 minutes, the temperature of the water in the kettle is 84°C. Use this information to find the values of k and A, giving your answer correct to two decimal places.

b At what time will the temperature of the water in the kettle be 70°C?

c Sketch the graph of T against t for $t \geq 0$.

d Find the average rate of change of temperature for the time interval $[0, 10]$.

e Find the instantaneous rate of change of temperature when:

i $t = 6$ **ii** $T = 60$

30 Large batches of similar components are delivered to a company. A sample of five articles is taken at random from each batch and tested. If at least four of the five articles are found to be good, the batch is accepted. Otherwise, the batch is rejected.

a If the fraction of defectives in a batch is $\frac{1}{2}$, find the probability of the batch being accepted.

b If the fraction of defectives in a batch is p, show that the probability of the batch being accepted is given by a function of the form

$$A(p) = (1 - p)^4(1 + bp), \quad 0 \leq p \leq 1$$

and find the value of b.

c Sketch the graph of A against p for $0 \leq p \leq 1$. (Using a calculator would be appropriate.)

d Find correct to two decimal places:

i the value of p for which $A(p) = 0.95$

ii the value of p for which $A(p) = 0.05$.

e **i** Find $A'(p)$, for $0 \leq p \leq 1$.

ii Sketch the graph of $A'(p)$ against p.

iii For what value of p is $A'(p)$ a minimum?

iv Describe what the result of **iii** means.

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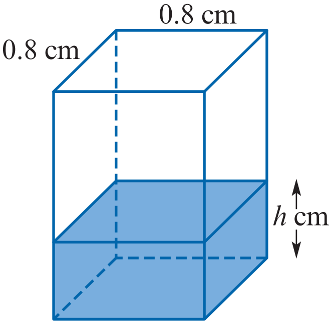
31 A liquid is contained in a tank which is a cuboid with square cross-section as shown in the diagram. The depth of liquid, h cm, in the tank at time t minutes is given by the function with the rule:

$$h(t) = (4.5 - 0.3t)^3$$



a State the depth of the liquid at time $t = 0$.

b State the practical domain for the function h.

c State the rule for the volume, V cm^3, of water in the tank at time t.

d Explain briefly why an inverse function h^{-1} exists and find its rule and domain.

e Draw graphs of both h and h^{-1} on the one set of axes.

32 A machine produces ball-bearings with a mean diameter of 3 mm. It is found that 6.3% of the production is being rejected as below the lower tolerance limit of 2.9 mm, and a further 6.3% is being rejected as above the upper tolerance limit of 3.1 mm. Assume that the diameters are normally distributed.

a Calculate the standard deviation of the distribution.

b A sample of eight ball-bearings is taken. Find the probability that:

i at least one is rejected **ii** two are rejected.

c The setting of the machine now 'wanders' such that the standard deviation remains the same, but the mean changes to 3.05 mm.

i Calculate the total percentage of the production that will now fall outside the given tolerance limits.

ii Find the value of c such that the probability that the diameter lies in the interval $(3.05 - c, 3.05 + c)$ is 0.9.

33 There is a probability of 0.8 that a boarding student will miss breakfast if he oversleeps. There is a probability of 0.3 that the student will miss breakfast even if he does not oversleep. The student has a probability of 0.4 of oversleeping.

a On a random day, what is the probability of:

i the student oversleeping and missing breakfast

ii the student not oversleeping and still missing breakfast

iii the student not missing breakfast?

b Given that the student misses breakfast, find the probability that he overslept.

c It is found that 10 students in the boarding house have identical probabilities for sleeping in and missing breakfast to the student mentioned above. Find the probability that:

i exactly two of the 10 students miss breakfast

ii at least one of the 10 students misses breakfast

iii at least eight of the students don't miss breakfast.

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34 **a** On the one set of axes, sketch the graphs of $y = \frac{1}{x}$ and $y = e^x$ for $x > 0$.

b Using addition of ordinates, sketch the graph of $y = \frac{1}{x} + e^x$ for $x > 0$.

(Do not attempt at this stage to find the coordinates of the turning points.)

c Find $\frac{dy}{dx}$ for $y = \frac{1}{x} + e^x$.

d **i** Show that $\frac{dy}{dx} = 0 \Leftrightarrow 2\log_e x = -x$, for $x > 0$.

ii Explain why this implies that the local minimum of $y = \frac{1}{x} + e^x$ lies in the interval $(0, 1)$.

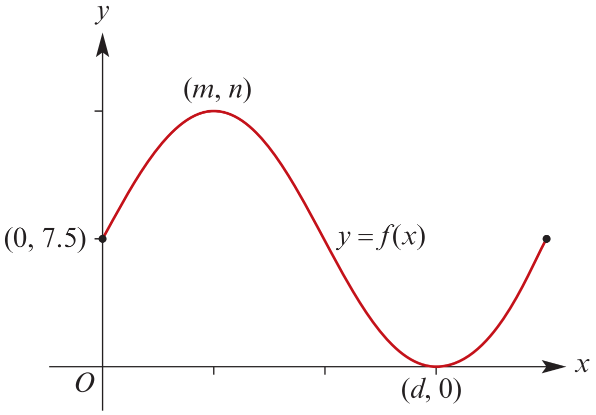
iii Using a calculator, show that the point of intersection of the graphs of $y = 2\log_e x$ and $y = -x$ is at $(0.70, -0.70)$, correct to two decimal places.

iv Hence find the coordinates of the local minimum of $y = \frac{1}{x} + e^x$, correct to one decimal place.

35 A section of a creek bank can be modelled by the function:

$$f\colon [0, 50] \to \mathbb{R},\ f(x) = a + b\sin\left(\frac{2\pi x}{50}\right)$$

where units are in metres.



a **i** Find the values of a, b, d, m and n.

ii The other bank of the creek can be modelled by the function $y = f(x) + 4$. Sketch the graph of this new function.

b Find the coordinates of the points on the first bank with y-coordinate 10.

c A particular river has a less severe bend than this creek. It is found that a section of the bank of the river can be modelled by the function:

$$g\colon [0, 250] \to \mathbb{R},\ g(x) = 2f\left(\frac{x}{5}\right)$$

Sketch the graph of this function; label the turning points with their coordinates.

d Over the years, the river bank moves. The shape of the bends are maintained, but there is a translation of 10 metres in the positive direction of the x-axis.

i Give the rule that describes this section of the river bank after the translation (relative to the original axes).

ii Sketch the graph of this new function.

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36 The continuous random variable X has probability density function f given by

$$f(x) = \begin{cases} k(5-2x) & \text{if } 2 < x \le \frac{5}{2} \\ 0 & \text{otherwise} \end{cases}$$

a Find the value of k.

b **i** Find $\mathrm{E}(X)$.

  ii Find the median of X.

  iii Find σ, the standard deviation of X, correct to two decimal places.

c Find $\Pr(X < \mu - \sigma)$, where $\mu = \mathrm{E}(X)$.

37 The lifetime, X days, of a particular type of computer component has a probability density function given by

$$f(x) = \begin{cases} k(a-x) & \text{if } 0 < x \le a \\ 0 & \text{if } x \le 0 \text{ or } x > a \end{cases}$$

where k and a are positive constants.

a Find k in terms of a.

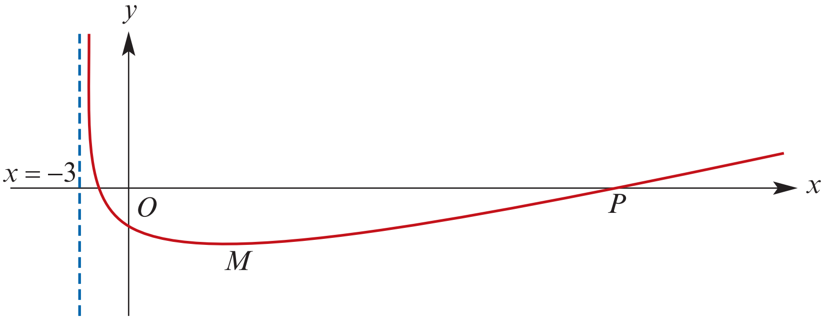
b Find the mean, μ, and the variance, σ^2, of X in terms of a.

c Find $\Pr(X > \mu + 2\sigma)$.

d Find the value of a if the median lifetime is 1000 days.

38 The diagram shows a sketch graph of

$$y = \frac{x}{10} - \log_e(x+3), \quad x > -3$$



a Find the x-coordinate of the local minimum at M.

b Show that the gradient of the curve is always less than $\frac{1}{10}$.

c Find the equation of the straight line through M with a gradient of $\frac{1}{10}$.

d **i** Hence show that the value of the x-axis intercept at P is greater than $10\log_e 10$.

  ii Find, correct to three decimal places, the value of the intercept at P.

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39 A particle is moving along a path with equation $y = \sqrt{x^2 + 24}$.

- **a** Find $\dfrac{dy}{dx}$.
- **b** Find the coordinates of the local minimum of the curve.
- **c** Does this rule define an even function?
- **d** As $x \to \infty$, $y \to x$ and as $x \to -\infty$, $y \to -x$. Sketch the graph of $y = \sqrt{x^2 + 24}$, showing the asymptotes.
- **e** Find the equation of the normal to the curve at the point with coordinates $(1, 5)$, and sketch the graph of this normal with the graph of **d**.
- **f** When the particle is at the point with coordinates $(5, 7)$, its y-coordinate is increasing at a rate of 10 units per second. At what rate is its x-coordinate increasing?
- **g** Show that

$$\frac{d}{dx}\left(12\log_e\left(\sqrt{x^2+24}+x\right)+\frac{x\sqrt{x^2+24}}{2}\right)=\sqrt{x^2+24}$$

  for $x > 0$.
- **h** Use this result to find the area of the region bounded by the curve, the x-axis and the lines $x = 2$ and $x = 5$.

40 The boxplot is a display used to describe the distribution of a data set. Located on the boxplot are the minimum, the lower quartile, the median, the upper quartile and the maximum. Boxplots also show outliers. These are values which are more than 1.5 interquartile ranges below the lower quartile or above the upper quartile.

- **a** Suppose that a random variable Z is normally distributed with a mean of 0 and a standard deviation of 1.
  - **i** Find the value of the median, i.e. find m such that $\Pr(Z \le m) = 0.5$.
  - **ii** Find the value of the lower quartile, i.e. find q_1 such that $\Pr(Z \le q_1) = 0.25$.
  - **iii** Find the value of the upper quartile, i.e. find q_3 such that $\Pr(Z \le q_3) = 0.75$.
  - **iv** Hence find the interquartile range (IQR) for this distribution.
  - **v** Find $\Pr(q_1 - 1.5 \times \text{IQR} < Z < q_3 + 1.5 \times \text{IQR})$.
  - **vi** What percentage of data values would you expect to be designated as outliers for this distribution?
- **b** Suppose that a random variable X is normally distributed with a mean of μ and a standard deviation of σ.
  - **i** Find the value of the median, i.e. find m such the $\Pr(X \le m) = 0.5$.
  - **ii** Find the value of the lower quartile, i.e. find q_1 such that $\Pr(X \le q_1) = 0.25$.
  - **iii** Find the value of the upper quartile, i.e. find q_3 such that $\Pr(X \le q_3) = 0.75$.
  - **iv** Hence find the interquartile range (IQR) for this distribution.
  - **v** Find $\Pr(q_1 - 1.5 \times \text{IQR} < X < q_3 + 1.5 \times \text{IQR})$.
  - **vi** What percentage of data values would you expect to be designated as outliers for this distribution?

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41 The random variable X has probability density function given by

$$f(x) = \begin{cases} kx^n & \text{if } 0 \le x \le 1 \\ 0 & \text{if } x < 0 \text{ or } x > 1 \end{cases}$$

where n and k are constants with $n > 0$. Find in terms of n:

a k

b $\mathrm{E}(X)$

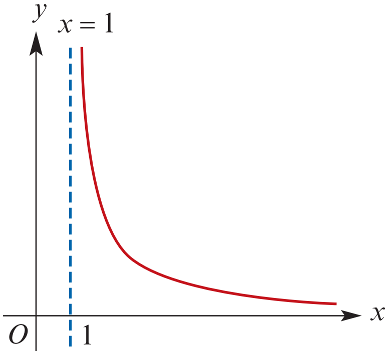
c $\mathrm{Var}(X)$

d the median of X

42 The diagram shows the graph of the function

$$g\colon (1, \infty) \to \mathbb{R}, \quad g(x) = \frac{1}{x-1}$$

The line segment AB is drawn from the point $A(2, 1)$ to the point $B(b, g(b))$, where $b > 2$.



a **i** What is the gradient of AB?

ii At what value of x between 1 and b does the tangent to the graph of g have the same gradient as AB?

b **i** Calculate $\int_2^{e+1} g(x)\,dx$.

ii Let c be a real number with $1 < c < 2$. Find the exact value of c such that $\int_c^{e+1} g(x)\,dx = 8$.

c **i** What is the area of the trapezium bounded by the line segment AB, the x-axis and the lines $x = 2$ and $x = b$?

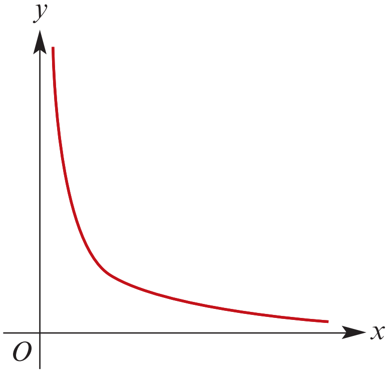
ii For what exact value of b does this area equal 8?

d Given that $\int_2^{mn+1} g(x)\,dx + \int_2^{\frac{m}{n}+1} g(x)\,dx = 2$, where $n > 0$, find the value of m.

43 The diagram shows the graph of the function

$$f\colon \mathbb{R}^+ \to \mathbb{R}, \quad f(x) = \frac{1}{x^2}$$

The line segment AB is drawn from the point $A(1, 1)$ to the point $B(b, f(b))$, where $b > 1$.



a **i** What is the gradient of AB?

ii At what value of x between 1 and b does the tangent to the graph of f have the same gradient as AB?

b **i** What is the area, $S(b)$, of the trapezium bounded by the line segment AB, the x-axis and the lines $x = 1$ and $x = b$?

ii For what exact value of b does this area equal $\frac{10}{9}$?

iii Show that $\int_1^b f(x)\,dx < 1$ for $b > 1$.

c Show that the function $D(b) = S(b) - \int_1^b f(x)\,dx$ is strictly increasing for $b > 1$.

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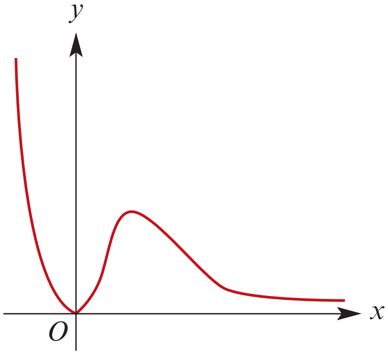
44 Define the function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^m e^{-nx+n}$, where m and n are positive integers. The graph of $y = f(x)$ is as shown.



a Find the coordinates of the stationary point not at the origin in terms of n, and state its nature.

b Find the coordinates of the point on the graph at which the tangent of f passes through the origin.

c Consider the continuous probability density function with rule

$$f(x) = \begin{cases} kx^2 e^{-2x+2} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where k is a positive real number.

i Find the value of k.

ii Find $\Pr(X < 1)$, where X is the associated random variable.

45 Let X be a continuous random variable with probability density function given by

$$f(x) = \begin{cases} ke^{-qx} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where q is a positive real number.

a **i** Find the value of k in terms of q.

ii Find $\mathrm{E}(X)$ in terms of q.

iii Find $\mathrm{Var}(X)$ in terms of q.

iv Show that the median of the distribution is $m = \frac{1}{q} \log_e 2$.

b Find $\Pr\left(X > \frac{1}{q} \log_e 3 \;\middle|\; X > \frac{1}{q} \log_e 2\right)$.

c The distance, X metres, between flaws in a certain type of yarn is a continuous random variable with probability density function $f(x) = 0.01e^{-0.01x}$ for $x \geq 0$.

i Sketch the graph of $y = f(x)$.

ii Find the probability, correct to two decimal places, that the distance between consecutive flaws is more than 100 m.

iii Find the median value of this distribution, correct to two decimal places.

46 A coin is tossed 1000 times, and 527 heads observed.

a Give a point estimate for p, the probability of observing a head when the coin is tossed.

b Determine an approximate 95% confidence interval for p.

c What level of confidence would be given by a confidence interval for p which is half the width of the approximate 95% confidence interval?

d What level of confidence would be given by a confidence interval for p which is twice the width of the approximate 95% confidence interval?

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Counting methods and the binomial theorem

A.1 Counting methods

▶ The addition rule

In general, to choose between alternatives simply add up the number of choices available for each alternative.

Example 1

At the library Alan is having trouble deciding which book to borrow. He has a choice of three mystery novels, three biographies or two science fiction books. How many choices of book does he have?

Solution

As he is choosing between alternatives (mystery novels *or* biographies *or* science fiction), he has a total of $3 + 3 + 2 = 8$ choices.

▶ The multiplication rule

When sequential choices are involved, the total number of possibilities is found by multiplying the number of options at each successive stage.

Example 2

Sandi has six choices of windcheaters or jackets, and seven choices of jeans or skirts. How many choices does she have for a complete outfit?

Solution

As Sandi will wear either a windcheater or a jacket *and* jeans or a skirt, we cannot consider these to be alternative choices. We could draw a tree diagram to list the possibilities, but this would be arduous. Using the multiplication rule, however, we can quickly determine the number of choices to be $6 \times 7 = 42$.

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Permutations or arrangements

The number of arrangements of n objects in groups of size r is denoted ${}^nP_r$ and given by

$$\begin{aligned}{}^nP_r &= \frac{n!}{(n-r)!}\\ &= n \times (n-1) \times (n-2) \times \cdots \times (n-r+1)\end{aligned}$$

Example 3

How many different four-digit numbers can be formed from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 if each digit may be used only once?

Solution

The number of arrangements of 9 digits in groups of size 4 is

$${}^9P_4 = \frac{9!}{5!} = 9 \times 8 \times 7 \times 6 = 3024$$

Combinations or selections

The number of combinations of n objects in groups of size r is

$${}^nC_r = \frac{{}^nP_r}{r!} = \frac{n!}{r!\,(n-r)!}$$

A commonly used alternative notation for ${}^nC_r$ is $\binom{n}{r}$.

Example 4

Four flavours of ice-cream – vanilla, chocolate, strawberry and caramel – are available at the school canteen. How many different double-scoop selections are possible if two different flavours must be used?

Solution

The number of combinations of 4 flavours in groups of size 2 is

$${}^4C_2 = \frac{4!}{2!\,2!} = \frac{4 \times 3 \times 2 \times 1}{2 \times 1 \times 2 \times 1} = 6$$

Example 5

A team of three boys and three girls is to be chosen from a group of eight boys and five girls. How many different teams are possible?

Solution

We can choose three boys from eight in ${}^8C_3$ ways *and* three girls from five in ${}^5C_3$ ways. Thus the total number of possible teams is

$${}^8C_3 \times {}^5C_3 = 56 \times 10 = 560$$

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Exercise A1

1 A student needs to select a two-unit study for her course: one unit in each semester. In Semester 1 she has a choice of two mathematics units, three language units and four science units. In Semester 2 she has a choice of two history units, three geography units and two art units. How many choices does she have for her two-unit study?

2 In order to travel from Melbourne to Brisbane, Dominic is given the following choices. He can fly directly from Melbourne to Brisbane on one of three airlines, or he can fly from Melbourne to Sydney on one of four airlines and then travel from Sydney to Brisbane with one of five bus lines, or he can go on one of three bus lines directly from Melbourne to Brisbane. In how many ways could he travel from Melbourne to Brisbane?

3 If there are eight swimmers in the final of the 1500 m freestyle event, in how many ways can the first three places be filled?

4 In how many ways can the letters of the word TROUBLE be arranged:

a if they are all used **b** in groups of three?

5 In how many ways can the letters of the word PANIC be arranged:

a if they are all used **b** in groups of four?

6 A student has the choice of three mathematics subjects and four science subjects. In how many ways can she choose to study one mathematics and two science subjects?

7 A survey is to be conducted, and eight people are to be chosen from a group of 30.

a How many different groups of eight people could be chosen?

b If the group contains 10 men and 20 women, how many groups of eight people containing exactly two men are possible?

8 From a standard 52-card deck, how many seven-card hands have exactly five spades and two hearts?

9 In how many ways can a committee of five be selected from eight women and four men:

a without restriction **b** if there must be exactly three women on the committee?

10 Six females and five males are interviewed for five positions. If all are found to be acceptable for any position, in how many ways could the following combinations be selected?

a three females and two males **b** four females and one male

c five females **d** five people regardless of sex

e at least four females

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A.2 Summation notation

Suppose that m and n are integers with $m < n$. Then

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_n$$

This notation, which is called **summation notation** or **sigma notation**, is very convenient for concisely representing sums. These sums will arise throughout the course. The notation uses the symbol Σ, which is the uppercase Greek letter *sigma*.

The notation

$$\sum_{i=m}^{n} a_i$$

is read 'the sum of the numbers a_i from i equals m to i equals n'.

The expression $a_m + a_{m+1} + a_{m+2} + \cdots + a_n$ is called the **expanded form** of $\sum\limits_{i=m}^{n} a_i$.

Example 6

Write $\sum\limits_{i=1}^{5} 2^i$ in expanded form and evaluate.

Solution

$$\begin{aligned}\sum_{i=1}^{5} 2^i &= 2^1 + 2^2 + 2^3 + 2^4 + 2^5 \\ &= 2 + 4 + 8 + 16 + 32 \\ &= 62\end{aligned}$$

Example 7

Write $1^2 + 2^2 + 3^2 + \cdots + 30^2$ using summation notation.

Solution

$$1^2 + 2^2 + 3^2 + \cdots + 30^2 = \sum_{k=1}^{30} k^2$$

Example 8

Write $x_1 + x_2 + x_3 + \cdots + x_{10}$ using summation notation.

Solution

$$x_1 + x_2 + x_3 + \cdots + x_{10} = \sum_{i=1}^{10} x_i$$

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Exercise A2

1 Write each of the following in expanded form and evaluate:

a $\sum_{i=1}^{4} i^3$ **b** $\sum_{k=1}^{5} k^3$ **c** $\sum_{i=1}^{5} (-1)^i\, i$ **d** $\frac{1}{5}\sum_{i=1}^{5} i$

e $\sum_{i=1}^{6} i$ **f** $\sum_{k=1}^{4} (k-1)^2$ **g** $\frac{1}{3}\sum_{i=1}^{4} (i-2)^2$ **h** $\sum_{i=1}^{6} i^2$

2 Write each of the following using summation notation:

a $1 + 2 + 3 + \cdots + n$ **b** $x_1 + x_2 + x_3 + \cdots + x_{11}$

c $\dfrac{x_1 + x_2 + x_3 + \cdots + x_{10}}{10}$ **d** $1^4 + 2^4 + 3^4 + \cdots + n^4 + (n+1)^4$

e $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$

3 Write each of the following in expanded form:

a $\sum_{i=1}^{n} x^i$ **b** $\sum_{i=0}^{5} x^i \cdot 2^{5-i}$ **c** $\sum_{i=0}^{6} (2x)^i \cdot 3^{6-i}$ **d** $\sum_{i=0}^{4} (x - x_i)^i$

4 Write each of the following using summation notation:

a $x^5 + 3x^4 + 9x^3 + 27x^2 + 81x + 243$ **b** $x^5 - 3x^4 + 9x^3 - 27x^2 + 81x - 243$

c $4x^2 + 2x + 1$ **d** $8x^3 + 12x^2 + 18x + 27$

A.3 The binomial theorem

Consider the expansions of binomial powers shown below:

$$\begin{aligned}
(x+b)^0 &= 1\\
(x+b)^1 &= 1x + 1b\\
(x+b)^2 &= 1x^2 + 2xb + 1b^2\\
(x+b)^3 &= 1x^3 + 3x^2b + 3xb^2 + 1b^3\\
(x+b)^4 &= 1x^4 + 4x^3b + 6x^2b^2 + 4xb^3 + 1b^4\\
(x+b)^5 &= 1x^5 + 5x^4b + 10x^3b^2 + 10x^2b^3 + 5xb^4 + 1b^5
\end{aligned}$$

The coefficients can be arranged in a triangle:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

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This array is known as **Pascal's triangle**, and can also be constructed from combinations:

Row 0: $\binom{0}{0}$

Row 1: $\binom{1}{0} \quad \binom{1}{1}$

Row 2: $\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$

Row 3: $\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$

Row 4: $\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$

Row 5: $\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$

Remember that $\binom{n}{r} = \dfrac{n!}{r!\,(n-r)!}$.

The expansion of $(x+b)^6$ can be written by using this observation:

$$(x+b)^6 = \binom{6}{0}x^6 + \binom{6}{1}x^5b + \binom{6}{2}x^4b^2 + \binom{6}{3}x^3b^3 + \binom{6}{4}x^2b^4 + \binom{6}{5}xb^5 + \binom{6}{6}b^6$$

In summation notation:

$$(x+b)^6 = \sum_{k=0}^{6}\binom{6}{k}x^{6-k}b^k$$

In general:

$$(x+b)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}b^k \qquad \text{and} \qquad (ax+b)^n = \sum_{k=0}^{n}\binom{n}{k}(ax)^{n-k}b^k$$

The first term of the expansion of $(ax+b)^n$ is $\binom{n}{0}(ax)^n$ and the second term is $\binom{n}{1}(ax)^{n-1}b$.

In general, the $(r+1)$st term is $\binom{n}{r}(ax)^{n-r}b^r$.

By convention, the expansion of $(ax+b)^n$ is written with decreasing powers of x.

Example 9

Expand $(2x+3)^5$.

Solution

$$\begin{aligned}(2x+3)^5 &= \sum_{k=0}^{5}\binom{5}{k}(2x)^{5-k}\,3^k \\ &= (2x)^5 + \binom{5}{1}(2x)^4\cdot 3 + \binom{5}{2}(2x)^3\cdot 3^2 + \binom{5}{3}(2x)^2\cdot 3^3 + \binom{5}{4}(2x)\cdot 3^4 + 3^5 \\ &= 32x^5 + 240x^4 + 720x^3 + 1080x^2 + 810x + 243\end{aligned}$$

Example 10

Find the eighth term in the expansion of $(2x-4)^{10}$.

Solution

The $(r+1)$st term is $\binom{10}{r}(2x)^{10-r}(-4)^r$.

Therefore the 8th term is

$$\binom{10}{7}(2x)^3(-4)^7 = -15\,728\,640x^3$$

Example 11

Find the coefficient of x^{20} in the expansion of $(x+2)^{30}$.

Solution

The $(r+1)$st term is $\binom{30}{r}x^{30-r}\,2^r$.

When $30-r=20$, $r=10$. Therefore the term with x^{20} is $\binom{30}{10}2^{10}x^{20}$.

Hence the coefficient of x^{20} is $\binom{30}{10}2^{10}$.

Exercise A3

1 Expand each of the following using the binomial theorem:

- **a** $(x+6)^6$
- **b** $(2x+1)^5$
- **c** $(2x-1)^5$
- **d** $(2x+3)^6$
- **e** $(2x-6)^6$
- **f** $(2x-3)^4$
- **g** $(x-2)^6$
- **h** $(x+1)^{10}$

2 Find the eighth term of each expansion (where descending powers of x are assumed):

- **a** $(2x-1)^{10}$
- **b** $(2x+1)^{10}$
- **c** $(1-2x)^{10}$
- **d** $(3x+1)^{12}$
- **e** $(x+3)^{12}$
- **f** $(2x-b)^{12}$

3 Find the third term in the expansion of $\left(2-\frac{1}{3}x\right)^9$, assuming descending powers of x.

4 Find the sixth term in the expansion of $(3x-1)^{11}$, assuming descending powers of x.

5 Expand $(1-x)^{11}$.

6 Find the coefficient of x^3 in the expansion of each of the following:

- **a** $(x+2)^5$
- **b** $(2x-1)^6$
- **c** $(1-2x)^5$
- **d** $(4x-3)^7$
- **e** $(3x+4)^4$
- **f** $(3x-2)^5$

7 Find the coefficient of x^{10} in the expansion of $(2x-3)^{14}$.

8 Find the coefficient of x^5 in the expansion of $(4-2x)^6$.

Glossary

A

Absolute maximum and minimum [p. 442]
For a continuous function f defined on an interval $[a, b]$:

- the *absolute maximum* is the value M of the function f such that $f(x) \leq M$ for all $x \in [a, b]$
- the *absolute minimum* is the value N of the function f such that $f(x) \geq N$ for all $x \in [a, b]$.

Absolute value function [p. 490]

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Acceleration [p. 424] the rate of change of a particle's velocity with respect to time

Acceleration, average [MM1&2] The average acceleration of a particle for the time interval $[t_1, t_2]$ is given by $\dfrac{v_2 - v_1}{t_2 - t_1}$, where v_2 is the velocity at time t_2 and v_1 is the velocity at time t_1.

Acceleration, instantaneous [MM1&2] $a = \dfrac{dv}{dt}$

Addition rule for choices [p. 759] To determine the total number of choices from disjoint alternatives, simply add up the number of choices available for each alternative.

Addition rule for probability [p. 553] The probability of A or B or both occurring is given by

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

Amplitude of circular functions [p. 266]
The distance between the mean position and the maximum position is called the amplitude.
The graph of $y = \sin x$ has an amplitude of 1.

Antiderivative [p. 484] To find the general antiderivative of $f(x)$: If $F'(x) = f(x)$, then

$$\int f(x)\,dx = F(x) + c$$

where c is an arbitrary real number.

Arrangements [p. 760] counted when order is important. The number of ways of selecting and arranging r objects from a total of n objects is

$$\frac{n!}{(n-r)!} = n \times (n-1) \times (n-2) \times \cdots \times (n-r+1)$$

Average value [p. 516] The average value of a continuous function f for an interval $[a, b]$ is defined as $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$.

B

Bernoulli sequence [p. 597] a sequence of repeated trials with the following properties:

- Each trial results in one of two outcomes, usually designated as a success or a failure.
- The probability of success on a single trial, p, is constant for all trials.
- The trials are independent. (The outcome of a trial is not affected by outcomes of other trials.)

Note: The glossary contains some terms which were introduced in Mathematical Methods Units 1 & 2, but which are not explicitly mentioned in the Mathematical Methods Units 3 & 4 study design. The reference for these is given as [MM1&2].

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Binomial distribution [p. 598] The probability of observing x successes in n independent trials, each with probability of success p, is given by

$$\Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \dots, n$$

where $\displaystyle\binom{n}{x} = \frac{n!}{x!\,(n-x)!}$

Binomial expansion [p. 764]

$$(x+a)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} a^k$$

$$= x^n + \binom{n}{1} x^{n-1} a + \binom{n}{2} x^{n-2} a^2 + \dots + a^n$$

The $(r+1)$st term is $\displaystyle\binom{n}{r} x^{n-r} a^r$.

Binomial experiment [p. 598] a Bernoulli sequence of n independent trials, each with probability of success p

C

Chain rule [p. 375] The chain rule can be used to differentiate a complicated function $y = f(x)$ by transforming it into two simpler functions, which are 'chained' together:

$$x \xrightarrow{h} u \xrightarrow{g} y$$

Using Leibniz notation, the chain rule is stated as

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Change of base [p. 231] $\log_a b = \dfrac{\log_c b}{\log_c a}$

Circle, general equation [p. 7] The general equation for a circle is $(x-h)^2 + (y-k)^2 = r^2$, where the centre is (h, k) and the radius is r.

Complement, A' [p. 553] the set of outcomes that are in the sample space, ε, but not in A. The probability of the event A' is $\Pr(A') = 1 - \Pr(A)$.

Composite function [p. 28] For functions f and g such that $\operatorname{ran} f \subseteq \operatorname{dom} g$, the composite function of g with f is defined by $g \circ f(x) = g(f(x))$, where $\operatorname{dom}(g \circ f) = \operatorname{dom} f$.

Conditional probability [p. 561] the probability of an event A occurring when it is known that some event B has occurred, given by

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

Confidence interval [p. 708] an interval estimate for the population proportion p based on the value of the sample proportion $\hat{p}$

Constant function [MM1&2] a function $f\colon \mathbb{R} \to \mathbb{R}, f(x) = a$

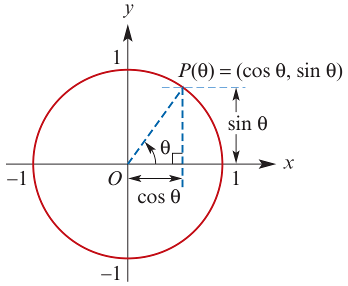
Continuous function [p. 404] A function f is continuous at the point $x = a$ if $f(x)$ is defined at $x = a$ and $\lim_{x \to a} f(x) = f(a)$.

Continuous random variable [p. 620] a random variable X that can take any value in an interval of the real number line

Coordinates [MM1&2] an ordered pair of numbers that identifies a point in the Cartesian plane; the first number identifies the position with respect to the x-axis, and the second number identifies the position with respect to the y-axis

Cosine and sine functions [p. 257]

- cosine θ is defined as the x-coordinate of the point P on the unit circle where OP forms an angle of θ radians with the positive direction of the x-axis
- sine θ is defined as the y-coordinate of the point P on the unit circle where OP forms an angle of θ radians with the positive direction of the x-axis



Cubic function [p. 182] a polynomial of degree 3. A cubic function f has a rule of the form $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$.

Cumulative distribution function [p. 645] gives the probability that the random variable X takes a value less than or equal to x; that is,

$$F(x) = \Pr(X \leq x) = \int_{-\infty}^{x} f(t)\, dt$$

D

Definite integral [pp. 480, 496] $\int_a^b f(x)\,dx$ denotes the signed area enclosed by the graph of $y = f(x)$ between $x = a$ and $x = b$.

Degree of a polynomial [p. 166] given by the highest power of x with a non-zero coefficient; e.g. the polynomial $2x^5 - 7x^2 + 4$ has degree 5.

Dependent trials [MM1&2] *see* sampling without replacement

Derivative function [p. 354] also called the gradient function. The derivative f' of a function f is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

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Derivatives, basic [pp. 379–391]

$f(x)$	$f'(x)$	
c	0	where c is a constant
x^a	ax^{a-1}	where $a \in \mathbb{R} \setminus \{0\}$
e^{kx}	ke^{kx}	
$\log_e(kx)$	$\frac{1}{x}$	
$\sin(kx)$	$k\cos(kx)$	
$\cos(kx)$	$-k\sin(kx)$	
$\tan(kx)$	$k\sec^2(kx)$	

Determinant of a 2×2 matrix [p. 139] If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det(\mathbf{A}) = ad - bc$.

Difference of two cubes [p. 176]
$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

Difference of two squares [MM1&2]
$x^2 - y^2 = (x - y)(x + y)$

Differentiable [p. 407] A function f is said to be differentiable at the point $x = a$ if $\displaystyle\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists.

Differentiation rules [p. 357]
- Sum: $f(x) = g(x) + h(x)$, $f'(x) = g'(x) + h'(x)$
- Multiple: $f(x) = k\,g(x)$, $f'(x) = k\,g'(x)$

see also chain rule, product rule, quotient rule

Dilation from the x-axis [p. 104] A dilation of factor b from the x-axis is described by the rule $(x, y) \to (x, by)$. The curve with equation $y = f(x)$ is mapped to the curve with equation $y = bf(x)$.

Dilation from the y-axis [p. 105] A dilation of factor a from the y-axis is described by the rule $(x, y) \to (ax, y)$. The curve with equation $y = f(x)$ is mapped to the curve with equation $y = f\left(\frac{x}{a}\right)$.

Dimension of a matrix [p. 76] the size of a matrix. A matrix with m rows and n columns is said to be an $m \times n$ matrix.

Discontinuity [p. 404] A function is said to be discontinuous at a point if it is not continuous at that point.

Discrete random variable [p. 569] a random variable X which can take only a countable number of values, usually whole numbers

Discriminant, Δ, of a quadratic [p. 157] the expression $b^2 - 4ac$, which is part of the quadratic formula. For the quadratic equation $ax^2 + bx + c = 0$:
- If $b^2 - 4ac > 0$, there are two solutions.
- If $b^2 - 4ac = 0$, there is one solution.
- If $b^2 - 4ac < 0$, there are no real solutions.

Disjoint [p. 2] Two sets A and B are disjoint if they have no elements in common, i.e. $A \cap B = \emptyset$.

Distance between two points [p. 70] The distance between points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Domain [p. 6] the set of all the first coordinates of the ordered pairs in a relation

E

Element [p. 2] a member of a set.
- If x is an element of a set A, we write $x \in A$.
- If x is *not* an element of a set A, we write $x \notin A$.

Empty set, $\emptyset$ [p. 2] the set that has no elements

Equating coefficients [p. 168] Two polynomials P and Q are equal only if their corresponding coefficients are equal. For example, two cubic polynomials $P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ and $Q(x) = b_3x^3 + b_2x^2 + b_1x + b_0$ are equal if and only if $a_3 = b_3$, $a_2 = b_2$, $a_1 = b_1$ and $a_0 = b_0$.

Euler's number, e [p. 216] the natural base for exponential and logarithmic functions:

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = 2.718\,281\ldots$$

Even function [p. 19] A function f is even if $f(-x) = f(x)$ for all x in the domain of f; the graph is symmetric about the y-axis.

Event [p. 551] a subset of the sample space (that is, a set of outcomes)

Expected value of a random variable, $\mathrm{E}(X)$ [pp. 577, 631] also called the mean, μ.
For a discrete random variable X:

$$\mathrm{E}(X) = \sum_x x \cdot \Pr(X = x) = \sum_x x \cdot p(x)$$

For a continuous random variable X:

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

Exponential function [p. 210] a function $f(x) = ka^x$, where k is a non-zero constant and the base a is a positive real number other than 1

F

Factor [MM1&2] a number or expression that divides another number or expression without remainder

Factor theorem [p. 174] If $\beta x + \alpha$ is a factor of $P(x)$, then $P\left(-\frac{\alpha}{\beta}\right) = 0$. Conversely, if $P\left(-\frac{\alpha}{\beta}\right) = 0$, then $\beta x + \alpha$ is a factor of $P(x)$.

Factorise [MM1&2] express as a product of factors

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Formula [MM1&2] an equation containing symbols that states a relationship between two or more quantities; e.g. $A = \ell w$ (area = length × width). The value of A, the subject of the formula, can be found by substituting given values of ℓ and w.

Function [p. 8] a relation such that for each x-value there is only one corresponding y-value. This means that, if (a, b) and (a, c) are ordered pairs of a function, then $b = c$.

Function, many-to-one [p. 17] a function that is not one-to-one

Function, one-to-one [p. 15] different x-values map to different y-values. For example, the function $y = x + 1$ is one-to-one. But $y = x^2$ is not one-to-one, as both 2 and −2 map to 4.

Fundamental theorem of calculus [pp. 496, 523] If f is a continuous function on an interval $[a, b]$, then

$$\int_a^b f(x)\,dx = G(b) - G(a)$$

where G is any antiderivative of f and $\int_a^b f(x)\,dx$ is the definite integral from a to b.

G

Gradient function *see* derivative function

Gradient of a line [p. 70] The gradient is

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

where (x_1, y_1) and (x_2, y_2) are the coordinates of two points on the line. The gradient of a vertical line (parallel to the y-axis) is undefined.

H

Horizontal-line test [p. 16] If a horizontal line can be drawn anywhere on the graph of a function and it only ever intersects the graph a maximum of once, then the function is *one-to-one*.

Hybrid function *see* piecewise-defined function

I

Implied domain *see* maximal domain

Indefinite integral *see* antiderivative

Independence [p. 564] Two events A and B are independent if and only if

$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$

Independent trials *see* sampling with replacement

Index laws [p. 220]

- To multiply two powers with the same base, add the indices: $a^x \times a^y = a^{x+y}$
- To divide two powers with the same base, subtract the indices: $a^x \div a^y = a^{x-y}$
- To raise a power to another power, multiply the indices: $(a^x)^y = a^{x \times y}$
- Rational indices: $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$
- For base $a \in \mathbb{R}^+ \setminus \{1\}$, if $a^x = a^y$, then $x = y$.

Inequality [MM1&2] a mathematical statement that contains an inequality symbol rather than an equals sign; e.g. $2x + 1 < 4$

Integers [p. 3] $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

Integrals, basic [pp. 488–494, 503]

$f(x)$	$\int f(x)\,dx$	
x^r	$\dfrac{x^{r+1}}{r+1} + c$	where $r \in \mathbb{Q} \setminus \{-1\}$
$\dfrac{1}{ax+b}$	$\dfrac{1}{a}\log_e(ax+b) + c$	for $ax + b > 0$
e^{kx}	$\dfrac{1}{k}e^{kx} + c$	
$\sin(kx)$	$-\dfrac{1}{k}\cos(kx) + c$	
$\cos(kx)$	$\dfrac{1}{k}\sin(kx) + c$	

Integration, properties [p. 486]

- $\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$
- $\int kf(x)\,dx = k\int f(x)\,dx$

Integration (definite), properties [p. 498]

- $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
- $\int_a^a f(x)\,dx = 0$
- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Intersection of sets [pp. 2, 551] The intersection of two sets A and B, written $A \cap B$, is the set of all elements common to A and B.

Interval [p. 4] a subset of the real numbers of the form $[a, b]$, $[a, b)$, (a, ∞), etc.

Inverse function [p. 32] For a one-to-one function f, the inverse function f^{-1} is defined by $f^{-1}(x) = y$ if $f(y) = x$, for $x \in \operatorname{ran} f$, $y \in \operatorname{dom} f$.

Irrational number [p. 3] a real number that is not rational; e.g. π and $\sqrt{2}$

K

Karnaugh map [p. 556] a probability table

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L

Law of total probability [p. 562] In the case of two events, A and B:
$\Pr(A) = \Pr(A \mid B)\Pr(B) + \Pr(A \mid B')\Pr(B')$

Limit [p. 400] The notation $\lim_{x \to a} f(x) = p$ says that the limit of $f(x)$, as x approaches a, is p. We can also say: 'As x approaches a, $f(x)$ approaches p.'

Limits, properties [p. 401]

- Sum: $\lim_{x\to a}(f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
- Multiple: $\lim_{x\to a}(kf(x)) = k\lim_{x\to a} f(x)$
- Product: $\lim_{x\to a}(f(x)\,g(x)) = \lim_{x\to a} f(x)\ \lim_{x\to a} g(x)$
- Quotient: $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, if $\lim_{x\to a} g(x) \neq 0$

Linear equation [p. 64] a polynomial equation of degree 1; e.g. $2x + 1 = 0$

Linear function [p. 74] a function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = mx + c$; e.g. $f(x) = 3x + 1$

Literal equation [MM1&2] an equation for the variable x in which the coefficients of x, including the constants, are pronumerals; e.g. $ax + b = c$

Logarithm [p. 222] If $a \in \mathbb{R}^+ \setminus \{1\}$ and $x \in \mathbb{R}$, then the statements $a^x = y$ and $\log_a y = x$ are equivalent.

Logarithm laws [p. 224]

- $\log_a(xy) = \log_a x + \log_a y$
- $\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$
- $\log_a\left(\dfrac{1}{x}\right) = -\log_a x$
- $\log_a(x^p) = p\log_a x$

M

Margin of error, *M* [p. 712] the distance between the sample estimate and the endpoints of the confidence interval

Matrices, addition [p. 77] Addition is defined for two matrices of the same dimension. The sum is found by adding corresponding entries. For example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 4 & 3 \end{bmatrix}$$

Matrices, equal [p. 77] Two matrices $\mathbf{A}$ and $\mathbf{B}$ are equal, and we can write $\mathbf{A} = \mathbf{B}$, when:

- they have the same number of rows and the same number of columns, and
- they have the same number or entry at corresponding positions.

Matrices, multiplication [p. 80] The product of two matrices $\mathbf{A}$ and $\mathbf{B}$ is only defined if the number of columns of $\mathbf{A}$ is the same as the number of rows of $\mathbf{B}$. If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is an $n \times r$ matrix, then the product $\mathbf{AB}$ is the $m \times r$ matrix whose entries are determined as follows:

> To find the entry in row i and column j of $\mathbf{AB}$, single out row i in matrix $\mathbf{A}$ and column j in matrix $\mathbf{B}$. Multiply the corresponding entries from the row and column and then add up the resulting products.

Matrix, identity [p. 81] For square matrices of a given dimension (e.g. 2×2), a multiplicative identity $\mathbf{I}$ exists.

For 2×2 matrices, the identity is $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
and $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$ for each 2×2 matrix $\mathbf{A}$.

Matrix, inverse [p. 139] If $\mathbf{A}$ is a square matrix and there exists a matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$, then $\mathbf{B}$ is called the inverse of $\mathbf{A}$. The inverse of a square matrix $\mathbf{A}$ is denoted by $\mathbf{A}^{-1}$. The inverse is unique. It does not exist for every square matrix.

Matrix, multiplication by a scalar [p. 77] If $\mathbf{A}$ is an $m \times n$ matrix and k is a real number, then $k\mathbf{A}$ is an $m \times n$ matrix whose entries are k times the corresponding entries of $\mathbf{A}$. For example:

$$3\begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -6 \\ 0 & 3 \end{bmatrix}$$

Matrix, regular [p. 139] A square matrix is said to be regular if its inverse exists.

Matrix, singular [p. 139] A square matrix is said to be singular if it does not have an inverse.

Matrix, square [p. 81] a matrix with the same number of rows and columns; e.g. 2×2

Matrix, zero [p. 77] The $m \times n$ matrix with all entries equal to zero is called the zero matrix.

Maximal domain [p. 17] When the rule for a relation is given and no domain is specified, then the domain taken is the largest for which the rule has meaning.

Mean of a random variable, μ [pp. 577, 631] *see* expected value of a random variable, $\mathrm{E}(X)$

Median of a random variable, *m* [p. 634] the middle value of the distribution. For a continuous random variable, the median is the value m such that $\int_{-\infty}^{m} f(x)\,dx = 0.5$.

Midpoint of a line segment [p. 70] If $P(x, y)$ is the midpoint of the line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$, then

$$x = \frac{x_1 + x_2}{2} \quad \text{and} \quad y = \frac{y_1 + y_2}{2}$$

Multiplication rule for choices [p. 759] When sequential choices are involved, the total number of possibilities is found by multiplying the number of options at each successive stage.

Multiplication rule for probability [p. 561] the probability of events A and B both occurring is $\Pr(A \cap B) = \Pr(A \mid B) \times \Pr(B)$

Multi-stage experiment [p. 562] an experiment that could be considered to take place in more than one stage; e.g. tossing two coins

Mutually exclusive [p. 553] Two events are said to be mutually exclusive if they have no outcomes in common.

N

$n!$ [p. 760] read as 'n factorial', the product of all the natural numbers from n down to 1:

$n! = n \times (n-1) \times (n-2) \times (n-3) \times \cdots \times 2 \times 1$

Natural numbers [p. 3] $\mathbb{N} = \{1, 2, 3, 4, \dots\}$

${}^nC_r$ [p. 760] the number of combinations of n objects in groups of size r:

$${}^nC_r = \frac{n!}{r!\,(n-r)!}$$

Normal distribution [p. 657] the distribution of a continuous random variable X with probability density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

where μ is the mean of X and σ is the standard deviation of X

Normal, equation of [p. 417] Let (x_1, y_1) be a point on the curve $y = f(x)$. If f is differentiable at $x = x_1$, the equation of the normal at (x_1, y_1) is

$$y - y_1 = \frac{-1}{f'(x_1)}(x - x_1)$$

O

Odd function [p. 19] A function f is odd if $f(-x) = -f(x)$ for all x in the domain of f; the graph has rotational symmetry about the origin.

Ordered pair [p. 6] a pair of elements, denoted (x, y), where x is the first coordinate and y is the second coordinate

P

Percentile [p. 633] For a continuous random variable X, the value p such that $\Pr(X \le p) = q\%$ is called the qth percentile of X, and is found by solving $\int_{-\infty}^{p} f(x)\,dx = \frac{q}{100}$.

Period of a function [p. 266] A function f with domain $\mathbb{R}$ is periodic if there is a positive constant a such that $f(x + a) = f(x)$ for all x. The smallest such a is called the period of f.

- Sine and cosine have period 2π.
- Tangent has period π.
- A function of the form $y = a\cos(nx + \varepsilon) + b$ or $y = a\sin(nx + \varepsilon) + b$ has period $\frac{2\pi}{n}$.

Piecewise-defined function [p. 18] a function which has different rules for different subsets of its domain

Point estimate [p. 708] If the value of the sample proportion $\hat{p}$ is used as an estimate of the population proportion p, then it is called a point estimate of p.

Polynomial function [p. 166] A polynomial has a rule of the type

$$y = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad n \in \mathbb{N} \cup \{0\}$$

where $a_0, a_1, \dots, a_n$ are numbers called coefficients.

Population [p. 687] the set of all eligible members of a group which we intend to study

Population parameter [p. 691] a statistical measure that is based on the whole population; the value is constant for a given population

Population proportion, p [p. 690] the proportion of individuals in the entire population possessing a particular attribute

Power function [pp. 43, 312] a function of the form $f(x) = x^r$, where r is a non-zero real number

Probability [p. 551] a numerical value assigned to the likelihood of an event occurring. If the event A is impossible, then $\Pr(A) = 0$; if the event A is certain, then $\Pr(A) = 1$; otherwise $0 < \Pr(A) < 1$.

Probability density function [p. 622] usually denoted $f(x)$; describes the probability distribution of a continuous random variable X such that $\Pr(a < X < b) = \int_a^b f(x)\,dx$

Probability function (discrete) [p. 570] denoted by $p(x)$ or $\Pr(X = x)$, a function that assigns a probability to each value of a discrete random variable X. It can be represented by a rule, a table or a graph, and must give a probability $p(x)$ for every value x that X can take.

Probability table [p. 556] a table used for illustrating a probability problem diagrammatically

Product of functions [p. 24] $(fg)(x) = f(x)\,g(x)$ and $\text{dom}(fg) = \text{dom}\, f \cap \text{dom}\, g$

Product rule [p. 393] Let $F(x) = f(x) \cdot g(x)$.
If $f'(x)$ and $g'(x)$ exist, then
$F'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$
In Leibniz notation:
If $y = uv$, then $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

Q

$\mathbb{Q}$ [p. 3] the set of all rational numbers

Quadratic formula [p. 156] $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is the solution of the quadratic equation $ax^2 + bx + c = 0$

Quadratic function [p. 151] A quadratic has a rule of the form $y = ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$.

Quadratic, turning point form [p. 152] The turning point form of a quadratic function is $y = a(x - h)^2 + k$, where (h, k) is the turning point.

Quartic function [p. 186] a polynomial of degree 4. A quartic function f has a rule of the form $f(x) = ax^4 + bx^3 + cx^2 + dx + e$, where $a \neq 0$.

Quotient rule [p. 397] Let $F(x) = \dfrac{f(x)}{g(x)}$, where $g(x) \neq 0$. If $f'(x)$ and $g'(x)$ exist, then
$$F'(x) = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}$$
In Leibniz notation:
If $y = \dfrac{u}{v}$, then $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

R

$\mathbb{R}^+$ [p. 4] $\{ x : x > 0 \}$, positive real numbers

$\mathbb{R}^-$ [p. 4] $\{ x : x < 0 \}$, negative real numbers

$\mathbb{R} \setminus \{0\}$ [p. 4] the set of real numbers excluding 0

$\mathbb{R}^2$ [p. 98] $\{ (x, y) : x, y \in \mathbb{R} \}$; i.e. $\mathbb{R}^2$ is the set of all ordered pairs of real numbers

Radian [p. 255] One radian (written 1^c) is the angle subtended at the centre of the unit circle by an arc of length 1 unit.

Random experiment [p. 551] an experiment, such as the rolling of a die, in which the outcome of a single trial is uncertain but observable

Random sample [p. 687] A sample of size n is called a *simple random sample* if it is selected from the population in such a way that every subset of size n has an equal chance of being chosen as the sample. In particular, every member of the population must have an equal chance of being included in the sample.

Random variable [p. 569] a variable that takes its value from the outcome of a random experiment; e.g. the number of heads observed when a coin is tossed three times

Range [p. 6] the set of all the second coordinates of the ordered pairs in a relation

Rational number [p. 3] a number that can be written as $\dfrac{p}{q}$, for some integers p and q with $q \neq 0$

Rational-root theorem [p. 176]
Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of degree n with all coefficients a_i integers. Let α and β be integers such that the highest common factor of α and β is 1.
If $\beta x + \alpha$ is a factor of $P(x)$, then β divides a_n and α divides a_0.

Rectangular hyperbola [p. 44] The basic rectangular hyperbola has equation $y = \dfrac{1}{x}$.

Reflection in the *x*-axis [p. 109] A reflection in the x-axis is described by the rule $(x, y) \to (x, -y)$. The curve with equation $y = f(x)$ is mapped to the curve with equation $y = -f(x)$.

Reflection in the *y*-axis [p. 109] A reflection in the y-axis is described by the rule $(x, y) \to (-x, y)$. The curve with equation $y = f(x)$ is mapped to the curve with equation $y = f(-x)$.

Relation [p. 6] a set of ordered pairs; e.g. $\{ (x, y) : y = x^2 \}$

Remainder theorem [p. 174]
When a polynomial $P(x)$ is divided by $\beta x + \alpha$, the remainder is $P\left(-\dfrac{\alpha}{\beta}\right)$.

S

Sample [p. 687] a subset of the population which we select in order to make inferences about the whole population

Sample proportion, $\hat{p}$ [p. 690] the proportion of individuals in a particular sample possessing a particular attribute. The sample proportions $\hat{p}$ are the values of a random variable $\hat{P}$.

Sample space, ε [p. 551] the set of all possible outcomes for a random experiment

Sample statistic [p. 691] a statistical measure that is based on a sample from the population; the value varies from sample to sample

Sampling distribution [p. 695] the distribution of a statistic which is calculated from a sample

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Sampling with replacement [p. 597] selecting individual objects sequentially from a group of objects, and replacing the selected object, so that the probability of obtaining a particular object does not change with each successive selection

Sampling without replacement [MM1&2] selecting individual objects sequentially from a group of objects, and not replacing the selected object, so that the probability of obtaining a particular object changes with each successive selection

Selections [p. 760] counted when order is not important. The number of ways of selecting r objects from a total of n objects is

$$^nC_r = \frac{n!}{r!\,(n-r)!}$$

Set difference [p. 3] The set difference of two sets A and B is $A \setminus B = \{\, x : x \in A \text{ and } x \notin B \,\}$.

Simulation [MM1&2] the process of finding an approximate solution to a probability problem by repeated trials using a simulation model

Simulation model [MM1&2] a simple model which is analogous to a real-world situation. For example, the outcomes from a toss of a coin (head, tail) could be used as a simulation model for the sex of a child (male, female) under the assumption that in both situations the probabilities are 0.5 for each outcome.

Simultaneous equations [pp. 83, 86, 197] equations of two or more lines or curves in the Cartesian plane, the solutions of which are the points of intersection of the lines or curves

Sine function [p. 257] *see* cosine and sine

Standard deviation of a random variable, σ [pp. 580, 638] a measure of the spread or variability, given by $\mathrm{sd}(X) = \sqrt{\mathrm{Var}(X)}$

Standard normal distribution [p. 655] a special case of the normal distribution where $\mu = 0$ and $\sigma = 1$

Stationary point [p. 428] A point $(a, f(a))$ on a curve $y = f(x)$ is a stationary point if $f'(a) = 0$.

Straight line, equation given two points [p. 70] $y - y_1 = m(x - x_1)$, where $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Straight line, gradient–intercept form [p. 70] $y = mx + c$, where m is the gradient and c is the y-axis intercept

Straight lines, perpendicular [p. 71] Two straight lines are perpendicular if and only if the product of their gradients is -1 (or if one is vertical and the other horizontal).

Strictly decreasing [pp. 43, 362] A function f is strictly decreasing on an interval if $x_2 > x_1$ implies $f(x_2) < f(x_1)$.

Strictly increasing [pp. 43, 362] A function f is strictly increasing on an interval if $x_2 > x_1$ implies $f(x_2) > f(x_1)$.

Subset [p. 2] A set B is called a subset of set A if every element of B is also an element of A. We write $B \subseteq A$.

Sum of functions [p. 24] $(f + g)(x) = f(x) + g(x)$ and $\mathrm{dom}(f + g) = \mathrm{dom}\, f \cap \mathrm{dom}\, g$

Sum of two cubes [p. 176] $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

T

Tangent, equation of [p. 417] Let (x_1, y_1) be a point on the curve $y = f(x)$. Then, if f is differentiable at $x = x_1$, the equation of the tangent at (x_1, y_1) is given by $y - y_1 = f'(x_1)(x - x_1)$.

Tangent function [p. 257] $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

Translation [p. 98] A translation of h units in the positive direction of the x-axis and k units in the positive direction of the y-axis is described by the rule $(x, y) \to (x + h, y + k)$, where $h, k > 0$. The curve with equation $y = f(x)$ is mapped to the curve with equation $y - k = f(x - h)$.

Tree diagram [p. 562] a diagram representing the outcomes of a multi-stage experiment

U

Union of sets [pp. 2, 551] The union of two sets A and B, written $A \cup B$, is the set of all elements which are in A or B or both.

V

Variance of a random variable, σ^2 [pp. 580, 638] a measure of the spread or variability, defined by $\mathrm{Var}(X) = \mathrm{E}[(X - \mu)^2]$. An alternative (computational) formula is $\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$

Velocity, average [MM1&2]

$$\text{average velocity} = \frac{\text{change in position}}{\text{change in time}}$$

Velocity, instantaneous [MM1&2] $v = \dfrac{dx}{dt}$

Vertical-line test [p. 8] If a vertical line can be drawn anywhere on the graph of a relation and it only ever intersects the graph a maximum of once, then the relation is a *function*.

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Answers

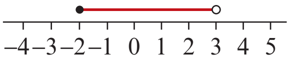
Chapter 1

Exercise 1A

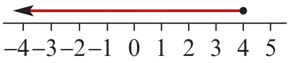
1 **a** $\{8, 11\}$ **b** $\{8, 11\}$
c $\{1, 3, 8, 11, 18, 22, 23, 24, 25, 30\}$
d $\{3, 8, 11, 18, 22, 23, 24, 25, 30, 32\}$
e $\{3, 8, 11, 18, 22, 23, 24, 25, 30, 32\}$
f $\{1, 8, 11, 25, 30\}$

2 **a** $\{3, 18, 22, 23, 24\}$ **b** $\{25, 30, 32\}$
c $\{3, 18, 22, 23, 24\}$ **d** $\{1, 25, 30\}$

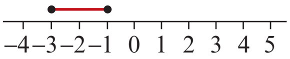
3 **a**


−4 −3 −2 −1 0 1 2 3 4 5

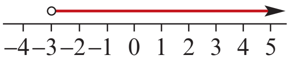
b


−4 −3 −2 −1 0 1 2 3 4 5

c


−4 −3 −2 −1 0 1 2 3 4 5

d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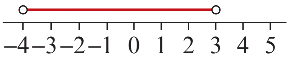
e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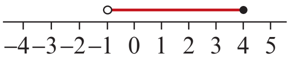
f


−4 −3 −2 −1 0 1 2 3 4 5

4 **a** $\{7, 9\}$ **b** $\{7, 9\}$
c $\{2, 3, 5, 7, 9, 11, 15, 19, 23\}$ **d** $\{2, 3, 5, 11\}$
e $\{2\}$ **f** $\{2, 7, 9\}$ **g** $\{2, 3, 5, 7\}$
h $\{7\}$ **i** $\{7, 9, 15, 19, 23\}$ **j** $(3, \infty)$

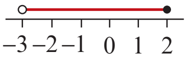
5 **a** $\{a, e\}$ **b** $\{a, b, c, d, e, i, o, u\}$
c $\{b, c, d\}$ **d** $\{i, o, u\}$

6 **a** $\{6\}$ **b** $\{2, 4, 8, 10\}$ **c** $\{1, 3, 5, 7, 9\}$
d $\{1, 2, 3, 4, 5, 7, 8, 9, 10\}$
e $\{1, 2, 3, 4, 5, 7, 8, 9, 10\}$ **f** $\{5, 7\}$
g $\{5, 7\}$ **h** $\{6\}$

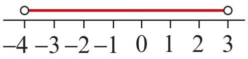
7 **a** $[-3, 1)$ **b** $(-4, 5]$ **c** $(-\sqrt{2}, 0)$
d $\left(-\frac{1}{\sqrt{2}}, \sqrt{3}\right)$ **e** $(-\infty, -3)$ **f** $(0, \infty)$
g $(-\infty, 0)$ **h** $[-2, \infty)$

8 **a** $(-2, 3)$ **b** $[-4, 1)$ **c** $[-1, 5]$ **d** $(-3, 2]$

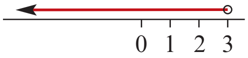
9 **a**



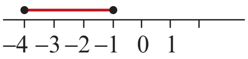
b



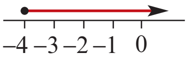
c



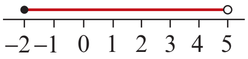
d



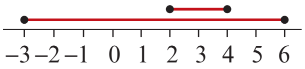
e



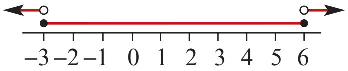
f



10 **a**


−3 −2 −1 0 1 2 3 4 5 6

b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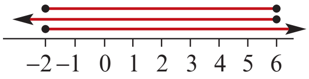
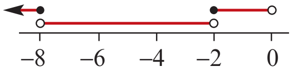
c



d



Exercise 1B

1 **a** Domain $= \mathbb{R}$ Range $= [-2, \infty)$
b Domain $= (-\infty, 2]$ Range $= \mathbb{R}$

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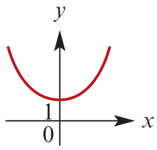
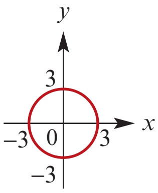
c Domain = $(-2, 3)$ Range = $[0, 9]$
d Domain = $(-3, 1)$ Range = $(-6, 2)$
e Domain = $[-4, 0]$ Range = $[0, 4]$
f Domain = $\mathbb{R}$ Range = $(-\infty, 2)$

2 a



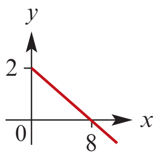
Domain = $\mathbb{R}$
Range = $[1, \infty)$

b



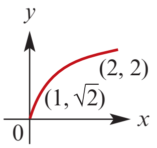
Domain = $[-3, 3]$
Range = $[-3, 3]$

c



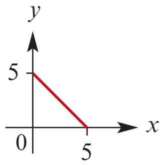
Domain = $[0, \infty)$
Range = $(-\infty, 2]$

d



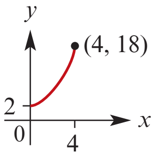
Domain = $[0, \infty)$
Range = $[0, \infty)$

e



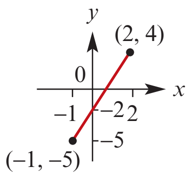
Domain = $[0, 5]$
Range = $[0, 5]$

f



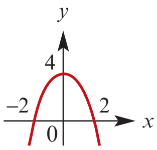
Domain = $[0, 4]$
Range = $[2, 18]$

g



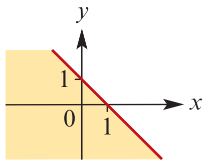
Domain = $[-1, 2]$
Range = $[-5, 4]$

h



Domain = $\mathbb{R}$
Range = $(-\infty, 4]$

i



Domain = $\mathbb{R}$
Range = $\mathbb{R}$

3 a Not a function; Domain = $\{-1, 1, 2, 3\}$;
Range = $\{1, 2, 3, 4\}$
b A function; Domain = $\{-2, -1, 0, 1, 2\}$;
Range = $\{-4, -1, 0, 3, 5\}$
c Not a function; Domain = $\{-2, -1, 2, 4\}$;
Range = $\{-2, 1, 2, 4, 6\}$
d A function; Domain = $\{-1, 0, 1, 2, 3\}$;
Range = $\{4\}$

4 a A function; Domain = $\mathbb{R}$; Range = $\{4\}$
b Not a function; Domain = $\{2\}$; Range = $\mathbb{Z}$
c A function; Domain = $\mathbb{R}$; Range = $\mathbb{R}$
d Not a function; Domain = $\mathbb{R}$; Range = $\mathbb{R}$
e Not a function; Domain = $[-4, 4]$;
Range = $[-4, 4]$

5 a $f(-1) = -2$, $f(2) = 16$, $f(-3) = 6$,
$f(2a) = 8a^2 + 8a$
b $g(-1) = -10$, $g(2) = 14$, $g(3) = 54$,
$g(a-1) = 2a^3 - 6a^2 + 8a - 10$

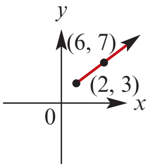
6 a $g(-2) = 10$, $g(4) = 46$
b i $12x^2 - 2$ **ii** $3x^2 - 12x + 10$
iii $3x^2 + 12x + 10$ **iv** $3x^4 - 2$

7 a 3 **b** 7 **c** $-\frac{3}{2}$ **d** $(3, \infty)$

8 a $x = -3$ **b** $x > -3$ **c** $x = \frac{2}{3}$

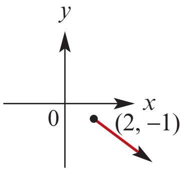
9 a $f\colon \mathbb{R} \to \mathbb{R}, f(x) = 2x + 3$
b $f\colon \mathbb{R} \to \mathbb{R}, f(x) = -\frac{4}{3}x + 4$
c $f\colon [0, \infty) \to \mathbb{R}, f(x) = 2x - 3$
d $f\colon \mathbb{R} \to \mathbb{R}, f(x) = x^2 - 9$
e $f\colon [0, 2] \to \mathbb{R}, f(x) = 5x - 3$

10 a



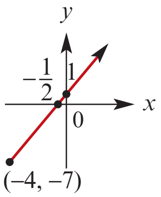
Range = $[3, \infty)$

b



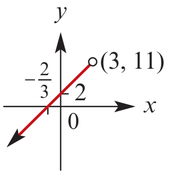
Range = $(-\infty, -1]$

c



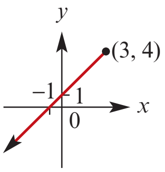
Range = $[-7, \infty)$

d



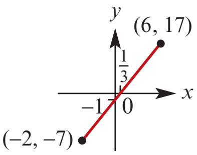
Range = $(-\infty, 11)$

e



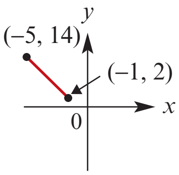
Range = $(-\infty, 4]$

f



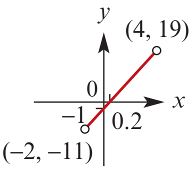
Range = $[-7, 17]$

g



Range = $[2, 14]$

h



Range = $(-11, 19)$

11 a $f(2) = -3$, $f(-3) = 37$, $f(-2) = 21$
b $g(-2) = 7$, $g(1) = 1$, $g(-3) = 9$
c i $f(a) = 2a^2 - 6a + 1$
ii $f(a+2) = 2a^2 + 2a - 3$
iii $g(-a) = 3 + 2a$ **iv** $g(2a) = 3 - 4a$
v $f(5-a) = 21 - 14a + 2a^2$
vi $f(2a) = 8a^2 - 12a + 1$
vii $g(a) + f(a) = 2a^2 - 8a + 4$
viii $g(a) - f(a) = 2 + 4a - 2a^2$

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12 a $\{\frac{2}{3}, -1\}$ **b** $\left\{-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right\}$ **c** $\{0, -\frac{1}{3}\}$

d $(-\infty, -1) \cup (\frac{2}{3}, \infty)$

e $\left(-\infty, -\sqrt{\frac{2}{3}}\right) \cup \left(\sqrt{\frac{2}{3}}, \infty\right)$ **f** $[-\frac{1}{3}, 0]$

13 a $f(-2) = 2$ **b** $f(2) = 6$

c $f(-a) = a^2 - a$ **d** $f(a) + f(-a) = 2a^2$

e $f(a) - f(-a) = 2a$ **f** $f(a^2) = a^4 + a^2$

14 a $\{2\}$ **b** $\{x : x > 2\}$ **c** $\left\{\frac{a+2}{3}\right\}$

d $\left\{-\frac{8}{3}\right\}$ **e** $\{1\}$ **f** $\left\{\frac{13}{18}\right\}$

15 a $\frac{4}{3}$ **b** 6 **c** $-\frac{7}{3}$ **d** 9 **e** $\frac{1}{3}$ **f** $-\frac{2}{9}$

16 a $\frac{6}{5}$ **b** $\frac{1}{5}$ **c** $\pm\frac{1}{3}$ **d** 1 **e** $-1, 2$

Exercise 1C

1 One-to-one functions: b, c

2 One-to-one functions: b, d, f

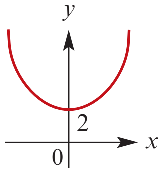
3 a Functions: i, iii, iv, vi, vii, viii

b One-to-one functions: iii, vii

4 $y = \sqrt{x+2},\ x \ge -2$; Range $= \mathbb{R}^+ \cup \{0\}$

$y = -\sqrt{x+2},\ x \ge -2$; Range $= \mathbb{R}^- \cup \{0\}$

5 a



b $g_1(x) = x^2 + 2,\ x \ge 0$

$g_2(x) = x^2 + 2,\ x < 0$

6 a Domain $= \mathbb{R}$ Range $= \mathbb{R}$

b Domain $= [0, \infty)$ Range $= [0, \infty)$

c Domain $= \mathbb{R}$ Range $= [-2, \infty)$

d Domain $= [-4, 4]$ Range $= [0, 4]$

e Domain $= \mathbb{R} \setminus \{0\}$ Range $= \mathbb{R} \setminus \{0\}$

f Domain $= \mathbb{R}$ Range $= (-\infty, 4]$

g Domain $= [3, \infty)$ Range $= [0, \infty)$

7 a Domain $= \mathbb{R}$ Range $= \mathbb{R}$

b Domain $= \mathbb{R}$ Range $= [-2, \infty)$

c Domain $= [-3, 3]$ Range $= [0, 3]$

d Domain $= \mathbb{R} \setminus \{1\}$ Range $= \mathbb{R} \setminus \{0\}$

8 a $\mathbb{R} \setminus \{3\}$ **b** $(-\infty, -\sqrt{3}] \cup [\sqrt{3}, \infty)$

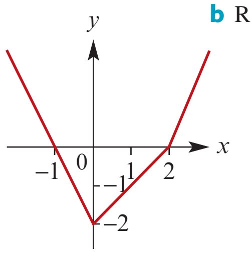
c $\mathbb{R}$ **d** $[4, 11]$ **e** $\mathbb{R} \setminus \{-1\}$

f $(-\infty, -1] \cup [2, \infty)$ **g** $\mathbb{R} \setminus \{-1, 2\}$

h $(-\infty, -2) \cup [1, \infty)$ **i** $\left[0, \frac{1}{3}\right]$

j $[-5, 5]$ **k** $[3, 12]$

9 a

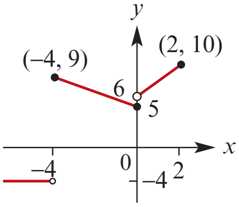


b Range $= [-2, \infty)$

10 Domain $= (-3, 0] \cup [1, 3)$; Range $= [-2, 3)$

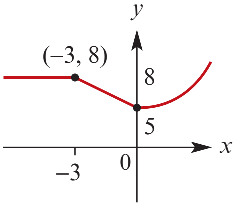
11 Domain $= [-5, 4]$; Range $= [-4, 0) \cup [2, 5]$

12 a



b Domain $= (-\infty, 2]$; Range $= [5, 10] \cup \{-4\}$

13 a



b Range $= [5, \infty)$

14 a $f(-4) = -8$ **b** $f(0) = 0$ **c** $f(4) = \frac{1}{4}$

d $f(a+3) = \begin{cases} \frac{1}{a+3}, & a > 0 \\ 2(a+3), & a \le 0 \end{cases}$

e $f(2a) = \begin{cases} \frac{1}{2a}, & a > \frac{3}{2} \\ 4a, & a \le \frac{3}{2} \end{cases}$

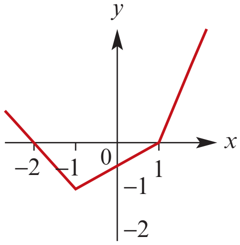
f $f(a-3) = \begin{cases} \frac{1}{a-3}, & a > 6 \\ 2(a-3), & a \le 6 \end{cases}$

15 a $f(0) = 4$ **b** $f(3) = \sqrt{2}$ **c** $f(8) = \sqrt{7}$

d $f(a+1) = \begin{cases} \sqrt{a}, & a \ge 0 \\ 4, & a < 0 \end{cases}$

e $f(a-1) = \begin{cases} \sqrt{a-2}, & a \ge 2 \\ 4, & a < 2 \end{cases}$

16



17 $y = \begin{cases} -x - 4, & x < -2 \\ \frac{1}{2}x - 1, & -2 \le x \le 3 \\ -\frac{1}{2}x + 2, & x > 3 \end{cases}$

18 a Even **b** Odd **c** Neither

d Even **e** Odd **f** Neither

19 a Even **b** Even **c** Odd

d Odd **e** Neither **f** Even

g Neither **h** Neither **i** Even

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Exercise 1D

1 a $(f+g)(x) = 4x+2$ **b** $(f+g)(x) = 1$

$(fg)(x) = 3x^2+6x$ $(fg)(x) = x^2 - x^4$

$\text{dom} = \mathbb{R}$ $\text{dom} = (0, 2]$

c $(f+g)(x) = \dfrac{x+1}{\sqrt{x}}$

$(fg)(x) = 1$

$\text{dom} = [1, \infty)$

d $(f+g)(x) = x^2 + \sqrt{4-x}$

$(fg)(x) = x^2\sqrt{4-x}$

$\text{dom} = [0, 4]$

2 a i Even **ii** Odd **iii** Even **iv** Odd

b $(f+h)(x) = x^2 + 1 + \dfrac{1}{x^2}$, even;

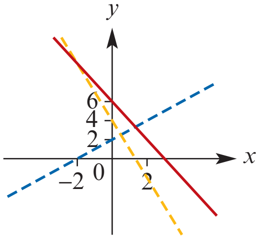
$(gk)(x) = 1$, even; $(fh)(x) = 1 + \dfrac{1}{x^2}$, even;

$(f+g)(x) = x^2 + x + 1$, neither;

$(g+k)(x) = x + \dfrac{1}{x}$, odd;

$(fg)(x) = x^3 + x$, odd

3



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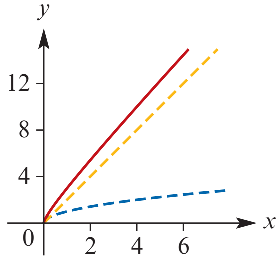

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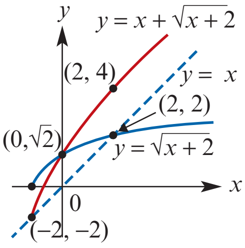

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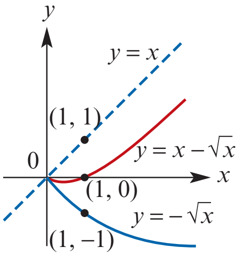

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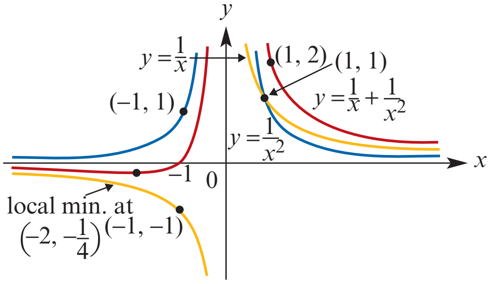

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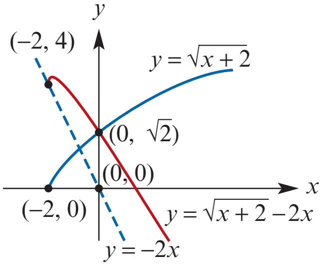

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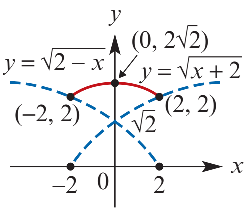

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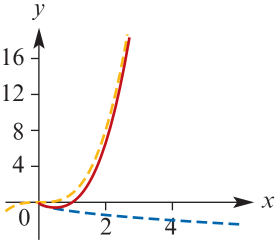

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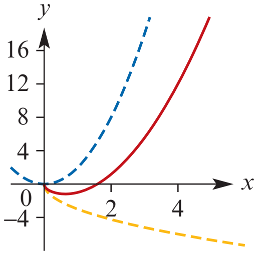

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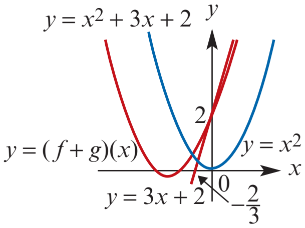

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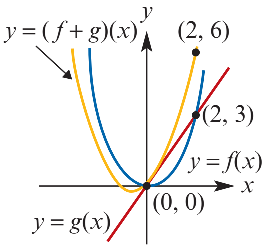
12 a



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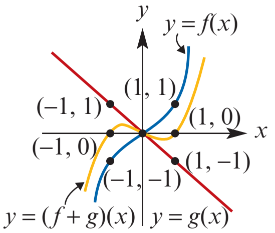
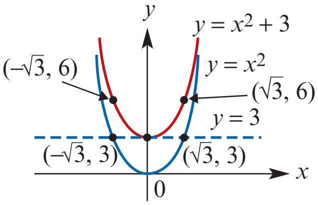


13 a



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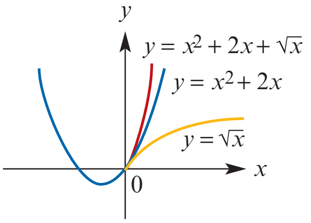

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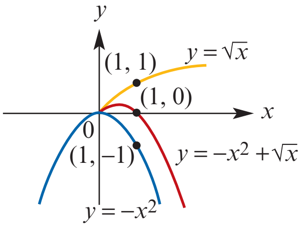


Exercise 1E

1 a $f(g(x)) = 4x - 1, \quad g(f(x)) = 4x - 2$
b $f(g(x)) = 8x + 5, \quad g(f(x)) = 8x + 3$
c $f(g(x)) = 4x - 7, \quad g(f(x)) = 4x - 5$
d $f(g(x)) = 2x^2 - 1, \quad g(f(x)) = (2x - 1)^2$
e $f(g(x)) = 2(x - 5)^2 + 1, \quad g(f(x)) = 2x^2 - 4$
f $f(g(x)) = 2x^2 + 1, \quad g(f(x)) = (2x + 1)^2$

2 a $f(h(x)) = 6x + 3$ **b** $h(f(x)) = 6x - 1$
c 15 **d** 11 **e** 21 **f** -7 **g** 3

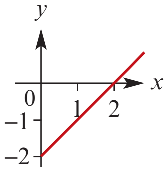
3 a $9x^2 + 12x + 3$ **b** $3x^2 + 6x + 1$
c 120 **d** 46 **e** 3 **f** 1

4 a $h(g(x)) = \dfrac{1}{(3x+2)^2}, \quad \text{dom}(h \circ g) = \mathbb{R}^+$
b $g(h(x)) = \dfrac{3}{x^2} + 2, \quad \text{dom}(g \circ h) = \mathbb{R} \setminus \{0\}$
c $\dfrac{1}{25}$ **d** 5

5 a $\text{ran}\, f = [-4, \infty), \quad \text{ran}\, g = [0, \infty)$
b $f \circ g(x) = x - 4, \quad \text{ran}(f \circ g) = [-4, \infty)$
c $\text{ran}\, f \not\subseteq \text{dom}\, g$

6 a $f \circ g(x) = x, \; \text{dom} = \mathbb{R} \setminus \{\tfrac{1}{2}\}, \; \text{ran} = \mathbb{R} \setminus \{\tfrac{1}{2}\}$
b $g \circ f(x) = x, \; \text{dom} = \mathbb{R} \setminus \{0\}, \; \text{ran} = \mathbb{R} \setminus \{0\}$

7 a $\text{ran}\, f = [-2, \infty) \not\subseteq \text{dom}\, g = \mathbb{R}^+ \cup \{0\}$
b $f \circ g(x) = x - 2, \; x \geq 0$



8 a $\text{ran}\, g = [-1, \infty) \not\subseteq \text{dom}\, f = (-\infty, 3]$
b $g^* : [-2, 2] \to \mathbb{R}, \; g^*(x) = x^2 - 1$
$f \circ g^* : [-2, 2] \to \mathbb{R}, \; f \circ g^*(x) = 4 - x^2$

9 a $\text{ran}\, g = \mathbb{R} \not\subseteq \text{dom}\, f = \mathbb{R}^+$
b $g_1 : (-\infty, 3) \to \mathbb{R}, \; g_1(x) = 3 - x$

10

	Domain	Range
f	$\mathbb{R}$	$[0, \infty)$
g	$(-\infty, 3]$	$[0, \infty)$

a $\text{ran}\, g \subseteq \text{dom}\, f$, so $f \circ g$ exists
b $\text{ran}\, f \not\subseteq \text{dom}\, g$, so $g \circ f$ does not exist

11 a $S = [-2, 2]$
b $\text{ran}\, f = [0, 2], \; \text{ran}\, g = [1, \infty)$
c $\text{ran}\, f \subseteq \text{dom}\, g$, so $g \circ f$ is defined
$\text{ran}\, g \not\subseteq \text{dom}\, f$, so $f \circ g$ is not defined

12 $a \in [2, 3]$

Exercise 1F

1 a $f^{-1}(x) = \dfrac{x - 3}{2}$ **b** $f^{-1}(x) = \dfrac{4 - x}{3}$
c $f^{-1}(x) = \dfrac{x - 3}{4}$

2 a $f^{-1}(x) = x + 4$ **b** $f^{-1}(x) = \dfrac{x}{2}$
c $f^{-1}(x) = \dfrac{4x}{3}$ **d** $f^{-1}(x) = \dfrac{4x + 2}{3}$

3 a $f^{-1}(x) = \dfrac{1}{2}(x + 4)$
dom $= [-8, 8]$
ran $= [-2, 6]$

b $g^{-1}(x) = 9 - \dfrac{1}{x}$
dom $= (-\infty, 0)$
ran $= (9, \infty)$

c $h^{-1}(x) = \sqrt{x - 2}$
dom $= [2, \infty)$
ran $= \mathbb{R}^+ \cup \{0\}$

d $f^{-1}(x) = \dfrac{1}{5}(x + 2)$
dom $= [-17, 28]$
ran $= [-3, 6]$

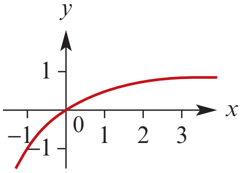
e $g^{-1}(x) = \sqrt{x + 1}$
dom $= (0, \infty)$
ran $= (1, \infty)$

f $h^{-1}(x) = x^2$
dom $= (0, \infty)$
ran $= (0, \infty)$

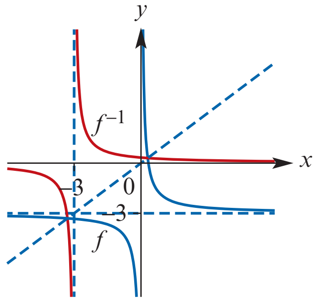
4 a $g^{-1}(x) = \sqrt{x+1} - 1$

dom $g^{-1} = [-1, \infty)$, ran $g^{-1} = [-1, \infty)$

b



5 $f^{-1}: \mathbb{R} \setminus \{-3\} \to \mathbb{R},\ f^{-1}(x) = \dfrac{1}{x+3}$



Intersection points: $\left(\dfrac{-3+\sqrt{13}}{2}, \dfrac{-3+\sqrt{13}}{2}\right)$

$\left(\dfrac{-3-\sqrt{13}}{2}, \dfrac{-3-\sqrt{13}}{2}\right)$

6 $f^{-1}(2) = \dfrac{1}{2}$, dom $f^{-1} = [-3, 3]$

7 a $f^{-1}(x) = \dfrac{x}{2}$,

dom $f^{-1} = [-2, 6]$, ran $f^{-1} = [-1, 3]$

b $f^{-1}(x) = \sqrt{\dfrac{x+4}{2}}$,

dom $f^{-1} = [-4, \infty)$, ran $f^{-1} = [0, \infty)$

c $\{(6, 1), (4, 2), (8, 3), (11, 5)\}$,

dom $= \{6, 4, 8, 11\}$, ran $= \{1, 2, 3, 5\}$

d $h^{-1}(x) = -x^2$, dom $h^{-1} = \mathbb{R}^+$, ran $h^{-1} = \mathbb{R}^-$

e $f^{-1}(x) = \sqrt[3]{x-1}$, dom $f^{-1} = \mathbb{R}$, ran $f^{-1} = \mathbb{R}$

f $g^{-1}(x) = -1 + \sqrt{x}$,

dom $g^{-1} = (0, 16)$, ran $g^{-1} = (-1, 3)$

g $g^{-1}(x) = x^2 + 1$,

dom $g^{-1} = \mathbb{R}^+ \cup \{0\}$, ran $g^{-1} = [1, \infty)$

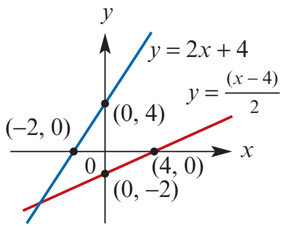
h $h^{-1}(x) = \sqrt{4 - x^2}$,

dom $h^{-1} = [0, 2]$, ran $h^{-1} = [0, 2]$

8 a $y = \dfrac{x-4}{2}$

dom $= \mathbb{R}$

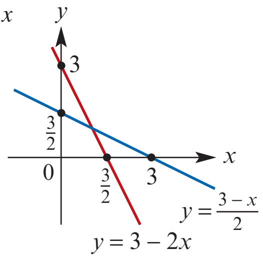
ran $= \mathbb{R}$



b $f^{-1}(x) = 3 - 2x$

dom $= \mathbb{R}$

ran $= \mathbb{R}$

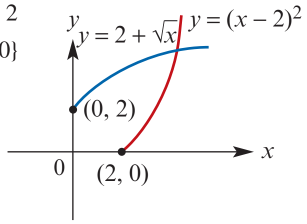


c

$f^{-1}(x) = \sqrt{x} + 2$

dom $= \mathbb{R}^+ \cup \{0\}$

ran $= [2, \infty)$

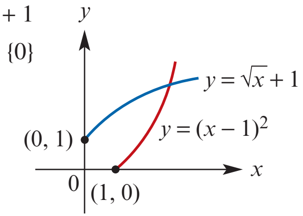


d

$f^{-1}(x) = \sqrt{x} + 1$

dom $= \mathbb{R}^+ \cup \{0\}$

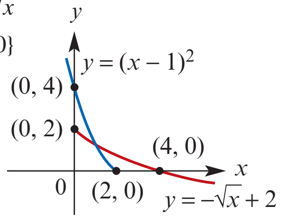
ran $= [1, \infty)$



e $f^{-1}(x) = 2 - \sqrt{x}$

dom $= \mathbb{R}^+ \cup \{0\}$

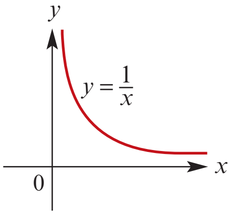
ran $= (-\infty, 2]$



f $f^{-1}(x) = \dfrac{1}{x}$

dom $= \mathbb{R}^+$

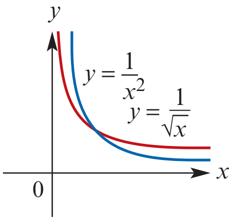
ran $= \mathbb{R}^+$



g $f^{-1}(x) = \dfrac{1}{\sqrt{x}}$

dom $= \mathbb{R}^+$

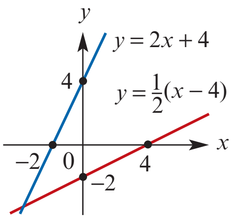
ran $= \mathbb{R}^+$



h $h^{-1}(x) = 2x + 4$

dom $= \mathbb{R}$

ran $= \mathbb{R}$



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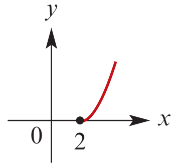
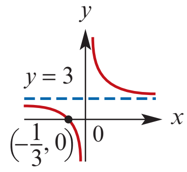
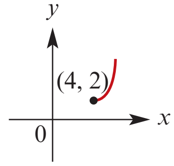
9 a $f^{-1}: [2, \infty) \to \mathbb{R}$,
$f^{-1}(x) = (x - 2)^2$



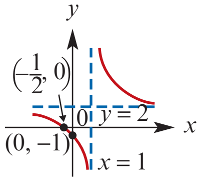
b $f^{-1}: \mathbb{R} \setminus \{0\} \to \mathbb{R}$,
$f^{-1}(x) = \dfrac{1}{x} + 3$



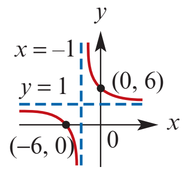
c $f^{-1}: [4, \infty) \to \mathbb{R}$,
$f^{-1}(x) = (x - 4)^2 + 2$



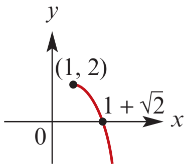
d $f^{-1}: \mathbb{R} \setminus \{1\} \to \mathbb{R}$,
$f^{-1}(x) = \dfrac{3}{x - 1} + 2$



e $f^{-1}: \mathbb{R} \setminus \{-1\} \to \mathbb{R}$,
$f^{-1}(x) = \dfrac{5}{x + 1} + 1$



f $f^{-1}: [1, \infty) \to \mathbb{R}$,
$f^{-1}(x) = 2 - (x - 1)^2$

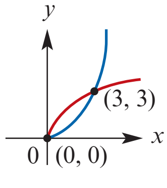


10 a $f^{-1}: \mathbb{R} \setminus \{1\} \to \mathbb{R}$, $f^{-1}(x) = \dfrac{x + 1}{x - 1}$

b $f^{-1}: \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$, $f^{-1}(x) = x^2 + 2$

c $f^{-1}: \mathbb{R} \setminus \{\frac{2}{3}\} \to \mathbb{R}$ $f^{-1}(x) = \dfrac{2x + 3}{3x - 2}$

11 a



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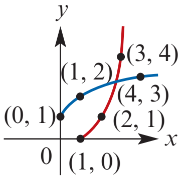

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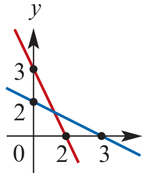

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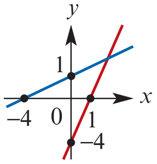
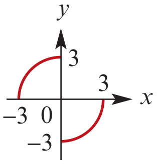


e



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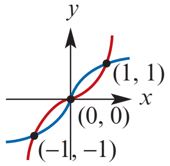

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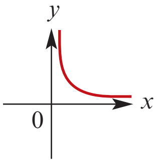
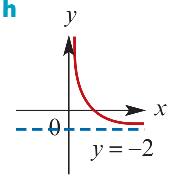


h



12 a C **b** B **c** D **d** A

13 a $A = (-\infty, 3]$

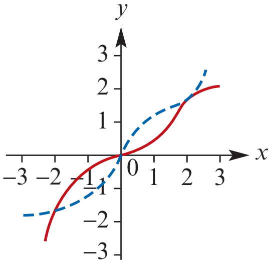
b $b = 0,\quad g^{-1}(x) = \sqrt{1 - x},\ x \in [-3, 1]$

14 $b = -2,\quad g^{-1}(x) = -2 + \sqrt{x + 4}$

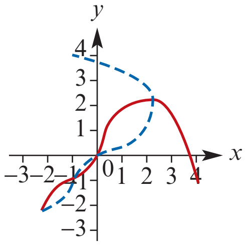
15 $a = 3,\quad f^{-1}(x) = 3 - \sqrt{x + 9}$

16 a	$y = \dfrac{3}{x}$	domain = $\mathbb{R} \setminus \{0\}$
b	$y = (x + 4)^3 - 2$	domain = $\mathbb{R}$
c	$y = (2 - x)^2$	domain = $(-\infty, 2]$
d	$y = \dfrac{3}{x - 1}$	domain = $\mathbb{R} \setminus \{1\}$
e	$y = \sqrt[3]{\dfrac{2}{5 - x}} + 6$	domain = $\mathbb{R} \setminus \{5\}$
f	$y = \dfrac{1}{(x - 2)^{\frac{4}{3}}} + 1$	domain = $(2, \infty)$

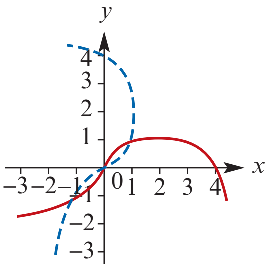
17 a Inverse is a function



b Inverse is not a function



c Inverse is not a function



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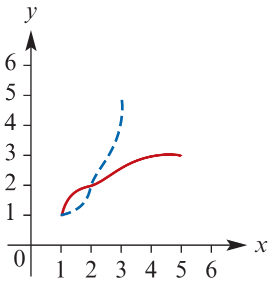
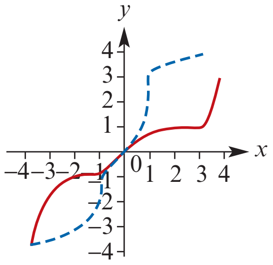
d Inverse is a function

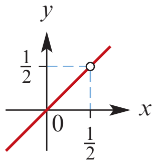


e Inverse is not a function



18 a dom $f = \mathbb{R} \setminus \{\frac{1}{2}\}$, ran $f = \mathbb{R} \setminus \{\frac{1}{2}\}$, $f \circ f$ is defined as ran $f \subseteq$ dom f

b $f \circ f(x) = x, \ x \in \mathbb{R} \setminus \{\frac{1}{2}\}$



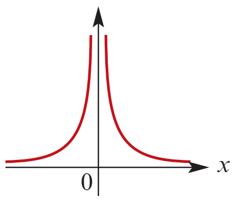
c $f^{-1} : \mathbb{R} \setminus \{\frac{1}{2}\} \to \mathbb{R}, f^{-1}(x) = \dfrac{x+3}{2x-1}$

Exercise 1G

1 a Maximal domain = $\mathbb{R} \setminus \{0\}$; Range = $\mathbb{R}^+$

b i $\dfrac{1}{16}$ **ii** $\dfrac{1}{16}$ **iii** 16 **iv** 16

c

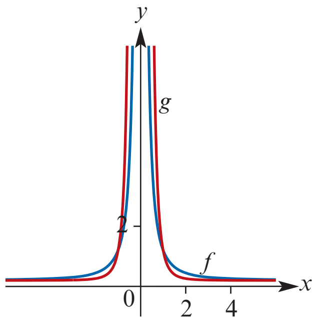


2 a Odd **b** Even **c** Odd

d Odd **e** Even **f** Odd

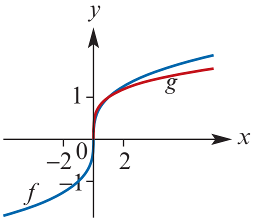
3 a $x = 1$ or $x = -1$

b



4 a $x = 1$ or $x = 0$

b



5 a $f^{-1} : \mathbb{R} \to \mathbb{R}, f^{-1}(x) = x^{\frac{1}{7}}$

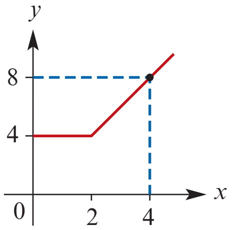
b $f^{-1} : [0, \infty) \to \mathbb{R}, f^{-1}(x) = -x^{\frac{1}{6}}$

c $f^{-1} : \mathbb{R} \to \mathbb{R}, f^{-1}(x) = \dfrac{1}{3}x^{\frac{1}{3}}$

d $f^{-1} : (16, \infty) \to \mathbb{R}, f^{-1}(x) = \dfrac{1}{2}x^{\frac{1}{4}}$

Exercise 1H

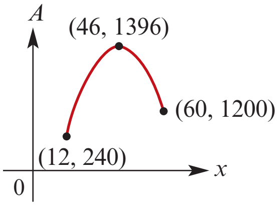
1 $f(x) = \begin{cases} 4 & \text{if } 0 \le x \le 2 \\ 2x & \text{if } x > 2 \end{cases}$



2 $V(x) = 4x(10 - x)(18 - x)$, domain = $[0, 10]$

3 a $A(x) = -x^2 + 92x - 720$ **b** $12 \le x \le 60$

c



d Maximum area 1396 m^2 occurs when $x = 46$ and $y = 34$

4 a i $S = 2x^2 + 6xh$ **ii** $S = 2x^2 + \dfrac{3V}{x}$

b Maximal domain = $(0, \infty)$

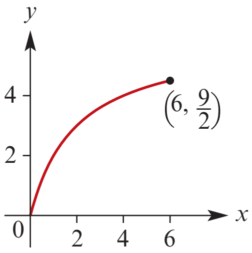
c Maximum value of S = 1508 m^2

5 Area = $x\sqrt{4a^2 - x^2}$, domain = $[0, 2a]$

6 a $A = \dfrac{6a}{a+2}$

b Domain = $[0, 6]$; Range = $\left[0, \dfrac{9}{2}\right]$

c $\dfrac{9}{2}$ **d**



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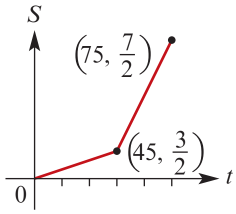
7 **a** $a = \frac{1}{30}$, $b = \frac{1}{15}$, $c = 45$, $d = -\frac{3}{2}$, $e = 75$

b



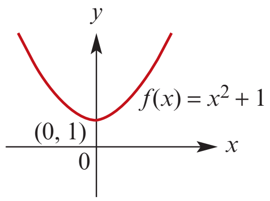
c Range $= \left[0, \frac{7}{2}\right]$

Chapter 1 review

Technology-free questions

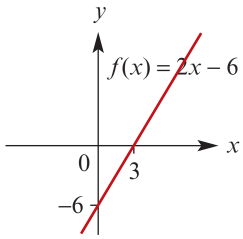
1 **a** Domain $= \mathbb{R}$

Range $= [1, \infty)$



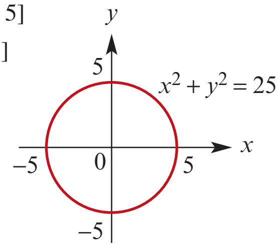
b Domain $= \mathbb{R}$

Range $= \mathbb{R}$



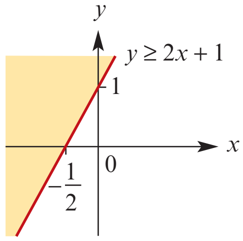
c Domain $= [-5, 5]$

Range $= [-5, 5]$



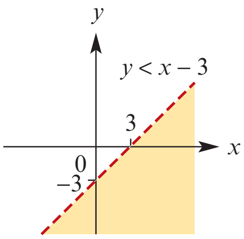
d Domain $= \mathbb{R}$

Range $= \mathbb{R}$

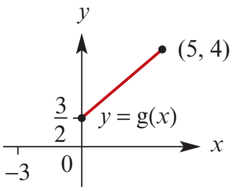


e Domain $= \mathbb{R}$

Range $= \mathbb{R}$



2 **a**



b $\text{ran}\, g = \left[\frac{3}{2}, 4\right]$

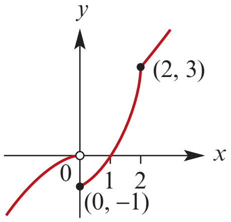
c $g^{-1} \colon \left[\frac{3}{2}, 4\right] \to \mathbb{R}$, $g^{-1}(x) = 2x - 3$

$\text{dom}\, g^{-1} = \left[\frac{3}{2}, 4\right]$, $\text{ran}\, g^{-1} = [0, 5]$

d $\{5\}$ **e** $\left\{\frac{7}{2}\right\}$

3 **a** $\left\{\frac{1}{5}\right\}$ **b** $\{11\}$ **c** $\left\{-\frac{1}{10}\right\}$

4



5 **a** $\mathbb{R} \setminus \{3\}$ **b** $\mathbb{R} \setminus [-\sqrt{5}, \sqrt{5}]$ **c** $\mathbb{R} \setminus \{1, -2\}$
d $[-5, 5]$ **e** $[5, 15]$ **f** $\mathbb{R} \setminus \{2\}$

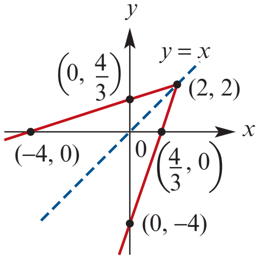
6 $(f + g)(x) = x^2 + 5x + 1$,
$(fg)(x) = (x - 3)(x + 2)^2$

7 $(f + g) \colon [1, 5] \to \mathbb{R}$, $(f + g)(x) = x^2 + 1$
$(fg) \colon [1, 5] \to \mathbb{R}$, $(fg)(x) = 2x(x - 1)^2$

8 $f^{-1} \colon [8, \infty) \to \mathbb{R}$, $f^{-1}(x) = \sqrt{x + 1}$

9 **a** $(f + g)(x) = -x^2 + 2x + 3$
b $(fg)(x) = -x^2(2x + 3)$ **c** $\{-1, 3\}$

10



11 **a** $f^{-1} \colon \mathbb{R} \to \mathbb{R}$, $f^{-1}(x) = \frac{1}{2}x^{\frac{1}{3}}$

b $f^{-1} \colon (-\infty, 0] \to \mathbb{R}$, $f^{-1}(x) = \frac{1}{2}x^{\frac{1}{5}}$

c $f^{-1} \colon [0, \infty) \to \mathbb{R}$, $f^{-1}(x) = \frac{1}{2}x^{\frac{1}{6}}$

d $f^{-1} \colon (10\,000, \infty) \to \mathbb{R}$, $f^{-1}(x) = \frac{1}{10}x^{\frac{1}{4}}$

12 **a** $f \circ g(x) = -2x^3 + 3$
b $g \circ f(x) = -(2x + 3)^3$
c $g \circ g(x) = x^9$ **d** $f \circ f(x) = 4x + 9$
e $f \circ (f + g)(x) = -2x^3 + 4x + 9$
f $f \circ (f - g)(x) = 2x^3 + 4x + 9$
g $f \circ (f \cdot g)(x) = -4x^4 - 6x^3 + 3$

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13 $x \geq -1$ or $x \leq -9$

14 $h^{-1}(x) = \left(\dfrac{x-64}{2}\right)^{\frac{1}{5}}$

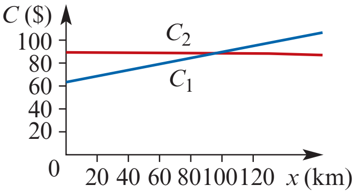
Multiple-choice questions

1 E **2** B **3** E **4** C **5** E **6** C
7 D **8** B **9** B **10** C **11** B **12** E
13 C **14** C **15** A **16** B **17** A **18** B
19 B **20** C **21** A **22** D

Extended-response questions

1 a $C_1 = 64 + 0.25x,\ C_2 = 89$

b

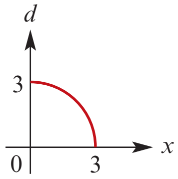


c $x > 100$ km

2 a $S = 6x^2$ **b** $S = 6V^{\frac{2}{3}}$

3 a $A = \dfrac{\sqrt{3}s^2}{4}$ **b** $A = \dfrac{\sqrt{3}h^2}{3}$

4 a $d(x) = \sqrt{9 - x^2}$ **b** dom $= [0, 3]$
ran $= [0, 3]$



5 $S(x) = \dfrac{160x}{x+80}$

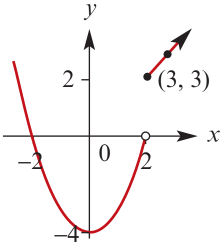
6 a $V_1 : (0, 12) \to \mathbb{R},\ V_1(h) = \pi h\left(36 - \dfrac{h^2}{4}\right)$

b $V_2 : (0, 6) \to \mathbb{R},\ V_2(r) = 2\pi r^2\sqrt{36 - r^2}$

7 a ran $f = \mathbb{R} =$ dom g, and so $g \circ f$ exists;
$g \circ f(x) = 2 + (1 + x)^3$

b $g \circ f$ is one-to-one and so $(g \circ f)^{-1}$ exists;
$(g \circ f)^{-1}(10) = 1$

8 a



b i -3 **ii** 3

c $S = (-\infty, 0]$

d $f(h(x)) = \begin{cases} 4x^2 - 4 & \text{if } x < 1 \\ 2x & \text{if } x \geq 1 \end{cases}$

$h(f(x)) = \begin{cases} 2x^2 - 8 & \text{if } x < 2 \\ 2x & \text{if } x \geq 2 \end{cases}$

9 $A(t) = \begin{cases} \dfrac{3t^2}{2}, & 0 < t \leq 1 \\ 3t - \dfrac{3}{2}, & t > 1 \end{cases}$

Domain $= (0, \infty)$; Range $= (0, \infty)$

10 a $f : \mathbb{R} \setminus \left\{\dfrac{a}{c}\right\} \to \mathbb{R},\ f^{-1}(x) = \dfrac{b - dx}{cx - a}$

b i $f^{-1} : \mathbb{R} \setminus \{1\} \to \mathbb{R},\ f^{-1}(x) = \dfrac{2-x}{3x-3}$

ii $f^{-1} : \mathbb{R} \setminus \left\{\dfrac{3}{2}\right\} \to \mathbb{R},\ f^{-1}(x) = \dfrac{3x+2}{2x-3}$

iii $f^{-1} : \mathbb{R} \setminus \{-1\} \to \mathbb{R},\ f^{-1}(x) = \dfrac{1-x}{x+1}$

iv $f^{-1} : \mathbb{R} \setminus \{-1\} \to \mathbb{R},\ f^{-1}(x) = \dfrac{1-x}{x+1}$

c For $a, b, c, d \in \mathbb{R} \setminus \{0\}$, $f = f^{-1}$ when $a = -d$

11 a i $YB = r$ **ii** $ZB = r$
iii $AZ = x - r$ **iv** $CY = 3 - r$

b $r = \dfrac{x + 3 - \sqrt{x^2 + 9}}{2}$

c i $r = 1$ **ii** $x = 1.25$

12 b $f(x) = \dfrac{q}{x}$

c i $f^{-1}(x) = \dfrac{3x+8}{x-3} = f(x)$

ii $x = 3 \pm \sqrt{17}$

13 a i $f(2) = 3,\ f(f(2)) = 2,\ f(f(f(2))) = 3$

ii $f(f(x)) = x$

b $f(f(x)) = \dfrac{-x-3}{x-1}$,
$f(f(f(x))) = x$, i.e. $f(f(x)) = f^{-1}(x)$

Chapter 2

Exercise 2A

1 a 10 **b** 1 **c** 4 **d** 28 **e** $8\frac{1}{2}$
f $\dfrac{17}{9}$ **g** $\dfrac{7}{5}$ **h** 21 **i** 2 **j** $\dfrac{7}{2}$

2 a $x = 12, y = 8$ **b** $x = 5, y = -8$
c $x = 3, y = 1$ **d** $x = 2, y = 1$
e $x = 17, y = -19$ **f** $x = 10, y = 6$

3 Width = 6 cm, length = 10 cm

4 John scored 4, David 8

5 a $w = 20n + 800$ **b** \$1400 **c** 41 units

6 a $V = 15t + 250$ **b** 1150 litres
c 5 hours, 16 minutes and 40 seconds

7 a $V = 10\,000 - 10t$ **b** 9400 litres
c 16 hours and 40 minutes

8 80 km

9 96 km

10 a $C = 25t + 100$
b i \$150 **ii** \$162.50
c i 11 hours **ii** 12 hours

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Exercise 2B

1 **a** $x = \frac{m-n}{a}$ **b** $x = \frac{b}{b-a}$ **c** $x = -\frac{bc}{a}$
d $x = \frac{5}{p-q}$ **e** $x = \frac{m+n}{n-m}$ **f** $x = \frac{ab}{1-b}$
g $x = 3a$ **h** $x = -mn$ **i** $x = \frac{a^2-b^2}{2ab}$
j $x = \frac{p-q}{p+q}$ **k** $x = \frac{3ab}{b-a}$ **l** $x = \frac{1}{3a-b}$
m $x = \frac{p^2+p^2t+t^2}{q(p+t)}$ **n** $x = -\frac{5a}{3}$

2 **a** $x = \frac{d-bc}{1-ab}, y = \frac{c-ad}{1-ab}$
b $x = \frac{a^2+ab+b^2}{a+b}, y = \frac{ab}{a+b}$
c $x = \frac{t+s}{2a}, y = \frac{t-s}{2b}$
d $x = a+b, y = a-b$
e $x = c, y = -a$
f $x = a+1, y = a-1$

3 **a** $s = a(2a+1)$ **b** $s = \frac{2a^2}{1-a}$
c $s = \frac{a^2+a+1}{a(a+1)}$ **d** $s = \frac{a}{(a-1)^2}$
e $s = 3a^3(3a+1)$ **f** $s = \frac{3a}{a+2}$
g $s = 2a^2 - 1 + \frac{1}{a^2}$ **h** $s = \frac{5a^2}{a^2+6}$

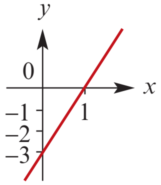
Exercise 2C

1 **a** $\sqrt{205}$ **b** $\left(1, -\frac{1}{2}\right)$ **c** $-\frac{13}{6}$
d $13x + 6y = 10$ **e** $13x + 6y = 43$
f $13y - 6x = -\frac{25}{2}$

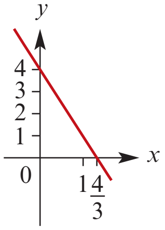
2 **a** $(3, 7\frac{1}{2})$ **b** $\left(-\frac{5}{2}, -2\right)$ **c** $\left(\frac{3}{2}, \frac{1}{2}\right)$

3 **a** $(4, 7)$ **b** $(5, -2)$ **c** $(2, 19)$ **d** $(-2, -9)$

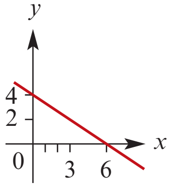
4 **a**



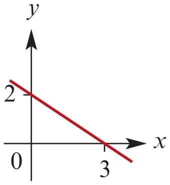
b



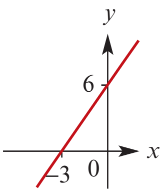
c



d



e



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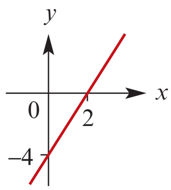
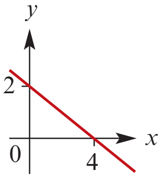
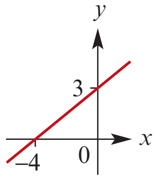


5 **a** $2x - 6 = y$ **b** $-3x - 5 = y$
c $5 = 3y - 4x$ **d** $y = 2x + 1$

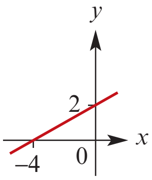
6 **a** $\frac{y}{2} - \frac{x}{3} = 1$ **b** $\frac{x}{4} + \frac{y}{6} = 1$
c $-\frac{x}{4} - \frac{y}{3} = 1$ **d** $\frac{x}{6} - \frac{y}{2} = 1$

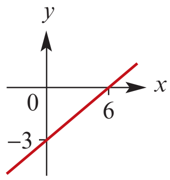
7 **a** $\frac{x}{4} + \frac{y}{2} = 1$ **b** $\frac{y}{3} - \frac{x}{4} = 1$





c $\frac{y}{2} - \frac{x}{4} = 1$ **d** $\frac{x}{6} - \frac{y}{3} = 1$





8 $C = \frac{11}{200}n + 2$, \$57

9 **a** $C = 5n + 175$ **b** Yes **c** \$175

10 **a** $\sqrt{5} \approx 2.236$ **b** $\sqrt{2} \approx 1.414$
c $\sqrt{29} \approx 5.385$ **d** $2\sqrt{82} \approx 18.111$
e $\sqrt{20} \approx 4.472$ **f** 5

11 **a** **i** $y = 2x + 4$ **ii** $2y + x = 13$
b **i** $y = -2x + 7$ **ii** $2y - x = 4$

12 $y = 2x - 3$ **13** $y = -5$ or $y = 3$

14 $y = 12$ or $y = 0$ **15** $y = 32$ or $y = -16$

16 **a** **i** $5y + 2x = 13$ **ii** $5y + 4x = 11$
b **i** $2y - 5x = 11$ **ii** $4y - 5x = 17$

17 **a** 32.01° **b** 153.43° **c** 56.31° **d** 120.96°

18 45° **19** $a = -12$ or $a = 8$

20 **a** $y = 3x - 6$ **b** $(2, 0)$

21 $k = 5$ and $h = 4$, or $k = -2$ and $h = -3$

22 **a** $a + 2$ **b** $\frac{4}{5}$

23 **a** $m = \frac{1}{2}$ **b** $(5, 7)$
c $AB = \sqrt{13}, AC = 2\sqrt{13}$

24 **a** $3y - x = 22$ **b** $(14, 12)$
c $(16, 6)$ **d** 80 square units

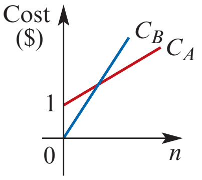
25 **a** $(2, 3)$
b $y + 5x = 13$
c **i** $2y = 3x - 13$ **ii** $(3, -2)$ **iii** $(1, 8)$

Exercise 2D

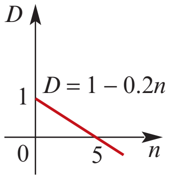
1 a $C_A = 0.4n + 1$

$C_B = 0.6n$

b



c $n = 5$ **d**



The difference in charges against kilometres travelled

2 a $4 - T$

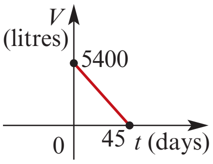
b i $90T$ **ii** $70(4 - T)$

c i $T = 1$

ii 90 km freeway; 210 km country roads

3 a $L = -120t + 5400$ **b** 5400 litres

c



d $[0, 45]$

e 45 days **f** 120 litres per day

4 a $y = \frac{9}{4}x$ **b** 24 622 m

c $y = -\frac{27}{26}x + \frac{855}{26}$ **d** $\frac{5393}{108}$

5 a i -4 **ii** $\frac{4}{9}$

b i $y = \frac{4}{9}x + \frac{10}{3}$ **ii** $y = -4x + 30$

c AC: $y = x$; BD: $y = 4$

d $(4, 4)$

6 a $M(7, 5)$, $N(11, 5)$

b i $y = \frac{5}{2}(x - 5)$ **ii** $y = -\frac{5}{2}(x - 13)$ **iii** $y = 5$

c $y - 5 = -\frac{2}{5}(x - 7)$ and $y - 5 = \frac{2}{5}(x - 11)$

Intersection point $\left(9, \frac{21}{5}\right)$

Exercise 2E

1 a $\begin{bmatrix} 3 & 2 \\ 0 & 2 \end{bmatrix}$ **b** $\begin{bmatrix} -2 \\ 0 \end{bmatrix}$

2 $\begin{bmatrix} -1 & -10 \\ 1 & -2 \end{bmatrix}$ **3** $\begin{bmatrix} 10 \\ 24 \end{bmatrix}$

4 a $\begin{bmatrix} 0 & -1 \\ 6 & 4 \end{bmatrix}$ **b** $\begin{bmatrix} -1 & -2 \\ 0 & -2 \end{bmatrix}$ **c** $\begin{bmatrix} -10 & -6 \\ 12 & 7 \end{bmatrix}$

d $\begin{bmatrix} 4 & 3 \\ 0 & 0 \end{bmatrix}$ **e** $\begin{bmatrix} 2k & k \\ 3k & 2k \end{bmatrix}$ **f** $\begin{bmatrix} -2 & -4 \\ 15 & 10 \end{bmatrix}$

g $\begin{bmatrix} 6 & 5 \\ -3 & -2 \end{bmatrix}$

5 a $\begin{bmatrix} 6 & 8 \\ -6 & -6 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & -12 \\ 15 & 3 \end{bmatrix}$

c $\begin{bmatrix} 6 & -4 \\ 9 & -3 \end{bmatrix}$ **d** $\begin{bmatrix} -6 & -20 \\ 21 & 9 \end{bmatrix}$

6 a $\begin{bmatrix} 0 & -4 \\ 5 & -4 \end{bmatrix}$ **b** $\begin{bmatrix} -2 & -12 \\ 15 & -4 \end{bmatrix}$ **c** $\begin{bmatrix} 3 & 0 \\ -4 & -7 \end{bmatrix}$

7 $\mathbf{X} = \begin{bmatrix} 2 & 3 \\ -\frac{4}{3} & -3 \end{bmatrix}$ and $\mathbf{Y} = \begin{bmatrix} -\frac{9}{2} & -8 \\ \frac{5}{2} & 7 \end{bmatrix}$

8 $\begin{bmatrix} 4 \\ -5 \end{bmatrix}$ $\begin{bmatrix} 6 \\ -1 \end{bmatrix}$ $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$ $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 3 & -2 \\ -3 & 3 \end{bmatrix}$

$\begin{bmatrix} 3 & -6 \\ -1 & 3 \end{bmatrix}$ $\begin{bmatrix} 3 & -8 \\ -4 & 11 \end{bmatrix}$ $\begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}$

Exercise 2F

1 a $x = 4,\ y = -3$ **b** $x = \frac{-3}{2},\ y = \frac{1}{2}$

c $x = \frac{51}{38},\ y = \frac{-31}{38}$ **d** $x = \frac{37}{10},\ y = \frac{7}{5}$

2 a one solution **b** infinitely many solutions

c no solutions

3 Their graphs are parallel straight lines that do not coincide

4 $x = t + 6,\ y = t,$ where $t \in \mathbb{R}$

5 a $m = -5$ **b** $m = 3$

6 $m = 9$

7 a i $m = -2$ **ii** $m = 4$

b $x = \frac{4}{m + 2},\ y = \frac{2(m + 4)}{m + 2}$

8 a $x = 2,\ y = 0,\ k \neq \frac{-3}{2}$ **b** $k = \frac{-3}{2}$

9 a $b \in \mathbb{R} \setminus \{10\}$ **b** $b = 10,\ c = 8$

c $b = 10,\ c \neq 8$

Exercise 2G

1 a $x = 2, y = 3, z = 1$ **b** $x = -3, y = 5, z = 2$

c $x = 5, y = 0, z = 7$ **d** $x = 6, y = 5, z = 1$

2 a $y = 4z - 2$

b $x = 8 - 5\lambda,\ y = 4\lambda - 2,\ z = \lambda$

3 a $-y + 5z = 15,\ -y + 5z = 15$

b The two equations are the same

c $y = 5\lambda - 15$ **d** $x = 43 - 13\lambda$

4 a $x = \lambda - 1,\ y = \lambda,\ z = 5$

b $x = \lambda + 3,\ y = 3\lambda,\ z = \lambda$

c $x = \frac{14 - 3\lambda}{6},\ y = \frac{10 - 3\lambda}{6},\ z = \lambda$

5 $z = t,\ y = \frac{-3(t + 2)}{4},\ x = \frac{26 - 3t}{4},\ w = \frac{t - 2}{2},$

where $t \in \mathbb{R}$; $w = 6, x = -4, y = -12, z = 14$

6 a $x = 1,\ y = 2,\ z = 3$

b $x = \frac{-5}{3},\ y = \frac{-(3\lambda + 5)}{3},\ z = \lambda$

c $z = t,\ y = -2(t - 1),\ x = \frac{2 - 3t}{2}$

Chapter 2 review

Technology-free questions

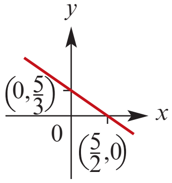
1 a -8 **b** $\frac{7}{5}$ **c** 30 **d** 7

2 a $x = -2,\ y = 2$ **b** $x = -44,\ y = -39$

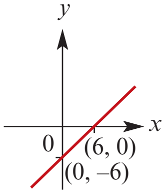
3 a $\frac{n + m}{b}$ **b** $\frac{b}{c + b}$ **c** d

d $\frac{6}{q - p}$ **e** $\frac{m + n}{m - n}$ **f** $\frac{a^2}{a - 1}$

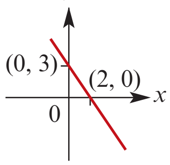
4 a



b



c



5 a $y = -2x + 5$ **b** $y = 2x + 2$

c $y = \frac{1}{2}x + \frac{1}{2}$ **d** $y = -2x + 3$

6 $\sqrt{13}$

7 $(1, 7)$

8 a $(22, 4)$ **b** $(5, -12)$

9 a $\begin{bmatrix} 6 & 1 \\ 4 & 4 \end{bmatrix}$ **b** $\begin{bmatrix} 9 & 2 \\ 14 & 4 \end{bmatrix}$ **c** $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$

d $\begin{bmatrix} -8 \\ 4 \end{bmatrix}$ **e** $\begin{bmatrix} -6 \\ 9 \end{bmatrix}$ **f** $\begin{bmatrix} 8 & 4 \\ 8 & 5 \end{bmatrix}$

g $\begin{bmatrix} -2 & 1 \\ 2 & 0 \end{bmatrix}$ **h** $\begin{bmatrix} 2k & k \\ 3k & 2k \end{bmatrix}$ **i** $\begin{bmatrix} 16 & 2 \\ 9 & 10 \end{bmatrix}$

j $\begin{bmatrix} -6 & 1 \\ 1 & -2 \end{bmatrix}$

10 $y = 24$ or $y = 0$

11 a $m = -2$ **b** $m \in \mathbb{R} \setminus \{-8, -2\}$

12 a $x = \frac{3\lambda - 1}{2},\ y = \lambda,\ z = 7$, where $\lambda \in \mathbb{R}$

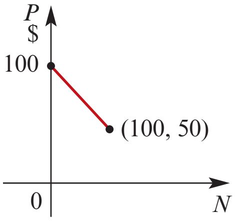
b $x = \frac{4 - \lambda}{2},\ y = \frac{3\lambda + 8}{2},\ z = \lambda$, where $\lambda \in \mathbb{R}$

Multiple-choice questions

1 E **2** E **3** D **4** C **5** B
6 A **7** A **8** C **9** B **10** C
11 D **12** A **13** D **14** C

Extended-response questions

1 a

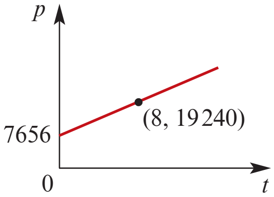


b $P = -\frac{1}{2}N + 100$

c i \$56 **ii** $N = 80$

2 a $p = 1448t + 7656$

b



c 22 136 **d** 1448 people per year

3 a $y = \frac{5}{3}x - 4$ **b** $\left(\frac{66}{7}, \frac{82}{7}\right)$

c $\frac{5}{3}$ **d** 15 **e** $\frac{629}{14}$ square units

4 a $y = \frac{4}{7}x + \frac{31}{14}$ **b** $\frac{59}{14}$ **c** $\sqrt{65}$

d $\frac{65}{28}$ square units

5 a $(1, -\frac{1}{2})$ **b** $\sqrt{269}$ **c** $\sqrt{269}$

d $y = -\frac{13}{10}x + \frac{4}{5}$ **e** $y = \frac{10}{13}x - \frac{33}{26}$

f $\left(\frac{7}{2}, -\frac{15}{4}\right)$ **g** $(26, -33)$

6 a 125 litres **b** $x = 291\frac{2}{3},\ y = 208\frac{1}{3}$

Chapter 3

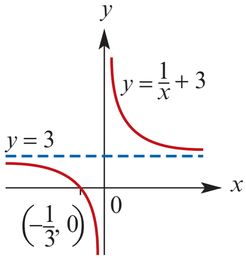
Exercise 3A

1 a $(-1, 3)$ **b** $(-5, 10)$ **c** $(-3, -1)$
d $(-5, 7)$ **e** $(-3, 6)$

2 a $y = \frac{1}{x - 2} - 3$ **b** $y = \frac{1}{x + 2} + 3$

c $y = \frac{1}{x - \frac{1}{2}} + 4 = \frac{2}{2x - 1} + 4$

3 a Domain $= \mathbb{R} \setminus \{0\}$
Range $= \mathbb{R} \setminus \{3\}$



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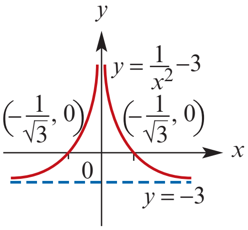
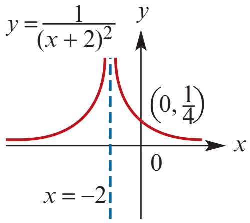
b Domain = $\mathbb{R} \setminus \{0\}$

Range = $(-3, \infty)$



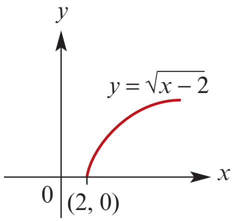
c Domain = $\mathbb{R} \setminus \{-2\}$

Range = $\mathbb{R}^+$



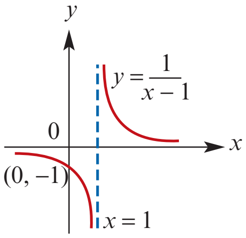
d Domain = $[2, \infty)$

Range = $\mathbb{R}^+ \cup \{0\}$



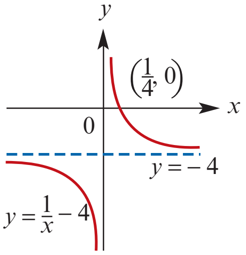
e Domain = $\mathbb{R} \setminus \{1\}$

Range = $\mathbb{R} \setminus \{0\}$



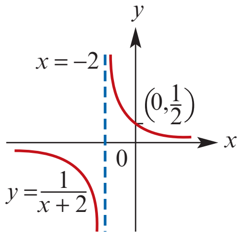
f Domain = $\mathbb{R} \setminus \{0\}$

Range = $\mathbb{R} \setminus \{-4\}$



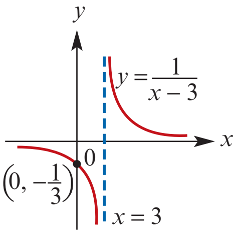
g Domain = $\mathbb{R} \setminus \{-2\}$

Range = $\mathbb{R} \setminus \{0\}$



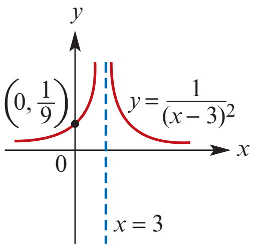
h Domain = $\mathbb{R} \setminus \{3\}$

Range = $\mathbb{R} \setminus \{0\}$



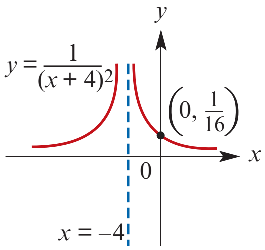
i Domain = $\mathbb{R} \setminus \{3\}$

Range = $\mathbb{R}^+$



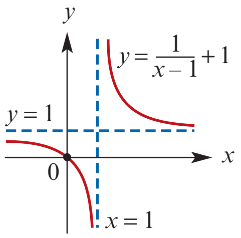
j Domain = $\mathbb{R} \setminus \{-4\}$

Range = $\mathbb{R}^+$



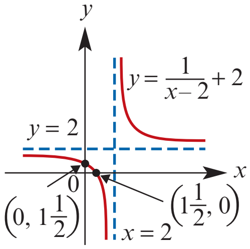
k Domain = $\mathbb{R} \setminus \{1\}$

Range = $\mathbb{R} \setminus \{1\}$

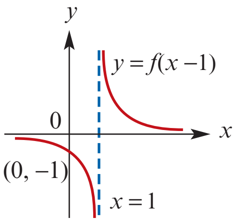


l Domain = $\mathbb{R} \setminus \{2\}$

Range = $\mathbb{R} \setminus \{2\}$



4 a



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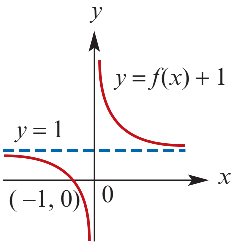

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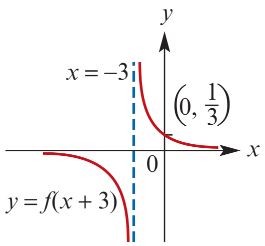

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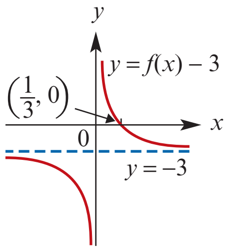

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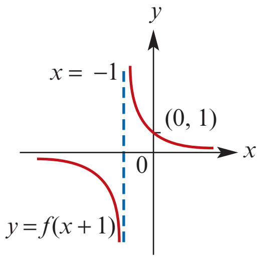

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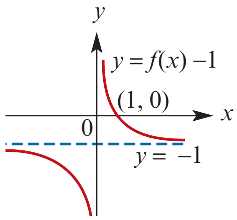


5 **a** Translation $(x, y) \rightarrow (x - 5, y)$
b Translation $(x, y) \rightarrow (x, y + 2)$
c Translation $(x, y) \rightarrow (x, y + 4)$
d Translation $(x, y) \rightarrow (x, y + 3)$
e Translation $(x, y) \rightarrow (x - 3, y)$

6 **a** **i** $y = (x - 7)^{\frac{1}{4}} + 1$ **ii** $y = (x + 2)^{\frac{1}{4}} - 6$
iii $y = (x - 2)^{\frac{1}{4}} - 3$ **iv** $y = (x + 1)^{\frac{1}{4}} + 4$
b **i** $y = \sqrt[3]{x - 7} + 1$ **ii** $y = \sqrt[3]{x + 2} - 6$
iii $y = \sqrt[3]{x - 2} - 3$ **iv** $y = \sqrt[3]{x + 1} + 4$
c **i** $y = \frac{1}{(x - 7)^3} + 1$ **ii** $y = \frac{1}{(x + 2)^3} - 6$
iii $y = \frac{1}{(x - 2)^3} - 3$ **iv** $y = \frac{1}{(x + 1)^3} + 4$
d **i** $y = \frac{1}{(x - 7)^4} + 1$ **ii** $y = \frac{1}{(x + 2)^4} - 6$
iii $y = \frac{1}{(x - 2)^4} - 3$ **iv** $y = \frac{1}{(x + 1)^4} + 4$

7 **a** $y = (x + 1)^2 + 5$ **b** $y = 2x^2$
c $y = \frac{1}{(x - 6)^2} + 1$ **d** $y = (x + 3)^2 + 2$
e $y = \sqrt[3]{x - 2} + 3$

8 **a** $(x, y) \rightarrow (x + 2, y + 3)$
b $(x, y) \rightarrow (x - 2, y - 3)$
c $(x, y) \rightarrow (x - 4, y + 2)$

Exercise 3B

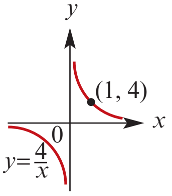
1 **a** $y = \frac{3}{x}$ **b** $y = \frac{3}{x}$

2 **a** $y = \frac{2}{x^2}$ **b** $y = \frac{4}{x^2}$

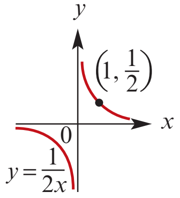
3 **a** $y = 2\sqrt{x}$ **b** $y = \sqrt{\frac{x}{2}}$

4 **a** $y = 2x^3$ **b** $y = \frac{x^3}{8}$

5 **a**



b



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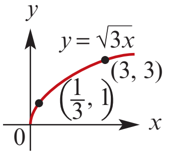

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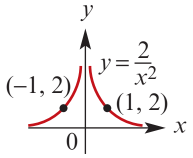
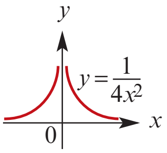


6 **a**



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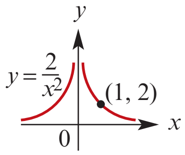

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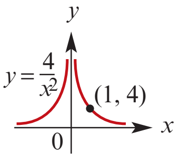

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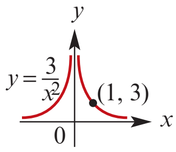

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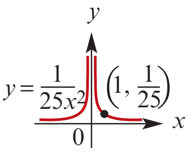

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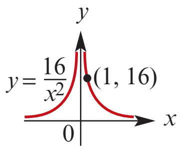

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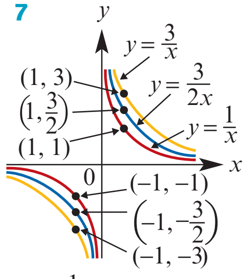

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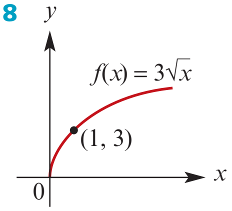

Range = $\mathbb{R}^+$

9 **a** $\frac{1}{5}$ **b** $\sqrt{5}$

10 **a** Dilation of factor 5 from the x-axis
b Dilation of factor 4 from the x-axis

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c Dilation of factor $\frac{1}{5}$ from the y-axis

d Dilation of factor $\frac{1}{3}$ from the y-axis

e Dilation of factor 2 from the y-axis

11 a i $y = 4x^2$ **ii** $y = \frac{2}{3}x^2$

iii $y = 4x^2$ **iv** $y = \frac{1}{25}x^2$

b i $y = \frac{4}{x^2}$ **ii** $y = \frac{2}{3x^2}$

iii $y = \frac{1}{4x^2}$ **iv** $y = \frac{25}{x^2}$

c i $y = 4\sqrt[3]{x}$ **ii** $y = \frac{2}{3} \times \sqrt[3]{x}$

iii $y = \sqrt[3]{2x}$ **iv** $y = \sqrt[3]{\frac{x}{5}}$

d i $y = \frac{4}{x^3}$ **ii** $y = \frac{2}{3x^3}$

iii $y = \frac{1}{8x^3}$ **iv** $y = \frac{125}{x^3}$

e i $y = \frac{4}{x^4}$ **ii** $y = \frac{2}{3x^4}$

iii $y = \frac{1}{16x^4}$ **iv** $y = \frac{625}{x^4}$

f i $y = 4\sqrt[4]{x}$ **ii** $y = \frac{2}{3} \times \sqrt[4]{x}$

iii $y = \sqrt[4]{2x}$ **iv** $y = \sqrt[4]{\frac{x}{5}}$

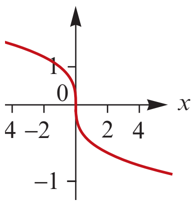
g i $y = 4x^{\frac{1}{5}}$ **ii** $y = \frac{2}{3}x^{\frac{1}{5}}$

iii $y = (2x)^{\frac{1}{5}}$ **iv** $y = \left(\frac{x}{5}\right)^{\frac{1}{5}}$

Exercise 3C

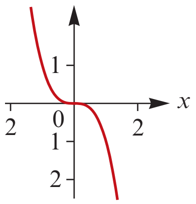
1 a $y = -(x-1)^2$ **b** $y = (x+1)^2$

2 a



Domain = $\mathbb{R}$

b



Domain = $\mathbb{R}$

3 Reflection in the y-axis

4 a i $y = -x^3$ **ii** $y = -x^3$

b i $y = -\sqrt[3]{x}$ **ii** $y = -\sqrt[3]{x}$

c i $y = \frac{-1}{x^3}$ **ii** $y = \frac{-1}{x^3}$

d i $y = \frac{-1}{x^4}$ **ii** $y = \frac{1}{x^4}$

e i $y = -x^{\frac{1}{3}}$ **ii** $y = -x^{\frac{1}{3}}$

f i $y = -x^{\frac{1}{5}}$ **ii** $y = -x^{\frac{1}{5}}$

g i $y = -x^{\frac{1}{4}}$ **ii** $y = (-x)^{\frac{1}{4}}$

Exercise 3D

1 a i $y = 2(x-2)^2 - 3$ **ii** $y = \left(\frac{x+2}{3}\right)^2 - 4$

iii $y = 2x^2$

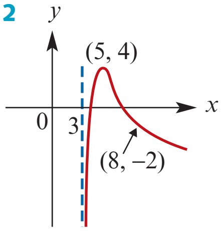
b i $y = 2\sqrt[3]{x-2} - 3$ **ii** $y = \sqrt[3]{\frac{x+2}{3}} - 4$

iii $y = -2\sqrt[3]{x}$

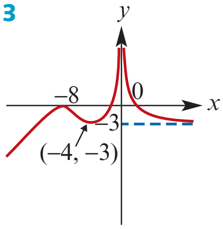
c i $y = \frac{2}{(x-2)^2} - 3$ **ii** $y = \frac{9}{(x+2)^2} - 4$

iii $y = \frac{2}{x^2}$

2



3



4 a i $y = -2(x-3)^2 - 4$

ii $y = -2(x-3)^2 + 4$

iii $y = -2(x-3)^2 - 4$

iv $y = -2(x-3)^2 - 8$

v $y = -2(x-3)^2 + 8$

vi $y = -2(x-3)^2 + 8$

b i $y = -2\sqrt[3]{x-3} - 4$

ii $y = -2\sqrt[3]{x-3} + 4$

iii $y = -2\sqrt[3]{x-3} - 4$

iv $y = -2\sqrt[3]{x-3} - 8$

v $y = -2\sqrt[3]{x-3} + 8$

vi $y = -2\sqrt[3]{x-3} + 8$

c i $y = \frac{-2}{(x-3)} - 4$ **ii** $y = \frac{-2}{(x-3)} + 4$

iii $y = \frac{-2}{(x-3)} - 4$ **iv** $y = \frac{-2}{(x-3)} - 8$

v $y = \frac{-2}{(x-3)} + 8$ **vi** $y = \frac{-2}{(x-3)} + 8$

d i $y = -2(x-3)^4 - 4$

ii $y = -2(x-3)^4 + 4$

iii $y = -2(x-3)^4 - 4$

iv $y = -2(x-3)^4 - 8$

v $y = -2(x-3)^4 + 8$

vi $y = -2(x-3)^4 + 8$

e i $y = \frac{-2}{(x-3)^3} - 4$ **ii** $y = \frac{-2}{(x-3)^3} + 4$

iii $y = \frac{-2}{(x-3)^3} - 4$ **iv** $y = \frac{-2}{(x-3)^3} - 8$

v $y = \dfrac{-2}{(x-3)^3} + 8$ **vi** $y = \dfrac{-2}{(x-3)^3} + 8$

f **i** $y = \dfrac{-2}{(x-3)^4} - 4$ **ii** $y = \dfrac{-2}{(x-3)^4} + 4$

iii $y = \dfrac{-2}{(x-3)^4} - 4$ **iv** $y = \dfrac{-2}{(x-3)^4} - 8$

v $y = \dfrac{-2}{(x-3)^4} + 8$ **vi** $y = \dfrac{-2}{(x-3)^4} + 8$

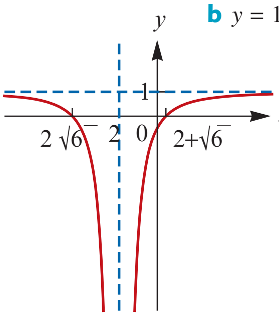
g **i** $y = \dfrac{-2}{(x-3)^2} - 4$ **ii** $y = \dfrac{-2}{(x-3)^2} + 4$

iii $y = \dfrac{-2}{(x-3)^2} - 4$ **iv** $y = \dfrac{-2}{(x-3)^2} - 8$

v $y = \dfrac{-2}{(x-3)^2} + 8$ **vi** $y = \dfrac{-2}{(x-3)^2} + 8$

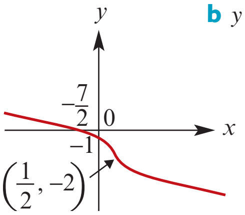
5 $y = -\sqrt{\dfrac{x+12}{3}}$

6 **a**



b $y = 1 - \dfrac{6}{(x+2)^2}$

7 **a**



b $y = (1-2x)^{\frac{1}{3}} - 2$

Exercise 3E

1 **a** **i** Dilation of factor 2 from the x-axis, then translation 1 unit to the right and 3 units up

ii Reflection in the x-axis, then translation 1 unit to the left and 2 units up

iii Dilation of factor $\frac{1}{2}$ from the y-axis, then translation $\frac{1}{2}$ unit to the left and 2 units down

b **i** Dilation of factor 2 from the x-axis, then translation 3 units to the left

ii Translation 3 units to the left and 2 units up

iii Translation 3 units to the right and 2 units down

c **i** Translation 3 units to the left and 2 units up

ii Dilation of factor $\frac{1}{3}$ from the y-axis and dilation of factor 2 from the x-axis

iii Reflection in the x-axis, then translation 2 units up

2 **a** Translation 1 unit to the left and 6 units down

b Dilation of factor $\frac{1}{2}$ from the x-axis, then translation $\frac{3}{2}$ units up and 1 unit to the left

c Translation 1 unit to the left and 6 units up

d Dilation of factor $\frac{1}{2}$ from the x-axis, then translation $\frac{5}{2}$ units up and 1 unit to the left

e Dilation of factor 2 from the y-axis, then translation of 1 unit to the left and 6 units down

3 **a** Dilation of factor $\frac{1}{5}$ from the x-axis, then translation $\frac{7}{5}$ units up and 3 units to the left

b Dilation of factor 3 from the y-axis, then translation 2 units to the right and 5 units down

c Reflection in the x-axis, dilation of factor $\frac{1}{3}$ from the x-axis, translation $\frac{7}{3}$ units up, dilation of factor 3 from the y-axis, translation 1 unit to the right

d Reflection in the y-axis, translation 4 units to the right, dilation of factor $\frac{1}{2}$ from the x-axis

e Reflection in the y-axis, translation 4 units to the right, reflection in the x-axis, dilation of factor $\frac{1}{2}$ from the x-axis, translation $\frac{15}{2}$ units up

4 **a** Dilation of factor 2 from the x-axis, then translation 1 unit to the right and 3 units up

b Dilation of factor 2 from the x-axis, then translation 4 units to the left and 7 units down

c Reflection in the y-axis and dilation of factor 4 from the x-axis (in either order), then translation 1 unit to the right and 5 units down

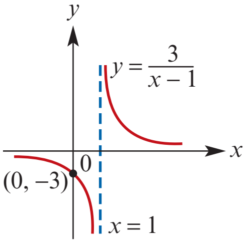
d Reflection in the x-axis, then translation 1 unit to the left and 2 units up

e Reflection in the y-axis and dilation of factor 2 from the x-axis (in either order), then translation 3 units up

f Translation 3 units to the left and 4 units down, then reflection in either axis and dilation of factor $\frac{1}{2}$ from the x-axis (in either order)

Exercise 3F

1 **a**



Range = $\mathbb{R} \setminus \{0\}$

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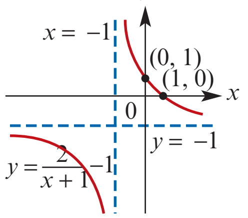
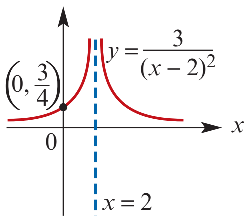
b



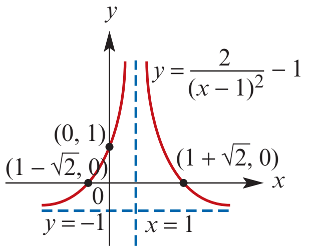
Range = $\mathbb{R} \setminus \{-1\}$

c



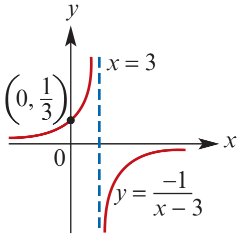
Range = $\mathbb{R}^+$

d



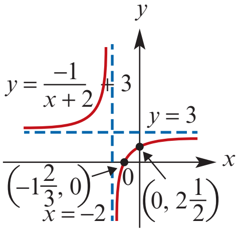
Range = $(-1, \infty)$

e



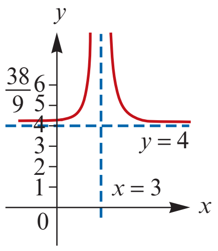
Range = $\mathbb{R} \setminus \{0\}$

f



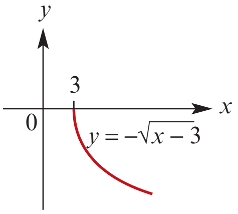
Range = $\mathbb{R} \setminus \{3\}$

g



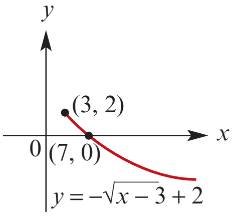
Range = $(4, \infty)$

2 a



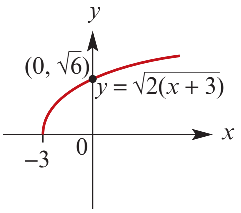
Range = $\mathbb{R}^- \cup \{0\}$

b



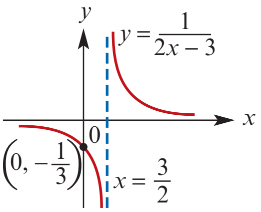
Range = $(-\infty, 2]$

c



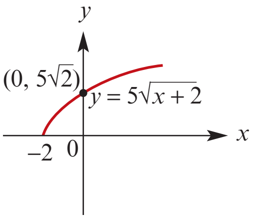
Range = $\mathbb{R}^+ \cup \{0\}$

d



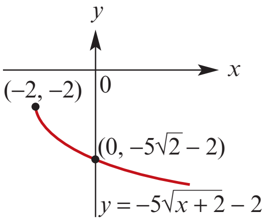
Range = $\mathbb{R} \setminus \{0\}$

e



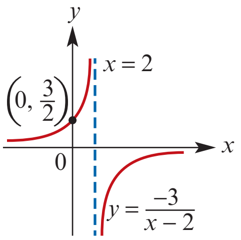
Range = $\mathbb{R}^+ \cup \{0\}$

f



Range = $(-\infty, -2]$

g



Range = $\mathbb{R} \setminus \{0\}$

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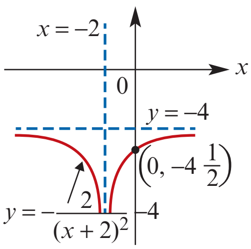
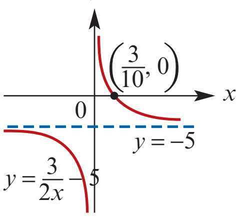
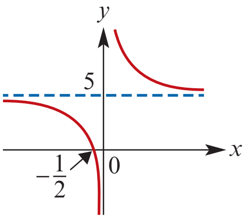


Range = $(-\infty, -4)$

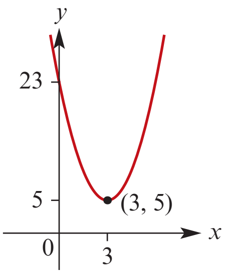
i Range = $\mathbb{R} \setminus \{-5\}$



j Range = $\mathbb{R} \setminus \{5\}$



k Range = $[5, \infty)$



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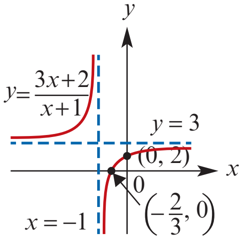

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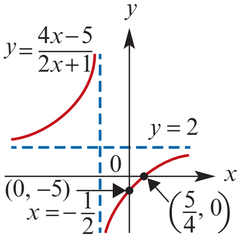
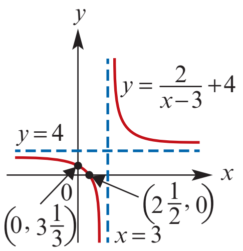


5 a Range = $\mathbb{R} \setminus \{4\}$



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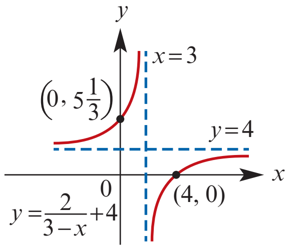
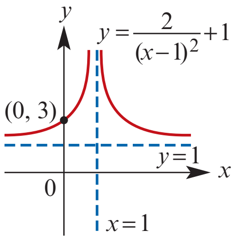
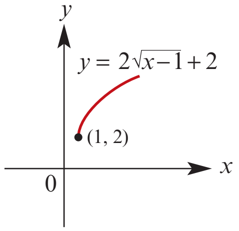


Range = $\mathbb{R} \setminus \{4\}$

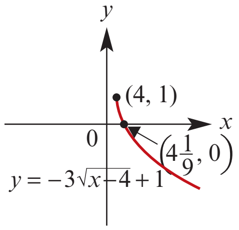
c Range = $(1, \infty)$



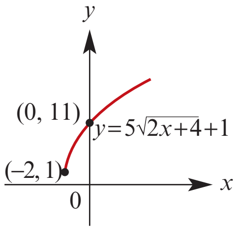
d Range = $[2, \infty)$



e Range = $(-\infty, 1]$



f Range = $[1, \infty)$



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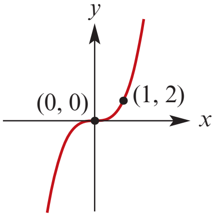
Exercise 3G

1 a



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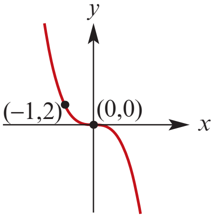
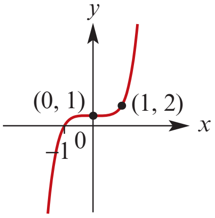


c



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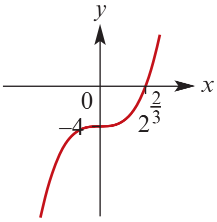

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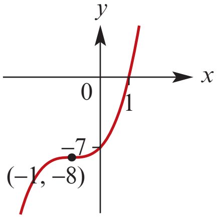

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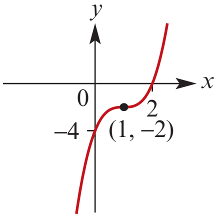

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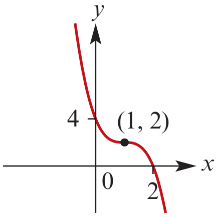

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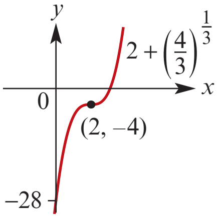

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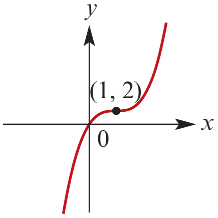

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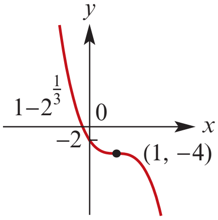

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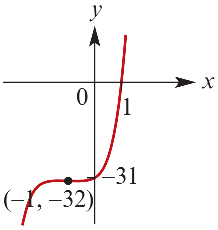

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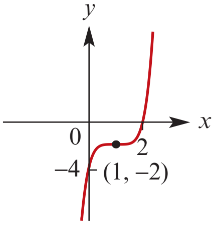

2 $a = -3,\ h = 0,\ k = 4$

3 a $y = 3x^3$ **b** $y = (x+1)^3 + 1$

c $y = -(x-2)^3 - 3$ **d** $y = 2(x+1)^3 - 2$

e $y = \dfrac{x^3}{27}$

4 a $y = \dfrac{(3-x)^3}{27} + 1$

b Dilation of factor 3 from the x-axis, reflection in the x-axis, then translation 1 unit to the left and 4 units up

5 $y = \dfrac{(x+2)^4}{16} - 1$

6 Dilation of factor 3 from the x-axis, reflection in the x-axis, then translation 1 unit to the left and 5 units up

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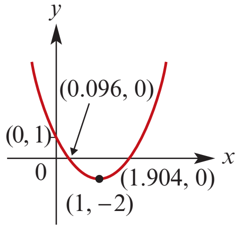
7 a



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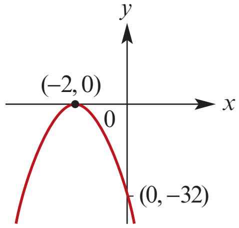

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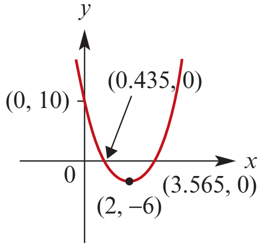

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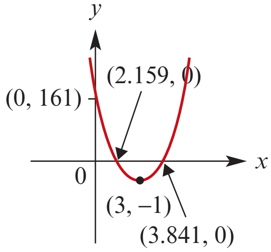
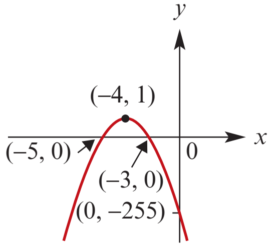


e



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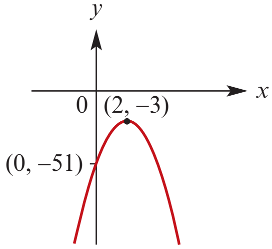

8 $a = -\frac{9}{16},\ h = -2,\ k = 3$

9 $a = 16,\ h = 1,\ k = 7$

Exercise 3H

1 $a = \frac{9}{2}, b = -\frac{1}{2}$ **2** $A = 1, b = -1, B = 2$

3 $a = \frac{5}{2}, b = -\frac{3}{2}$ **4** $A = 2, B = 3$

5 $A = 2, B = -1$ **6** $A = 8, b = 2, B = -3$

7 $a = -2, b = 1$ **8** $a = -6, b = -2$

Exercise 3I

1 a $(1, -6)$ **b** $(2, -2)$ **c** $(1, 2)$

d $(-1, -2)$ **e** $(-2, 1)$

2 a $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **c** $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

d $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ **e** $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

3 $(-6, 7)$

4 $(14, -7)$

5 $\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$, $(-3, -4)$

6 $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$

$x = x' - 2,\ y = \frac{y' - 1}{3}$

7 $y = 2\left(\frac{x^2}{16} - \frac{x}{4} + 2\right) = \frac{x^2}{8} - \frac{x}{2} + 4$

8 $y = -2(x^3 + 2x)$ **9** $y = -\frac{x}{3} + 3$

10 $y = \frac{3}{2}\left(\frac{x}{2} - 4\right)$ **11** $y = \frac{3}{2}x - 9$

12 $y = \frac{1}{6}x + \frac{11}{2}$

13 $\frac{y - 4}{2} = -2\left(\frac{x + 1}{4}\right)^3 + 6\left(\frac{x + 1}{4}\right)$

14 $\frac{2 - y}{2} = -2\left(\frac{x + 1}{3}\right)^3 + 6\left(\frac{x + 1}{3}\right)^2 + 2$

15 $y = -1 - \frac{1}{8}(x - 3)^2$ **16** $y = -1 - \frac{9}{(x + 2)^2}$

17 $y = \frac{3}{4}(x - 7)^2$

Exercise 3J

1 a 1 **b** $\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$ **c** 2 **d** $\begin{bmatrix} 1 & 1 \\ -\frac{3}{2} & -1 \end{bmatrix}$

2 a $\begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix}$ **b** $\begin{bmatrix} \frac{2}{7} & -\frac{1}{14} \\ \frac{1}{7} & \frac{3}{14} \end{bmatrix}$ **c** $\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{k} \end{bmatrix}$

3 a $\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & -1 \end{bmatrix}$, $\mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

b $\mathbf{AB} = \begin{bmatrix} 5 & 1 \\ -3 & -1 \end{bmatrix}$, $(\mathbf{AB})^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & -\frac{5}{2} \end{bmatrix}$

c $\mathbf{A}^{-1}\mathbf{B}^{-1} = \begin{bmatrix} -1 & \frac{1}{2} \\ 3 & -1 \end{bmatrix}$, $\mathbf{B}^{-1}\mathbf{A}^{-1} = (\mathbf{AB})^{-1}$

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4 a $\begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ 1 & -2 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & 7 \\ 1 & -8 \end{bmatrix}$ **c** $\begin{bmatrix} \frac{5}{2} & -\frac{7}{2} \\ \frac{11}{2} & -\frac{21}{2} \end{bmatrix}$

5 a $\frac{1}{16}\begin{bmatrix} -6 & 22 \\ 1 & 7 \end{bmatrix}$ **b** $\frac{1}{16}\begin{bmatrix} -11 & 17 \\ -4 & 12 \end{bmatrix}$

6 $y = \frac{x^2}{9} - \frac{2x}{3}$ **7** $y = -\frac{x^3}{9} - 12$

8 $10y + x = 12$ **9** $3x - 10y = 105$

10 $y + x = -12$ **11** $6y - x = 33$

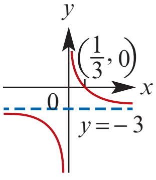
12 $y = \frac{1}{16}(-x^3 - 3x^2 + 45x + 111)$

13 $y = -\frac{2}{27}(2x^3 + 24x^2 + 42x + 47)$

Chapter 3 review

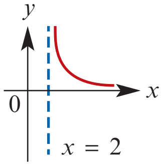
Technology-free questions

1 a



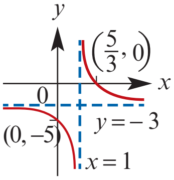
Range = $\mathbb{R} \setminus \{-3\}$

b



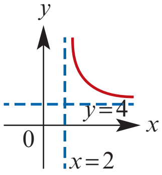
Range = $(0, \infty)$

c



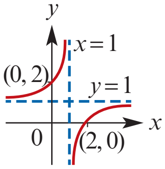
Range = $\mathbb{R} \setminus \{-3\}$

d



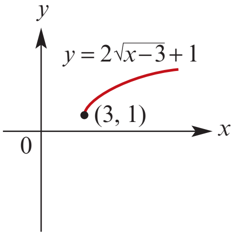
Range = $(4, \infty)$

e



Range = $\mathbb{R} \setminus \{1\}$

2 a



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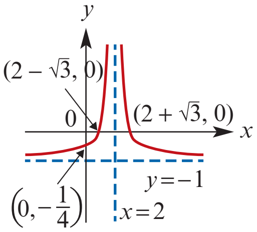
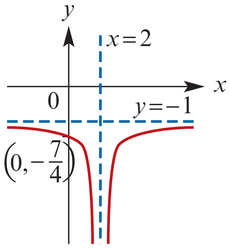
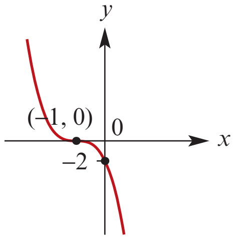


c

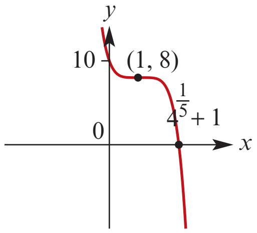


3 a



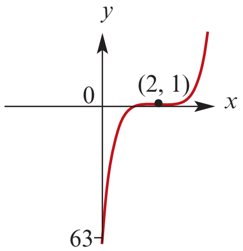
Point of zero gradient $(-1, 0)$;
Axis intercepts $(-1, 0)$, $(0, -2)$

b



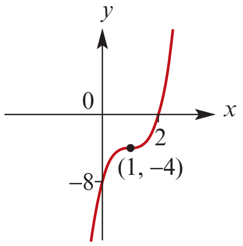
Point of zero gradient $(1, 8)$;
Axis intercepts $(4^{\frac{1}{5}}, 0)$, $(0, 10)$

c



Point of zero gradient $(2, 1)$;
Axis intercepts $(-\frac{1}{2})^{\frac{1}{5}} + 2, 0)$, $(0, -63)$

d



Point of zero gradient $(1, -4)$;
Axis intercepts $(2, 0)$, $(0, -8)$

4 $a = 2, b = 4$

5 $y = -2\sqrt{x + 4} - 1$

6 $y = -\frac{7}{2} - 2\sqrt{\frac{x - 8}{3}}$

7 $a = -6, b = 9$

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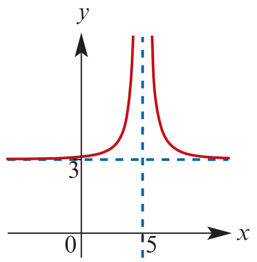
8 **a** $y = 6 - \dfrac{(x-4)^2}{4}$

b Reflection in the x-axis, dilation of factor 4 from the x-axis, then translate 1 unit to the left and 6 units up.

9 Dilation of factor 3 from the x-axis, then translation 5 units to the right and 3 units up



Asymptotes $x = 5$, $y = 3$; Intercept $\left(0, \dfrac{78}{25}\right)$

10 Dilation of factor $\frac{1}{2}$ from the x-axis, then translation $\frac{3}{2}$ units up

11 Dilation of factor $\frac{1}{2}$ from the x-axis, then translation 3 units to the left and 2 units down

Multiple-choice questions

1 B	**2** B	**3** B	**4** E	**5** D
6 A	**7** D	**8** A	**9** A	**10** A

Extended-response questions

1 **a** $\mathbb{R} \setminus \{-2\}$

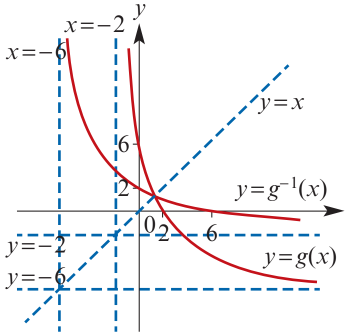
b Dilation of factor 24 from the x-axis, then translation 2 units to the left and 6 units down

c $(0, 6)$, $(2, 0)$

d $g^{-1}(x) = \dfrac{24}{x+6} - 2$

e Domain of g^{-1} = range of $g = (-6, \infty)$

f



g $x = -4 + 2\sqrt{7}$

2 **a** $[-3, \infty)$

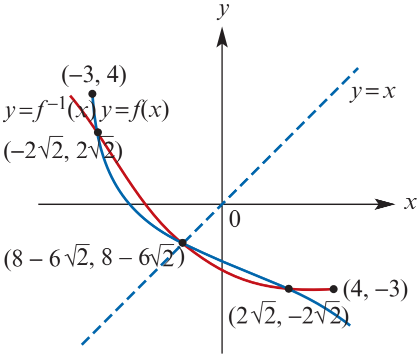
b Dilation of factor $\frac{1}{2}$ from the y-axis, dilation of factor 2 from the x-axis, reflection in the x-axis, then translation 3 units to the left and 4 units up

c $(0, 4 - 2\sqrt{6})$, $(-1, 0)$

d $f^{-1}(x) = \dfrac{(x-4)^2}{8} - 3$

e Domain of f^{-1} = range of $f = (-\infty, 4]$

f



g $x = 8 - 6\sqrt{2}$ or $x = 2\sqrt{2}$ or $x = -2\sqrt{2}$

3 **a** **i** $\dfrac{3}{125}$ **ii** $(x, y) \to (x, -y)$

iii $(x, y) \to (x + 25, y + 15)$

iv $(x, y) \to \left(x + 25, \dfrac{-3}{125}y + 15\right)$

b **i** $y = \dfrac{-3}{125}(x - 25)^2 + 15$

ii $(x, y) \to (x + 50, y)$

iii $y = \dfrac{-3}{125}(x - 75)^2 + 15$

c **i** $(x, y) \to \left(x + \dfrac{m}{2}, -\dfrac{4n}{m^2}y + n\right)$

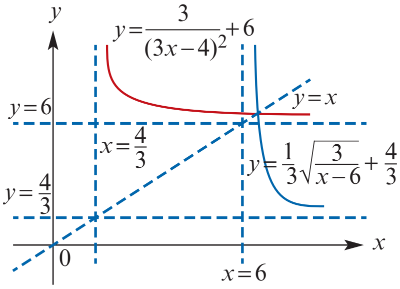
ii $y = -\dfrac{4n}{m^2}\left(x - \dfrac{m}{2}\right)^2 + n$

iii $y = -\dfrac{4n}{m^2}\left(x - \dfrac{3m}{2}\right)^2 + n$

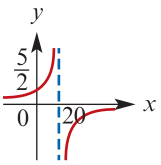
4 **a** $\mathbb{R} \setminus \left\{\dfrac{4}{3}\right\}$ **b** $\dfrac{4}{3}$

c $f^{-1}(x) = \dfrac{4}{3} + \dfrac{1}{3}\sqrt{\dfrac{3}{x-6}}$ **d** $x = 6.015$

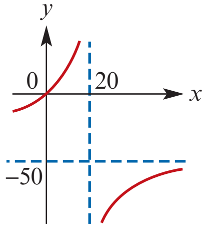
e



5 **a**



b



c $g^{-1}(x) = \dfrac{20x}{50 + x}$

6 a i $y = f^{-1}(x-5)+3$

ii $y = f^{-1}(x-3)+5$

iii $y = 5f^{-1}\left(\frac{x}{3}\right)$ **iv** $y = 3f^{-1}\left(\frac{x}{5}\right)$

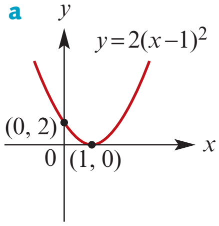
b $y = cf^{-1}\left(\frac{x-b}{a}\right)+d$

Reflection in the line $y = x$, then dilation of factor c from the x-axis and factor a from the y-axis, and a translation b units to the right and d units up

Chapter 4

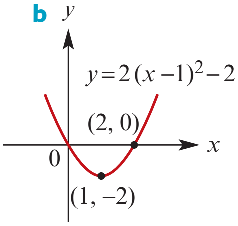
Exercise 4A

1 a



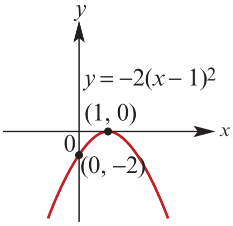
y
y=2(x−1)²
(0, 2)
0
(1, 0)
x

b



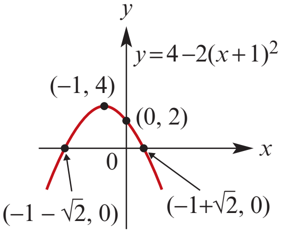
y
y=2(x−1)²−2
(2, 0)
0
x
(1, −2)

c



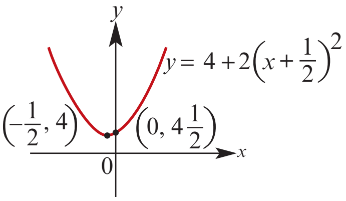
y
y=−2(x−1)²
(1, 0)
0
x
(0, −2)

d



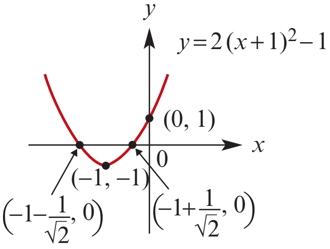
y
y=4−2(x+1)²
(−1, 4)
(0, 2)
x
0
(−1 − √2, 0)
(−1+√2, 0)

e



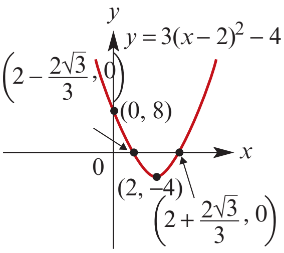
y
y = 4 + 2(x + 1/2)²
(−1/2, 4)
(0, 4 1/2)
x
0

f



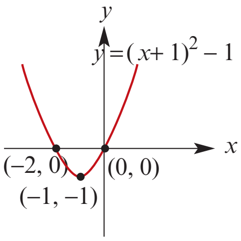
y
y=2(x+1)²−1
(0, 1)
x
0
(−1, −1)
(−1−1/√2, 0)
(−1+1/√2, 0)

g



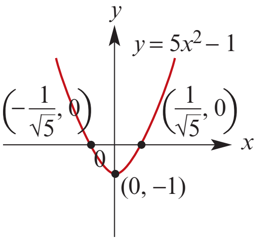
y
y=3(x−2)²−4
(2 − 2√3/3, 0)
(0, 8)
x
0
(2, −4)
(2 + 2√3/3, 0)

h



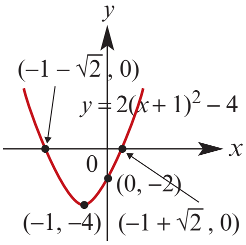
y
y=(x+1)²−1
x
(−2, 0)
(0, 0)
(−1, −1)

i



y
y = 5x² − 1
(−1/√5, 0)
(1/√5, 0)
x
0
(0, −1)

j



y
(−1 − √2, 0)
y = 2(x + 1)² − 4
x
0
(0, −2)
(−1, −4)
(−1 + √2 , 0)

2 a $f(x) = (x+\frac{3}{2})^2 - 4\frac{1}{4}$

Minimum $= -4\frac{1}{4}$; Range $= [-4\frac{1}{4}, \infty)$

b $f(x) = (x-3)^2 - 1$

Minimum $= -1$; Range $= [-1, \infty)$

c $f(x) = 2(x+2)^2 - 14$

Minimum $= -14$; Range $= [-14, \infty)$

d $f(x) = 4(x+1)^2 - 11$

Minimum $= -11$; Range $= [-11, \infty)$

e $f(x) = 2\left(x - \frac{5}{4}\right)^2 - \frac{25}{8}$

Minimum $= -\frac{25}{8}$; Range $= \left[-\frac{25}{8}, \infty\right)$

f $f(x) = -3\left(x + \frac{1}{3}\right)^2 + \frac{22}{3}$

Maximum $= \frac{22}{3}$; Range $= \left(-\infty, \frac{22}{3}\right]$

g $f(x) = -2\left(x - \frac{9}{4}\right)^2 + \frac{169}{8}$

Maximum $= \frac{169}{8}$; Range $= \left(-\infty, \frac{169}{8}\right]$

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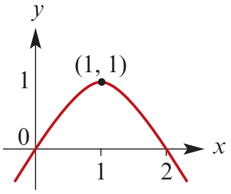
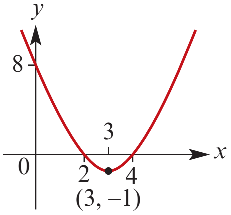
3 a



b



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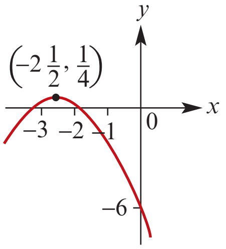
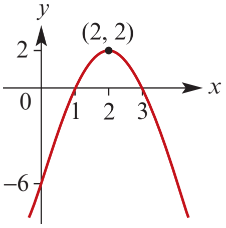
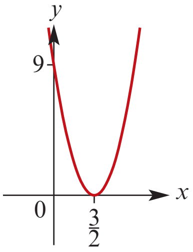

y
$\left(-2\frac{1}{2}, \frac{1}{4}\right)$
x
−3
−2
−1
0
−6

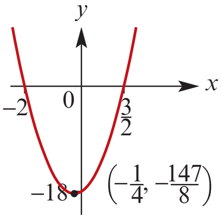
d


y
(2, 2)
2
0
1
2
3
x
−6

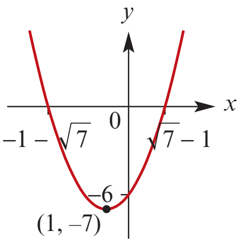
e


y
9
x
0
$\frac{3}{2}$

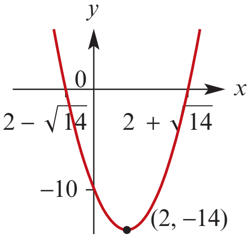
f


y
x
−2
0
$\frac{3}{2}$
−18
$\left(-\frac{1}{4}, -\frac{147}{8}\right)$

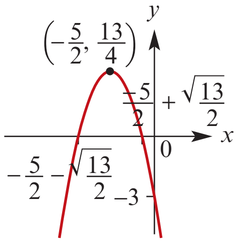
4 a $y = (x + 1)^2 - 7$


y
x
$-1-\sqrt{7}$
0
$\sqrt{7}-1$
−6
(1, −7)

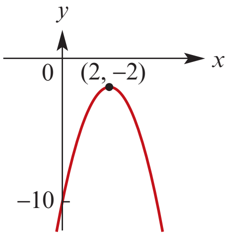
b $y = (x - 2)^2 - 14$


y
0
x
$2-\sqrt{14}$
$2+\sqrt{14}$
−10
(2, −14)

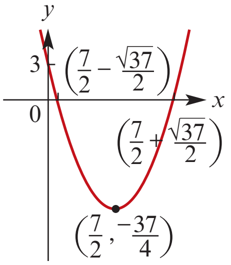
c $y = \frac{13}{4} - \left(x + \frac{5}{2}\right)^2$


y
$\left(-\frac{5}{2}, \frac{13}{4}\right)$
$-\frac{5}{2}+\frac{\sqrt{13}}{2}$
x
0
$-\frac{5}{2}-\frac{\sqrt{13}}{2}$
−3

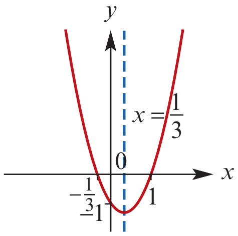
d $y = -2(x - 2)^2 - 2$



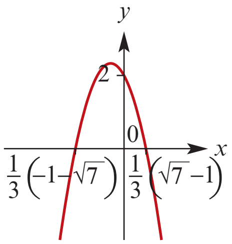
e $y = \left(x - \frac{7}{2}\right)^2 - \frac{37}{4}$


y
3
$\left(\frac{7}{2}-\frac{\sqrt{37}}{2}\right)$
x
0
$\left(\frac{7}{2}+\frac{\sqrt{37}}{2}\right)$
$\left(\frac{7}{2}, -\frac{37}{4}\right)$

5


y
$x = \frac{1}{3}$
0
x
$-\frac{1}{3}$
−1
1

6


y
2
0
x
$\frac{1}{3}\left(-1-\sqrt{7}\right)$
$\frac{1}{3}\left(\sqrt{7}-1\right)$

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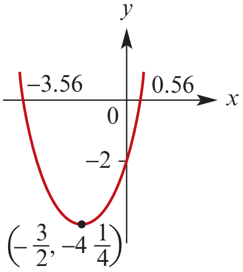
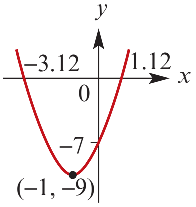
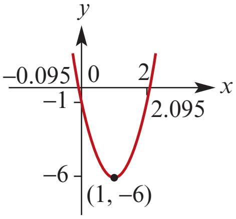
7 a



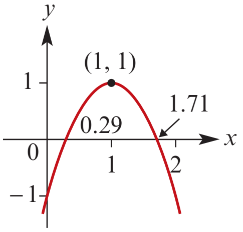
b



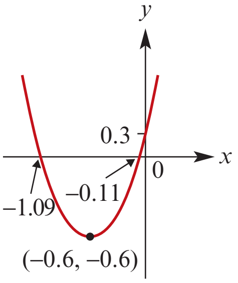
c



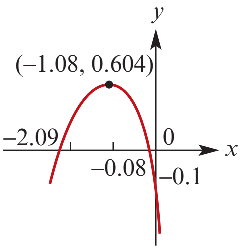
d



e



f



8 a B **b** D

9 a C **b** B **c** D **d** A

10 a Crosses the x-axis
b Does not cross the x-axis
c Just touches the x-axis
d Crosses the x-axis
e Does not cross the x-axis
f Does not cross the x-axis

11 a $m > 3$ or $m < 0$ **b** $m = 3$

12 $m = 2$ or $m = -\frac{2}{9}$

13 $a < -6$

14 Show that $\Delta > 0$ for all a

15 Show that $\Delta \geq 0$ for all k

16 a $k < -5$ or $k > 0$ **b** $k = -5$

17 a $k > -6$ **b** $k = -6$

18 Show that $\Delta \geq 0$ for all a, b

Exercise 4B

1 $y = -2(x + 3)(x + 2)$ **2** $y = (x + 3)(2x + 3)$
3 $y = \frac{3}{2}(x + 2)^2 + 4$ **4** $y = -2(x + 2)^2 - 3$
5 $y = -5x^2 + 6x + 18$ **6** $y = -2x^2 - 8x + 10$
7 a $y = 4 - \frac{4}{25}x^2$ **b** $y = -x^2$
c $y = x^2 + 2x$ **d** $y = 2x - x^2$
e $y = x^2 - 5x + 4$ **f** $y = x^2 - 4x - 5$
g $y = x^2 - 2x - 1$ **h** $y = x^2 - 4x + 6$
8 $y = -\frac{1}{8}x^2 + x + 1,\ y = \frac{1}{8}x^2 + x - 5$
9 $A = 1, b = 2, B = 4$

Exercise 4C

1 a 3 **b** −5 **c** 7 **d** −21 **e** $\frac{17}{8}$ **f** $-\frac{9}{8}$

2 a 6 **b** 6 **c** 18 **d** 12
e $a^3 + 3a^2 - 4a + 6$ **f** $8a^3 + 12a^2 - 8a + 6$

3 a $a = -5$ **b** $a = \frac{40}{9}$ **c** $c = 8$
d $a = -23,\ b = -4$ **e** $a = -17,\ b = 42$

4 a $2x^3 - x^2 + 2x + 2$ **b** $2x^3 + 5x$
c $2x^3 - x^2 + 4x - 2$ **d** $6x^3 - 3x^2 + 9x$
e $-2x^4 + 5x^3 - 5x^2 + 6x$ **f** $4x - x^3$
g $2x^3 + 4x + 2$
h $2x^5 + 3x^4 + x^3 + 6x^2$

5 a $x^3 - 5x^2 + 10x - 8$ **b** $x^3 - 7x^2 + 13x - 15$
c $2x^3 - x^2 - 7x - 4$
d $x^2 + (b + 2)x^2 + (2b + c)x + 2c$
e $2x^3 - 9x^2 - 2x + 3$

6 a $x^3 + (b + 1)x^2 + (c + b)x + c$
b $b = -2$ and $c = -4$
c $(x + 1)(x + \sqrt{5} - 1)(x - \sqrt{5} - 1)$

7 a $a = 2$ and $b = 5$
b $a = -2,\ b = -2$ and $c = -3$

8 $A = 1,\ B = 3$

9 a $A = 1,\ B = -2,\ C = 6$
b $A = 4,\ B = -\frac{3}{2},\ C = 5$
c $A = 1,\ B = -3,\ C = 5$

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Exercise 4D

1 a $x^2 - 5x + 6$ **b** $2x^2 + 7x - 4$

2 a $x^2 - 4x - 3 + \dfrac{34}{x+3}$

b $2x^2 + 6x + 14 + \dfrac{54}{x-3}$

3 a $x^2 - \dfrac{5}{2}x - \dfrac{15}{4} + \dfrac{145}{4(2x+3)}$

b $2x^2 + 6x + 7 + \dfrac{33}{2x-3}$

4 a $2x^2 - x + 12 + \dfrac{33}{x-3}$

b $5x^4 + 8x^3 - 8x^2 + 6x - 6$

5 a $x^2 - 9x + 27 - \dfrac{26(x-2)}{x^2-2}$ **b** $x^2 + x + 2$

6 a -16 **b** $a = 4$

7 a 28 **b** 0 **c** $(x+2)(3x+1)(2x-3)$

8 b $k = \dfrac{11}{2}$

9 a $a = 3,\ b = 8$ **b** $2x - 1,\ x - 1$

10 $a = \dfrac{-92}{9},\ b = 9$

11 81

12 b $6x - 4$

13 $x - 3,\ 2x - 1$

14 b $x^2 - 3,\ x^2 + x + 2$

15 a $(2a+3b)(4a^2 - 6ab + 9b^2)$
b $(4-a)(a^2 + 4a + 16)$
c $(5x+4y)(25x^2 - 20xy + 16y^2)$
d $2a(a^2 + 3b^2)$

16 a $(2x-1)(2x+3)(3x+2)$
b $(2x-1)(2x^2+3)$

17 a $(2x-3)(2x^2+3x+6)$
b $(2x-3)(2x-1)(2x+1)$

18 a $-4, 2, 3$ **b** $0, 2$ **c** $\dfrac{1}{2}, 2$ **d** $-2, 2$
e $0, -2, 2$ **f** $0, -3, 3$
g $1, -2, -\dfrac{1}{4}, \dfrac{1}{3}$ **h** $1, -2$ **i** $1, -2, \dfrac{1}{3}, \dfrac{3}{2}$

19 a $(-1, 0),\ (0, 0),\ (2, 0)$
b $(-2, 0),\ (0, 6),\ (1, 0),\ (3, 0)$
c $(-1, 0),\ (0, 6),\ (2, 0),\ (3, 0)$
d $\left(-\dfrac{1}{2}, 0\right),\ (0, 2),\ (1, 0),\ (2, 0)$
e $(-2, 0),\ (-1, 0),\ (0, -2),\ (1, 0)$
f $(-1, 0),\ \left(-\dfrac{2}{3}, 0\right),\ (0, -6),\ (3, 0)$
g $(-4, 0),\ (0, -16),\ \left(-\dfrac{2}{5}, 0\right),\ (2, 0)$
h $\left(-\dfrac{1}{2}, 0\right),\ (0, 1),\ \left(\dfrac{1}{3}, 0\right),\ (1, 0)$
i $(-2, 0),\ \left(-\dfrac{3}{2}, 0\right),\ (0, -30),\ (5, 0)$

20 $p = 1, q = -6$

21 -33

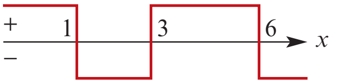
22 a $(x-9)(x-13)(x+11)$
b $(x+11)(x-9)(x-11)$
c $(x+11)(2x-9)(x-11)$
d $(x+11)(2x-13)(2x-9)$

23 a $(x-1)(x+1)(x-7)(x+6)$
b $(x-3)(x+4)(x^2+3x+9)$

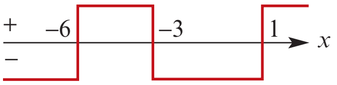
24 a $(x-9)(x-5)(2x^2+3x+9)$
b $(x+5)(x+9)(x^2-x+9)$
c $(x-3)(x+5)(x^2+x+9)$
d $(x-4)(x-3)(x+5)(x+6)$

Exercise 4E

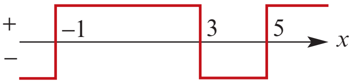
1 a



b



c



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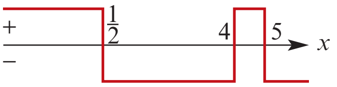

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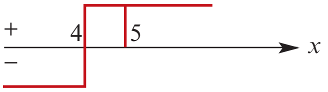

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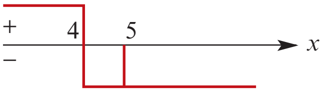
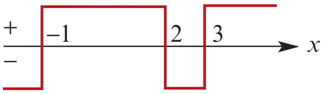
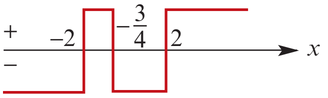


2 a



b



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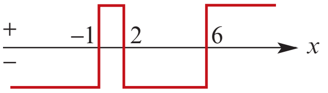
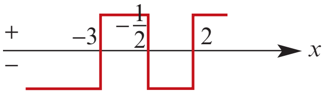


d



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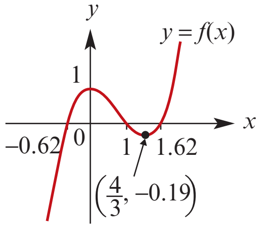
3 a



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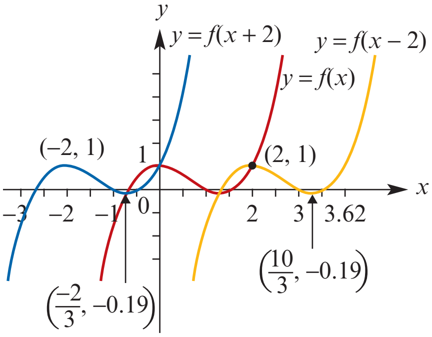
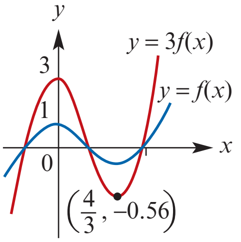
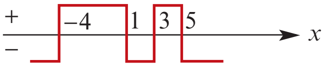


For clarity the graph of $y = 3f(x)$ is shown on separate axes:

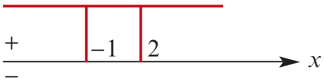


Exercise 4F

1 a



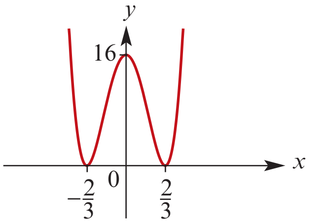
b



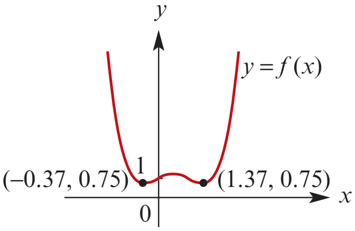
2 (0, 16)

$\left(\frac{2}{3}, 0\right)$

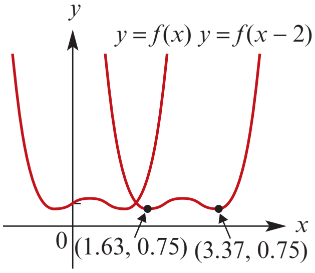
$\left(-\frac{2}{3}, 0\right)$



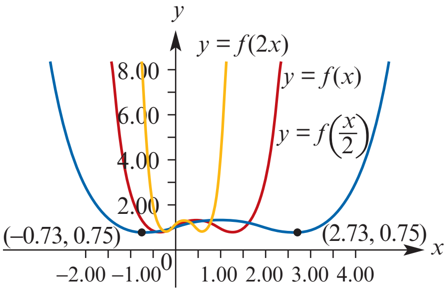
3 a



b

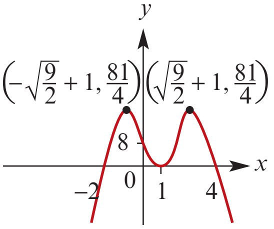


Graphs of dilations shown on separate axes for clarity:



Turning points for $y = f(2x)$ are at $(-0.18, 0.75)$ and $(0.68, 0.75)$

4 a



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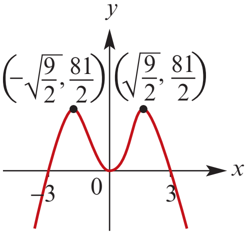

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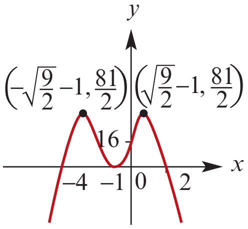
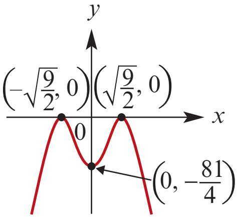


d



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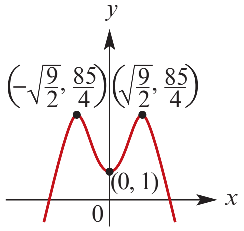
e



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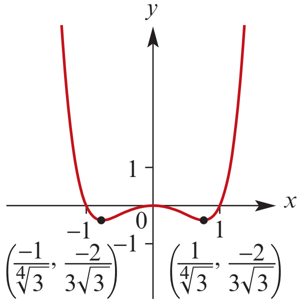

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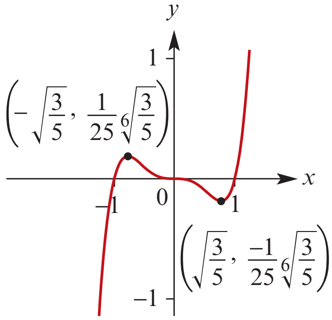

Exercise 4G

1 **a** $a = -2$ **b** $a = 3$ **c** $a = \frac{10}{3}$, $b = -\frac{70}{3}$

2 $a = -3$, $b = 2$, $c = -4$, $d = 5$

3 $y = \frac{11}{60}(x+5)(x+2)(x-6)$

4 $y = \frac{5}{9}(x+1)(x-3)^2$

5 **a** $y = x^3 + x + 1$ **b** $y = x^3 - x + 1$
c $y = 2x^3 - x^2 + x - 2$

6 **a** $y = (2x+1)(x-1)(x-2)$
b $y = \frac{1}{4}x(x^2+2)$ **c** $y = x^2(x+1)$
d $y = x^3 + 2x^2 - x - 2$
e $y = (x+2)(x-3)^2$

7 **a** $y = -2x^3 - 25x^2 + 48x + 135$
b $y = 2x^3 - 30x^2 + 40x + 13$

8 **a** $y = -2x^4 + 22x^3 - 10x^2 - 37x + 40$
b $y = x^4 - x^3 + x^2 + 2x + 8$
c $y = \frac{31}{36}x^4 + \frac{5}{4}x^3 - \frac{157}{36}x^2 - \frac{5}{4}x + \frac{11}{2}$

Exercise 4H

1 **a** $x = \frac{-1 \pm \sqrt{1-4k^2}}{2k}$, for $k \in [-\frac{1}{2}, \frac{1}{2}] \setminus \{0\}$
b $x = 4a, 3a$ or 0 **c** $x = 0$ or $x = a^{\frac{1}{3}}$
d $x = \frac{k \pm \sqrt{k^2-4k}}{2}$, for $k \geq 4$ or $k \leq 0$
e $x = 0$ or $x = \pm\sqrt{a}$ if $a > 0$ **f** $x = \pm a$
g $x = a$ or $x = b$
h $x = a$ or $x = a^{\frac{1}{3}}$ or $x = \pm\sqrt{a}$ if $a \geq 0$

2 **a** $x = \sqrt[3]{\frac{2c-b}{a}}$ **b** $x = \pm\sqrt{\frac{c+b}{a}}$
c $x = \pm\sqrt{\frac{a-c}{b}}$ **d** $x = a^3$
e $x = (a-c)^n$ **f** $x = 2b + \sqrt[3]{\frac{c}{a}}$
g $x = \left(\frac{b}{a}\right)^3$ **h** $x = (c+d)^{\frac{1}{3}}$

3 **a** $(0, 0)$, $(1, 1)$ **b** $(\frac{1}{2}, \frac{1}{2})$, $(0, 0)$
c $\left(\frac{3+\sqrt{13}}{2}, \sqrt{13}+4\right)$, $\left(\frac{3-\sqrt{13}}{2}, 4-\sqrt{13}\right)$

4 **a** $(13, 3)$, $(3, 13)$ **b** $(10, 5)$, $(5, 10)$
c $(11, 8)$, $(-8, -11)$ **d** $(9, 4)$, $(4, 9)$
e $(9, 5)$, $(-5, -9)$

5 **a** $(17, 11)$, $(11, 17)$ **b** $(37, 14)$, $(14, 37)$
c $(14, 9)$, $(-9, -14)$

6 $(2, 4)$, $(0, 0)$

7 $\left(\frac{\sqrt{5}+5}{2}, \frac{\sqrt{5}+5}{2}\right)$, $\left(\frac{5-\sqrt{5}}{2}, \frac{5-\sqrt{5}}{2}\right)$

8 $\left(\frac{-130-80\sqrt{2}}{41}, \frac{60-64\sqrt{2}}{41}\right)$,
$\left(\frac{80\sqrt{2}-130}{41}, \frac{64\sqrt{2}+60}{41}\right)$

9 $\left(\frac{1+\sqrt{21}}{2}, \frac{-1+\sqrt{21}}{2}\right)$, $\left(\frac{1-\sqrt{21}}{2}, \frac{-1+\sqrt{21}}{2}\right)$

10 $\left(\frac{4}{9}, 2\right)$ **11** $\left(\frac{-6\sqrt{5}}{5}, \frac{3\sqrt{5}}{5}\right)$

12 $\left(-2, \frac{1}{2}\right)$ **13** $(3, 2)$, $(0, -1)$

14 **a** $(3, 2)$, $\left(\frac{-8}{5}, \frac{-15}{4}\right)$ **b** $\left(\frac{27}{2}, \frac{10}{3}\right)$, $(5, 9)$
c $(6, 4)$, $\left(\frac{-12}{5}, -10\right)$

15 $c^2 - ac + b = 0$

16 $(-1-\sqrt{161}, 1-\sqrt{161})$, $(\sqrt{161}-1, \sqrt{161}+1)$

17 $y = -7x + 14$, $y = 5x + 2$

18 $m < -7$ or $m > 1$

19 $c = -8$ or $c = 4$

20 **a** $x = \frac{5 \pm \sqrt{4m+25}}{2m}$ **b** $m = \frac{-25}{4}$, $\left(-\frac{2}{5}, \frac{5}{2}\right)$
c $m < \frac{-25}{4}$ and $m \neq 0$

21 $y = 3x + 3$, $y = -x + 3$

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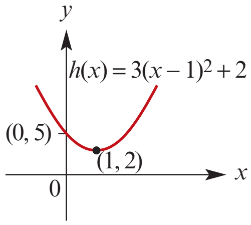
Chapter 4 review

Technology-free questions

1 a



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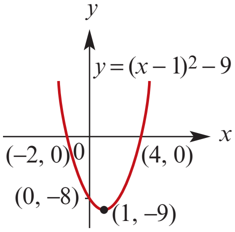

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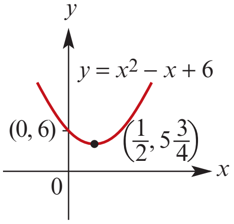

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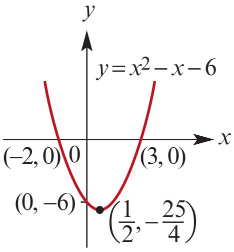

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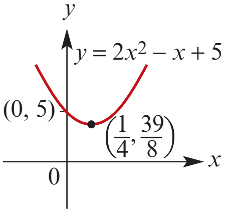

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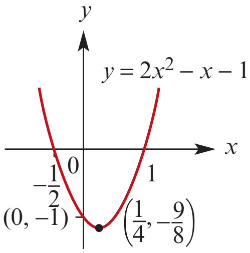
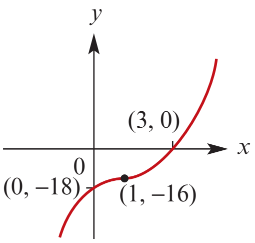


2 $y = \frac{4}{3}x^2 - \frac{1}{3}$; $a = \frac{4}{3}$, $b = -\frac{1}{3}$

3 $\frac{1}{3}(1 \pm \sqrt{31})$

4 a



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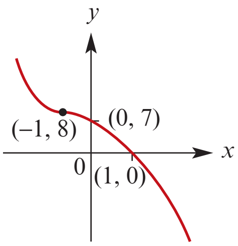

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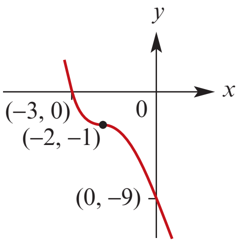

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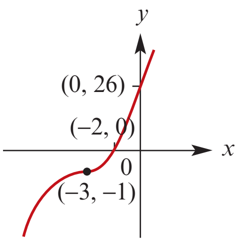

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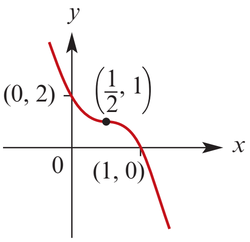
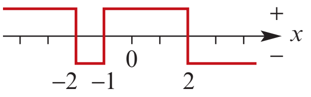


5 a $(x+2)^2 - 4$ **b** $3(x+1)^2 - 3$

c $(x-2)^2 + 2$ **d** $2\left(x - \frac{3}{2}\right)^2 - \frac{17}{2}$

e $2\left(x - \frac{7}{4}\right)^2 - \frac{81}{8}$ **f** $-\left(x - \frac{3}{2}\right)^2 - \frac{7}{4}$

6 a



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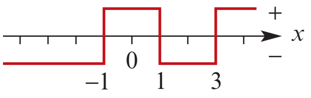

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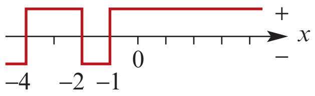

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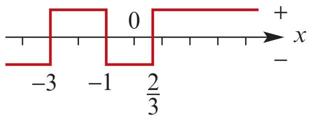


7 **a** 8 **b** 0 **c** 0

8 $y = (x - 7)(x + 3)(x + 2)$

9 **a** $(x - 2)(x + 1)(x + 3)$

b $(x - 1)(x + 1)(x - 3)$

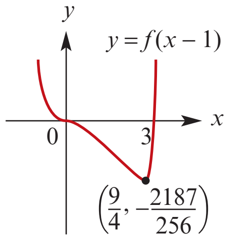
c $(x - 1)(x + 1)(x - 3)(x + 2)$

d $\frac{1}{4}(x - 1)(2x + 3 + \sqrt{13})(2x + 3 - \sqrt{13})$

10 $x^2 + 4 = 1 \times (x^2 - 2x + 2) + 2x + 2$

11 $a = -6$

12 **a**



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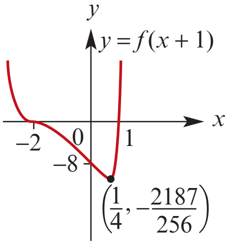

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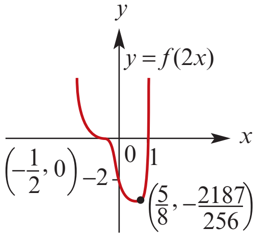

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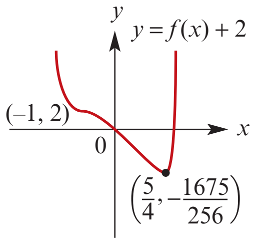


13 $k = \pm 8$

14 $(4, -5)$, $(3, 9)$

15 $a = 3$, $b = \frac{5}{6}$, $c = -\frac{13}{12}$

16 $64x^3 + 144x^2 + 108x + 27$

17 $a = 1$, $b = -1$, $c = 4$

18 $-2 < p < 6$

19 $y = -x^3 + 7x^2 - 11x + 6$

Multiple-choice questions

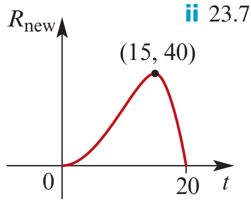
1 E	**2** D	**3** E	**4** C	**5** E	**6** A
7 C	**8** E	**9** C	**10** C	**11** C	**12** B

Extended-response questions

1 **a** $k = \frac{4}{3375}$

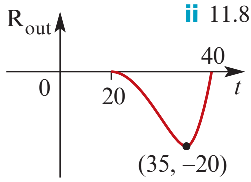
b 11.852 mL/min

c **i**



ii 23.704 mL/min

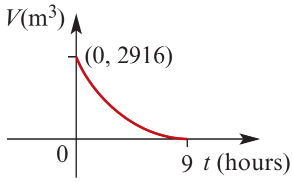
d **i**



ii 11.852 mL/min out

2 **a** **i** 2916 m^3 **ii** 0 m^3

b

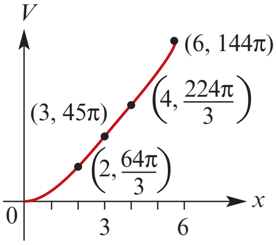


c 3.96 hours

3 **a** **i** $\frac{64\pi}{3}$ cm^3 **ii** 45π cm^3 **iii** $\frac{224\pi}{3}$ cm^3

b 144π cm^3

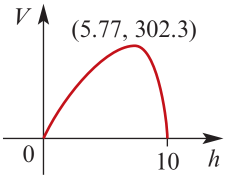
c



d $x = 5$; depth is 5 cm

4 **a** $r = \sqrt{25 - \frac{h^2}{4}}$

b



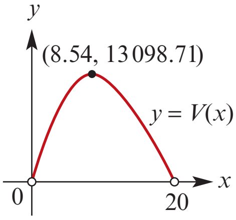
c $V = 96\pi$ cm^3

d $h = 2$, $r = 2\sqrt{6}$, i.e. height = 2 cm and radius = $2\sqrt{6}$ cm, or $h = 8.85$ and $r = 2.33$

5 a $V = (84 - 2x)(40 - 2x)x$

b $(0, 20)$

c



d i $x = 2, V = 5760$

ii $x = 6, V = 12\,096$

iii $x = 8, V = 13\,056$

iv $x = 10, V = 12\,800$

e $x = 13.50$ or $x = 4.18$

f $13\,098.71\text{ cm}^3$

6 a i $A = 2x(16 - x^2)$ **ii** $(0, 4)$

b i 42 **ii** $x = 0.82$ or $x = 3.53$

c i $V = 2x^2(16 - x^2)$ **ii** $x = 2.06$ or $x = 3.43$

7 a $A = \frac{\pi}{2}x^2 + yx$

b i $y = 100 - \pi x$ **ii** $A = 100x - \frac{\pi}{2}x^2$

iii $\left(0, \frac{100}{\pi}\right)$

c $x = 12.43$

d i $V = \frac{x^2}{50}\left(100 - \frac{\pi}{2}x\right), \ x \in \left(0, \frac{100}{\pi}\right)$

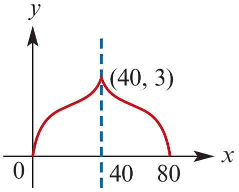
ii 248.5 m^3 **iii** $x = 18.84$

8 a $y = \frac{1}{12000}x^3 - \frac{1}{200}x^2 + \frac{17}{120}x$

b $x = 20$

d $y = -\frac{1}{6000}x^3 + \frac{29}{3000}x^2 - \frac{1}{20}x$

e i

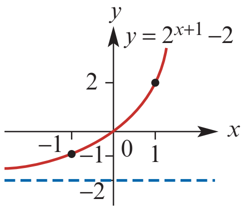


ii Second section of graph is formed reflecting the graph of $y = f(x)$, $x \in [0, 40]$, in the line $x = 40$

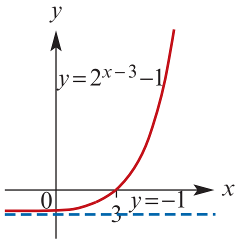
Chapter 5

Exercise 5A

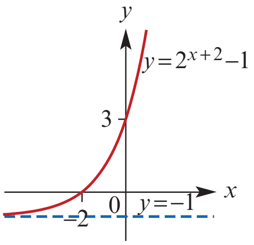
1 a Range $= (-2, \infty)$



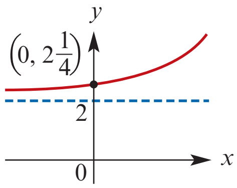
b Range $= (-1, \infty)$



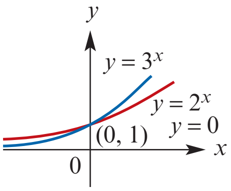
c Range $= (-1, \infty)$



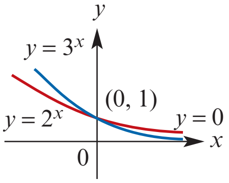
d Range $= (2, \infty)$



2 a



b



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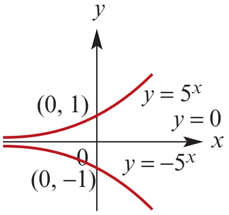
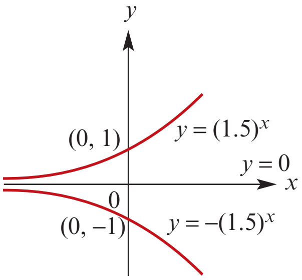
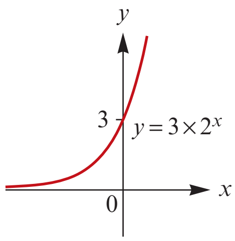


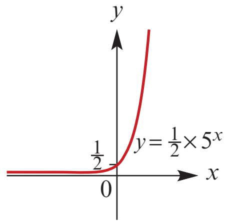
d



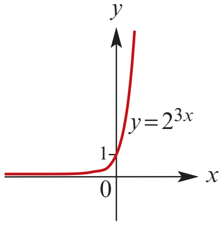
3 a Range = $(0, \infty)$



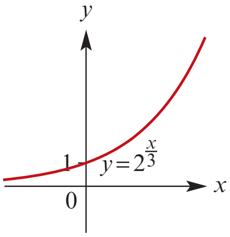
b Range = $(0, \infty)$



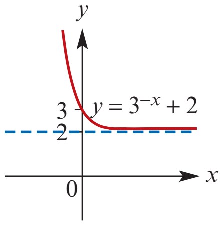
c Range = $(0, \infty)$



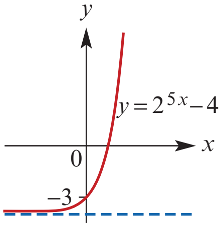
d Range = $(0, \infty)$



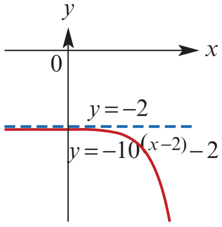
4 a Range = $(2, \infty)$



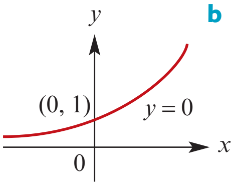
b Range = $(-4, \infty)$



c Range = $(-\infty, -2)$

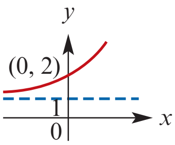


5 a



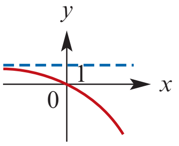
Range = $\mathbb{R}^+$

b



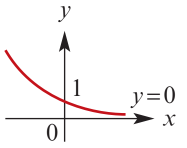
Range = $(1, \infty)$

c



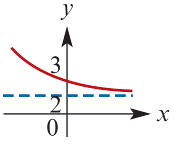
Range = $(-\infty, 1)$

d



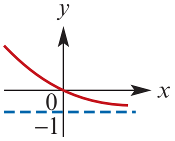
Range = $\mathbb{R}^+$

e



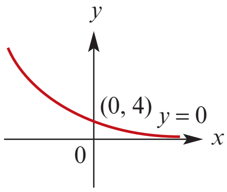
Range = $(2, \infty)$

f

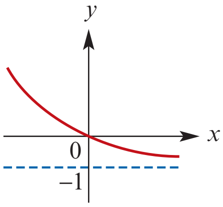


Range = $(-1, \infty)$

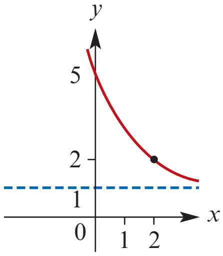
6 a Range = $\mathbb{R}^+$



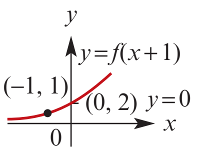
b Range = $(-1, \infty)$



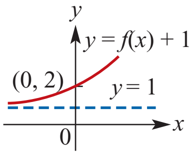
c Range = $(1, \infty)$



7 a



b



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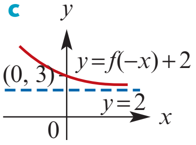

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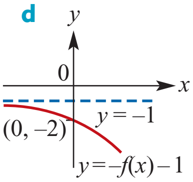

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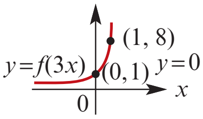
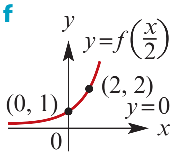
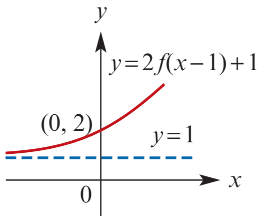
e



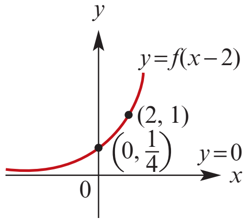
f



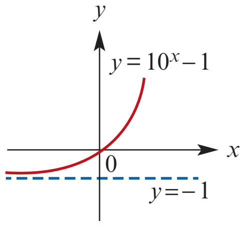
g



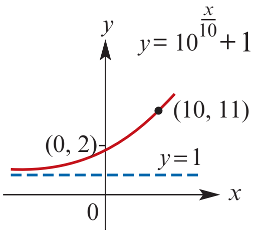
h



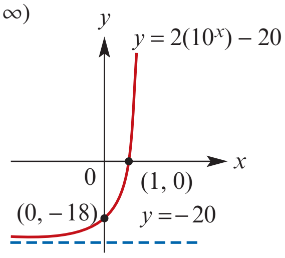
8 a Range $= (-1, \infty)$



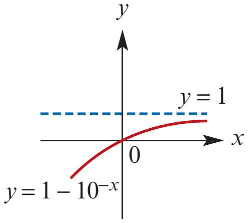
b Range $= (1, \infty)$



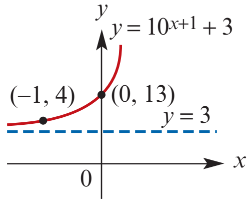
c Range $= (-20, \infty)$



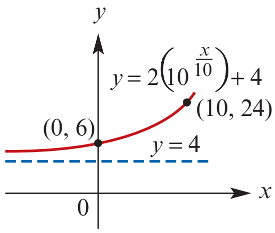
d Range $= (-\infty, 1)$



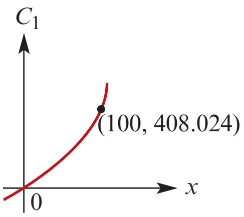
e Range $= (3, \infty)$



f Range $= (4, \infty)$



9 a



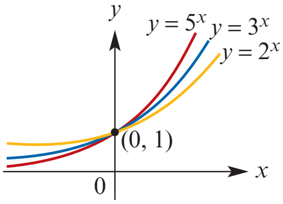
b i \$408.02 **ii** \$1274.70

c 239 days

d ii 302 days

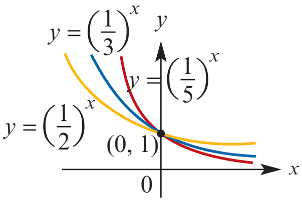
10 36 days

11 a i



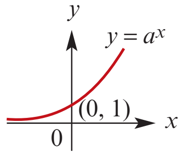
ii $x < 0$ **iii** $x > 0$ **iv** $x = 0$

b i

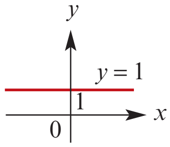


ii $x > 0$ **iii** $x < 0$ **iv** $x = 0$

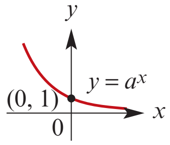
c i $a > 1$



ii $a = 1$



iii $0 < a < 1$



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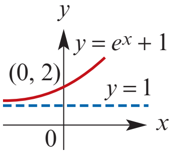
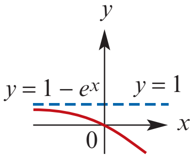
Exercise 5B

1 a



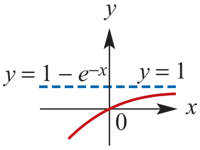
Range = $(1, \infty)$

b



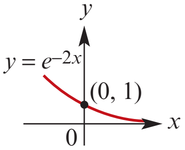
Range = $(-\infty, 1)$

c



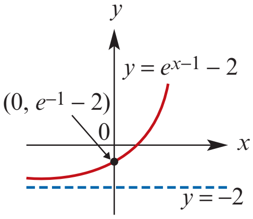
Range = $(-\infty, 1)$

d



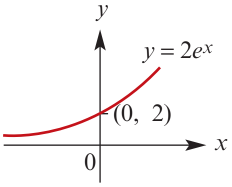
Range = $(0, \infty)$

e



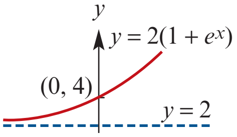
Range = $(-2, \infty)$

f



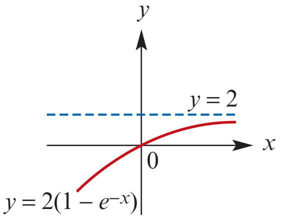
Range = $(0, \infty)$

g



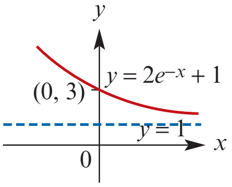
Range = $(2, \infty)$

h



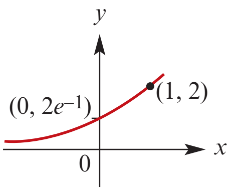
Range = $(-\infty, 2)$

i



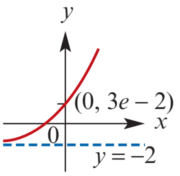
Range = $(1, \infty)$

j



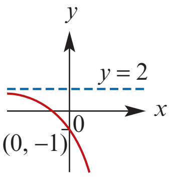
Range = $(0, \infty)$

k



Range = $(-2, \infty)$

l



Range = $(-\infty, 2)$

2 a Translation 2 units to the left and 3 units down

b Dilation of factor 3 from the x-axis, then translation 1 unit to the left and 4 units down

c Dilation of factor 5 from the x-axis and factor $\frac{1}{2}$ from the y-axis, then translation $\frac{1}{2}$ unit to the left

d Reflection in the x-axis, then translation 1 unit to the right and 2 units up

e Dilation of factor 2 from the x-axis, reflection in the x-axis, then translation 2 units to the left and 3 units up

f Dilation of factor 4 from the x-axis and factor $\frac{1}{2}$ from the y-axis, then translation 1 unit down

3 a $y = -2e^{x-3} - 4$ **b** $y = 4 - 2e^{x-3}$

c $y = -2e^{x-3} - 4$ **d** $y = -2e^{x-3} - 8$

e $y = 8 - 2e^{x-3}$ **f** $y = -2e^{x-3} + 8$

4 a Translation 2 units to the right and 3 units up

b Translation 1 unit to the right and 4 units up, then dilation of factor $\frac{1}{3}$ from the x-axis

c Translation $\frac{1}{2}$ unit to the right, then dilation of factor $\frac{1}{5}$ from the x-axis and factor 2 from the y-axis

d Translation 1 unit to the left and 2 units down, then reflection in the x-axis

e Translation 2 units to the right and 3 units down, then dilation of factor $\frac{1}{2}$ from the x-axis and reflection in the x-axis

f Translation 1 unit up, then dilation of factor $\frac{1}{4}$ from the x-axis and factor 2 from the y-axis

5 a $x = 1.146$ or $x = -1.841$

b $x = -0.443$

c $x = -0.703$

d $x = 1.857$ or $x = 4.536$

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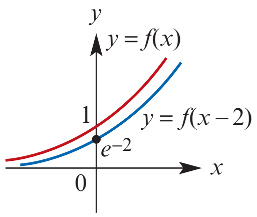
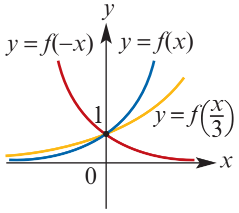
6 a



b



Exercise 5C

1 **a** $6x^6y^9$ **b** $3x^6$ **c** $\frac{6y^2}{x^2}$ **d** 8
e 16 **f** $\frac{5x^{28}}{y^6}$ **g** $24x^5y^{10}$ **h** $2xy^2$
i x^2y^2

2 **a** 4 **b** $\frac{1}{2}$ **c** 8 **d** $\frac{1}{4}$ **e** $\frac{3}{5}$ **f** 3
g $\frac{5}{2}$ **h** 6 **i** 4

3 **a** 1 **b** 1 **c** $-\frac{3}{2}$ **d** 3 **e** -2 **f** 4
g $-\frac{10}{3}$ **h** $-\frac{3}{2}$ **i** 6 **j** $\frac{3}{5}$ **k** $\pm\frac{1}{2}$

4 **a** 1 **b** 2 **c** 1 **d** 1, 2 **e** 0, 1
f 2, 4 **g** 0, 1 **h** −1, 2 **i** −1, 0

Exercise 5D

1 **a** 3 **b** −4 **c** −3 **d** 6 **e** 6 **f** −7

2 **a** $\log_e 6$ **b** $\log_e 4$
c $\log_e(10^6) = 6\log_e 10$ **d** $\log_e 7$
e $\log_e \frac{1}{60} = -\log_e 60$
f $\log_e(u^3v^6) = 3\log_e(uv^2)$
g $\log_e(x^7) = 7\log_e x$ **h** $\log_e 1 = 0$

3 **a** $x = 100$ **b** $x = 16$ **c** $x = 6$ **d** $x = 64$
e $x = e^3 - 5 \approx 15.086$ **f** $x = \frac{1}{2}$ **g** $x = -1$
h $x = 10^{-3} = \frac{1}{1000}$ **i** $x = 36$

4 **a** $x = 15$ **b** $x = 5$ **c** $x = 4$
d $x = 1$ ($x = -\frac{1}{2}$ is not an allowable solution)
e $x = \frac{3}{2}$

5 **a** $\log_{10} 27$ **b** $\log_2 4 = 2$
c $\frac{1}{2}\log_{10}\left(\frac{a}{b}\right) = \log_{10}\sqrt{\frac{a}{b}}$ **d** $\log_{10}\left(\frac{10a}{b^{\frac{1}{3}}}\right)$
e $\log_{10}\left(\frac{1}{8}\right) = -3\log_{10} 2$

6 **a** 1 **b** 1 **c** $2\frac{1}{2}$ **d** 3 **e** 0

7 **a** $-x$ **b** $2\log_2 x$ **c** 0

8 **a** $x = 4$ **b** $x = \frac{3e}{5 + 2e} \approx 0.7814$

9 **a** $x = \frac{-1 + \sqrt{1 + 12e}}{6}$, i.e. $x \approx 0.7997$
b $x = \log_e 2 \approx 0.6931$

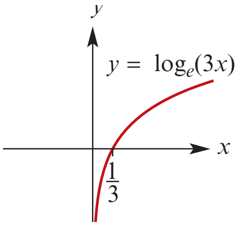
10 **a** $x = 3$ **b** $x = \frac{1}{2}$

11 $\frac{1}{4}$, 2

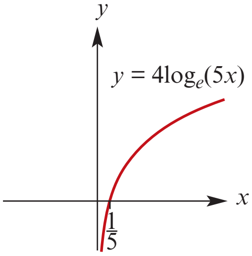
12 $N = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$

Exercise 5E

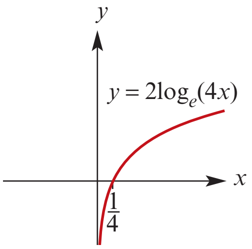
1 **a**



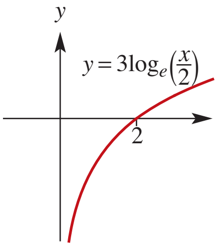
b



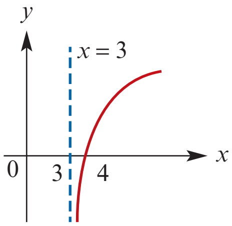
c



d



2 **a** Domain = $(3, \infty)$
Range = $\mathbb{R}$



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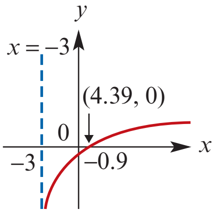
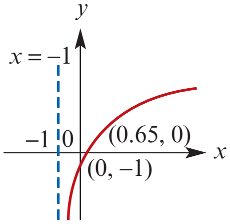
b Domain = $(-3, \infty)$

Range = $\mathbb{R}$



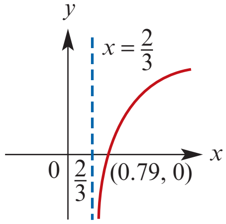
c Domain = $(-1, \infty)$

Range = $\mathbb{R}$



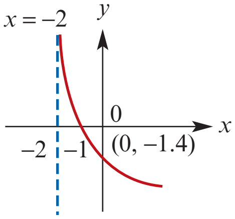
d Domain = $\left(\frac{2}{3}, \infty\right)$

Range = $\mathbb{R}$



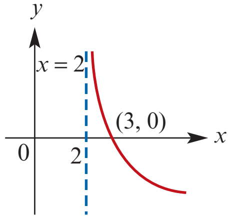
e Domain = $(-2, \infty)$

Range = $\mathbb{R}$



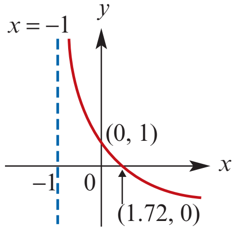
f Domain = $(2, \infty)$

Range = $\mathbb{R}$



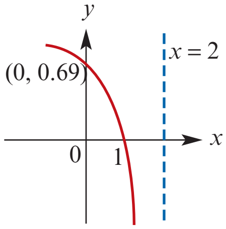
g Domain = $(-1, \infty)$

Range = $\mathbb{R}$



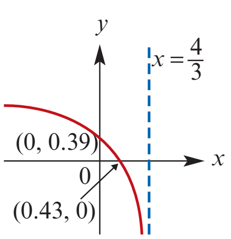
h Domain = $(-\infty, 2)$

Range = $\mathbb{R}$

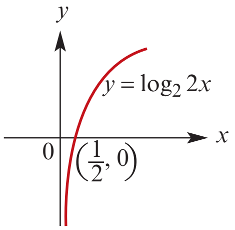


i Domain = $\left(-\infty, \frac{4}{3}\right)$

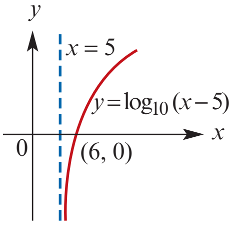
Range = $\mathbb{R}$



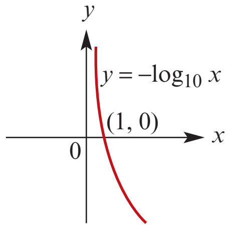
3 **a** Domain = $\mathbb{R}^+$



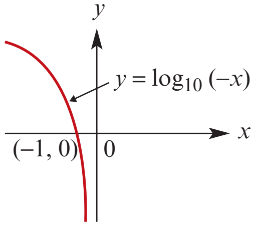
b Domain = $(5, \infty)$



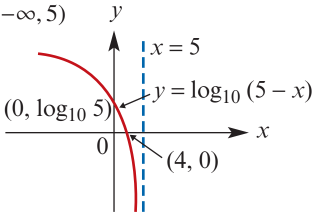
c Domain = $\mathbb{R}^+$



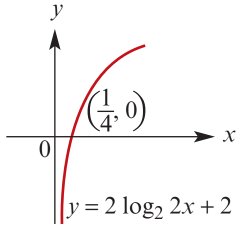
d Domain = $\mathbb{R}^-$



e Domain = $(-\infty, 5)$



f Domain = $\mathbb{R}^+$



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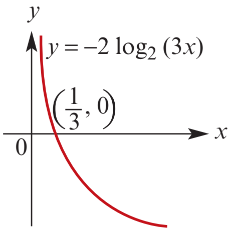
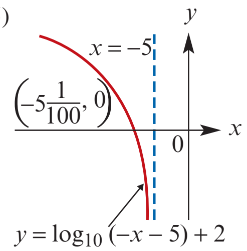
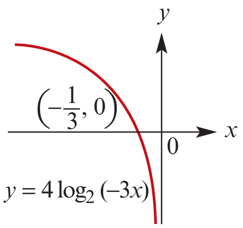
g Domain = $\mathbb{R}^+$



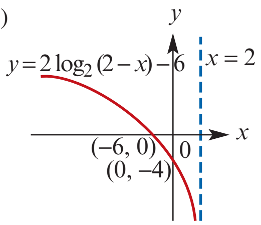
h Domain = $(-\infty, -5)$



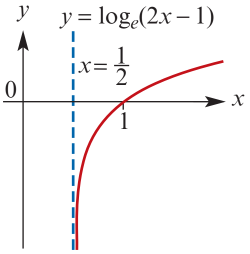
i Domain = $\mathbb{R}^-$



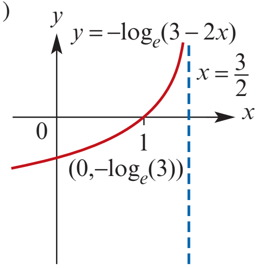
j Domain = $(-\infty, 2)$



k Domain = $(\frac{1}{2}, \infty)$

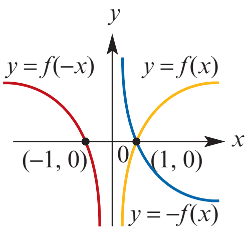


l Domain = $(-\infty, \frac{3}{2})$

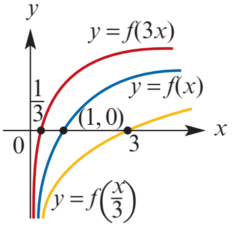


4 a $x = 1.557$
b $x = 1.189$

5 a



b



6 A dilation of factor $\log_e 3$ from the y-axis

7 A dilation of factor $\frac{1}{\log_e 2}$ from the y-axis

Exercise 5F

1 $a = \frac{6}{e^4-1}$, $b = \frac{5e^4-11}{e^4-1}$

2 $a = \frac{2}{\log_e 6}$, $b = -4$ **3** $a = 2$, $b = 4$

4 $a = \frac{14}{e-1}$, $b = \frac{14}{1-e}$ $(a \approx 8.148, b \approx -8.148)$

5 $a = 250$, $b = \frac{1}{3}\log_e 5$

6 $a = 200$, $b = 500$ **7** $a = 2$, $b = 4$

8 $a = 3$, $b = 5$ **9** $a = 2$, $b = \frac{1}{3}\log_e 5$

10 $a = 2$, $b = 3$

11 $b = 1$, $a = \frac{2}{\log_e 2}$, $c = 8$ $(a \approx 2.885)$

12 $a = \frac{2}{\log_e 2}$, $b = 4$

Exercise 5G

1 a $k = \frac{1}{\log_2 7}$ **b** $x = \frac{\log_2 7 - 4}{\log_2 7}$

c $x = \frac{\log_e 7 - 1}{\log_e 14}$

2 a 2.58 **b** −0.32 **c** 2.18 **d** 1.16
e −2.32 **f** −0.68 **g** −2.15 **h** −1.38
i 2.89 **j** −1.70 **k** −4.42 **l** 5.76
m −6.21 **n** 2.38 **o** 2.80

3 a $x < 2.81$ **b** $x > 1.63$ **c** $x < -0.68$
d $x \leq 3.89$ **e** $x \geq 0.57$

4 a $\log_2 5$ **b** $\frac{1}{2}(\log_3 8 + 1)$
c $\frac{1}{3}(\log_7 20 - 1)$ **d** $\log_3 7$ **e** $\log_3 6$
f $\log_5 6$ **g** $x = \log_3 8$ or $x = 0$ **h** $x = 1$

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5 **a** $x > \log_7 52$ **b** $x < \frac{1}{2}\log_3 120$

c $x \geq \frac{1}{6}\log_2\left(\frac{5}{4}\right)$ **d** $x \leq \log_3 7290$

e $x < \log_3 106$ **f** $x < \log_5\left(\frac{3}{5}\right)$

6 **a** 0.544 **b** 549.3

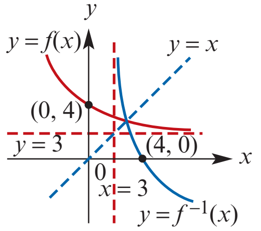
8 **a** 9^u **b** $u + \frac{1}{2}$ **c** $\frac{2}{u}$

9 $625, \frac{1}{625}$ **10** $\frac{2}{p}$

Exercise 5H

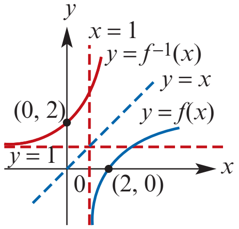
1 $f^{-1}: (-2, \infty) \to \mathbb{R}$, $f^{-1}(x) = \log_e(x + 2)$

2 $f(x) = e^{-x} + 3$, $f^{-1}(x) = -\log_e(x - 3)$


y
y = f(x)
y = x
(0, 4)
y = 3
(4, 0)
x
0
x = 3
y = f⁻¹(x)

3 $f(x) = \log_e(x - 1)$

$f^{-1}(x) = e^x + 1$


y
x = 1
y = f⁻¹(x)
y = x
y = f(x)
(0, 2)
y = 1
0
(2, 0)
x

4 $x = e^{\frac{y+4}{3}}$

5 **a** $f^{-1}(x) = \frac{1}{2}e^x$, dom $= \mathbb{R}$, ran $= \mathbb{R}^+$

b $f^{-1}(x) = \frac{1}{2}e^{\frac{x-1}{3}}$, dom $= \mathbb{R}$, ran $= \mathbb{R}^+$

c $f^{-1}(x) = \log_e(x - 2)$, dom $= (2, \infty)$, ran $= \mathbb{R}$

d $f^{-1}(x) = \log_e(x) - 2$, dom $= \mathbb{R}^+$, ran $= \mathbb{R}$

e $f^{-1}(x) = \frac{1}{2}(e^x - 1)$,

dom $= \mathbb{R}$, ran $= \left(-\frac{1}{2}, \infty\right)$

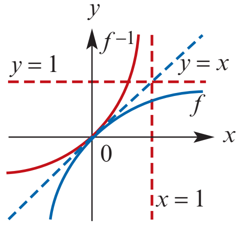
f $f^{-1}(x) = \frac{1}{3}\left(e^{\frac{x}{4}} - 2\right)$,

dom $= \mathbb{R}$, ran $= \left(-\frac{2}{3}, \infty\right)$

g $f^{-1}(x) = 10^x - 1$, dom $= \mathbb{R}$, ran $= (-1, \infty)$

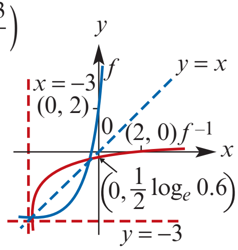
h $f^{-1}(x) = \log_e\left(\frac{x}{2}\right) + 1$, dom $= \mathbb{R}^+$, ran $= \mathbb{R}$

6 $f^{-1}(x) = -\log_e(1 - x)$, dom $f^{-1} = (-\infty, 1)$


y
f⁻¹
y = 1
y = x
f
x
0
x = 1

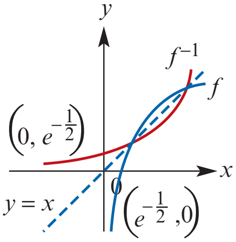
7 $f^{-1}(x) = \frac{1}{2}\log_e\left(\frac{x + 3}{5}\right)$

dom $f^{-1} = (-3, \infty)$


y
f
y = x
x = −3
(0, 2)
0
(2, 0)
f⁻¹
x
(0, ½ logₑ 0.6)
y = −3

8 $f^{-1}(x) = e^{\frac{x-1}{2}}$

ran $f^{-1} = \mathbb{R}^+$


y
f⁻¹
f
(0, e^(−½))
x
y = x
0
(e^(−½), 0)

9 $t = \frac{-1}{k}\log_e\left(\frac{P - b}{A}\right)$

10 **a** $x = e^{\frac{y-5}{2}}$ **b** $x = -\frac{1}{6}\log_e\left(\frac{P}{A}\right)$

c $n = \frac{\log_e\left(\frac{y}{a}\right)}{\log_e x}$ **d** $x = \log_{10}\left(\frac{y}{5}\right)$

e $x = \frac{1}{2}e^{\frac{5-y}{3}}$ **f** $n = \frac{1}{2}\left(\frac{\log_e\left(\frac{y}{6}\right)}{\log_e x}\right)$

g $x = \frac{1}{2}(e^y + 1)$ **h** $x = \log_e\left(\frac{5}{5 - y}\right)$

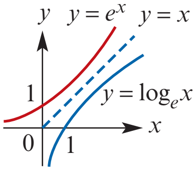
11 **a** $f^{-1}(x) = \log_e\left(\frac{x + 4}{2}\right)$

b $(0.895, 0.895)$, $(-3.962, -3.962)$

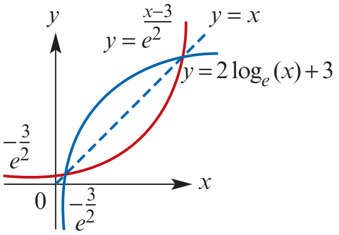
12 **a** $f^{-1}(x) = e^{\frac{x-4}{3}} - 3$

b $(8.964, 8.964)$, $(-2.969, -2.969)$

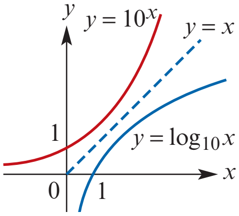
13 **a** **i**


y
y = eˣ
y = x
1
y = logₑx
0
1
x

ii


y
y = e^((x−3)/2)
y = x
y = 2 logₑ(x) + 3
e^(−3/2)
0
e^(−3/2)
x

iii


y
y = 10ˣ
y = x
1
y = log₁₀x
0
1
x

b f and g are inverse functions

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Exercise 5I

1 a $N = 1000 \times 2^{\frac{t}{15}}$ **b** 50 minutes

2 $d_0 = 52\left(\frac{13}{20}\right)^{\frac{1}{2}}$, $m = \frac{1}{2}\log_{10}\left(\frac{20}{13}\right)$

3 a i $N_0 = 20\,000$ **ii** -0.223
b 6.2 years

4 a $M_0 = 10$, $k = 4.95 \times 10^{-3}$
b 7.07 grams **c** 325 days

5 a $k = \frac{1}{1690}\log_e 2$ **b** 3924 years

6 55 726 years

7 7575 years

8 a 16 471 **b** 35 years on from 2002

9 18.4 years

10 a 607 millibars **b** 6.389 km

11 21.82 hours

12 6.4°C

13 $k = 0.349$, $N_0 = 50.25$

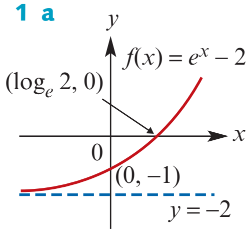
14 a $k = \log_e\left(\frac{5}{4}\right)$ **b** 7.21 hours

15 a $a = 1000$, $b = 15^{\frac{1}{5}}$ **b** 3 hours
c 13 hours **d** 664 690

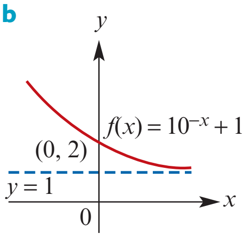
Chapter 5 review

Technology-free questions

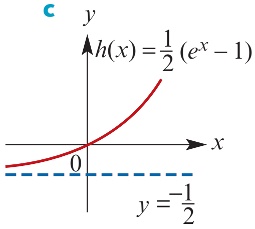
1 a



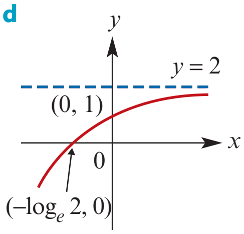
b



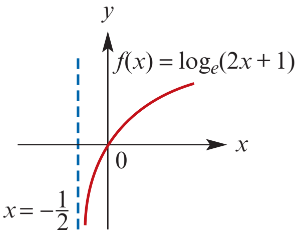
c



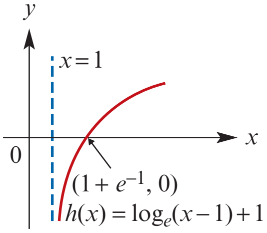
d



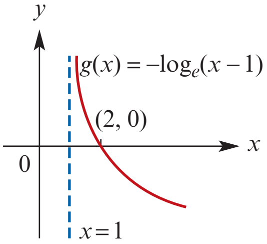
e



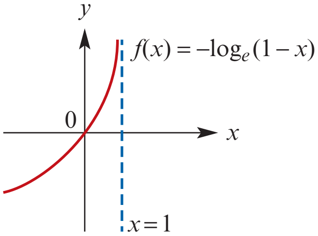
f



g



h



2 a $f^{-1}\colon (-1, \infty) \to \mathbb{R}$, $f^{-1}(x) = \frac{1}{2}\log_e(x + 1)$
b $f^{-1}\colon \mathbb{R} \to \mathbb{R}$, $f^{-1}(x) = e^{\frac{x}{3}} + 2$
c $f^{-1}\colon \mathbb{R} \to \mathbb{R}$, $f^{-1}(x) = 10^x - 1$
d $f^{-1}\colon (2, \infty) \to \mathbb{R}$, $f^{-1}(x) = \log_2(x - 1)$

3 a $y = e^2 x$ **b** $y = 10x$ **c** $y = 16x^3$
d $y = \frac{x^5}{10}$ **e** $y = \frac{e^3}{x}$ **f** $y = e^{2x-3}$

4 a $x = \frac{\log_e 11}{\log_e 3}$ **b** $x = \frac{\log_e 0.8}{\log_e 2}$
c $x = \frac{\log_e 3}{\log_e\left(\frac{2}{3}\right)}$

5 a $x = 1$ **b** $x = \frac{2}{3}$ **c** $x = \frac{1}{20}$
d $x = \log_{10} 3$ or $x = \log_{10} 4$

6 $a = 2$, $b = 2^{-\frac{2}{3}} - 1$ **7** $10^{\frac{6}{5}} - 1$

8 $\frac{1}{3}\log_e\left(\frac{287}{4}\right)$ **9** $2a$ **10** $\mathbb{R}$

11 $a = \log_e 5$, $b = 5$, $k = 2$

12 a $f^{-1}(x) = \frac{1}{3}\log_e(x + 4)$, dom $f^{-1} = (-4, \infty)$
b $\frac{1}{3x + 4} - 4$

13 a $f(-x) = f(x)$ **b** $2(e^u + e^{-u})$
c 0 **d** $e^{2u} + e^{-2u}$
e $g(-x) = -g(x)$ **f** $2e^x, 2e^{-x}, e^{2x} - e^{-2x}$

Multiple-choice questions

1 C **2** D **3** B **4** E **5** A **6** C
7 B **8** A **9** C **10** D **11** A **12** C
13 C **14** D **15** B **16** D

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Extended-response questions

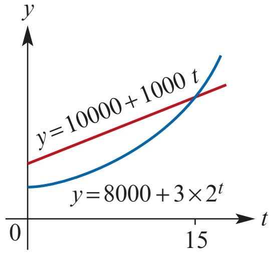
1 a 73.5366°C **b** 59.5946

2 a 770 **b** 1840

3 a $k = 22\,497,\ \lambda = 0.22$ **b** \$11 627

4 a $A = 65\,000,\ p = 0.064$ **b** \$47 200

5 a



b **i** (12.210, 22 209.62)
ii $t = 12.21$ **iii** 22 210

c **ii** (12.21, 12.21)

d $c = 0.52$

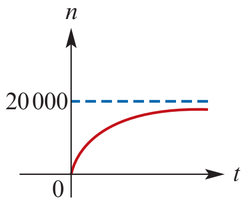
6 a **iii** $a = \frac{1}{2}$ or $a = 1$

iv If $a = 1$, then $e^{-2B} = 1$, and so $B = 0$;

If $a = \frac{1}{2}$, then $B = \frac{1}{2}\log_e 2$

v $A = 20\,000$

b



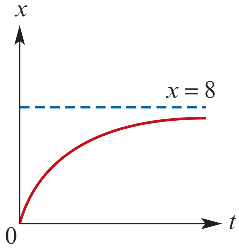
c $\dfrac{\log_e 0.1}{\frac{1}{2}\log_e(\frac{1}{2})} = \dfrac{2\log_e 10}{\log_e 2} \approx 6.644$

After 6.65 hours, the population is 18 000

7 a 75 **b** 2.37 **c** 0.646

8 $k = -0.5,\ A_0 = 100$

9 a

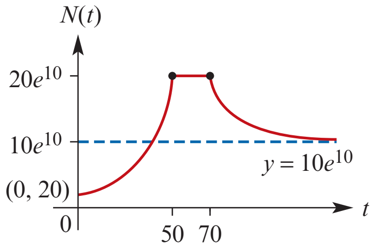


b **i** 0 grams **ii** 2.64 grams **iii** 6.92 grams

c 10.4 minutes

10 a $k = 0.235$ **b** 22.7°C **c** 7.17 minutes

11 a



b **i** $N(10) = 147.78$
ii $N(40) = 59\,619.16$
iii $N(60) = 20e^{10} = 440\,529.32$
iv $N(80) = 220\,274.66$

c **i** 25 days **ii** 35 days

Chapter 6

Exercise 6A

1 a $\frac{5\pi}{18}$ **b** $\frac{34\pi}{45}$ **c** $\frac{25\pi}{18}$
d $\frac{17\pi}{9}$ **e** $\frac{7\pi}{3}$ **f** $\frac{49\pi}{18}$

2 a 60° **b** 150° **c** 240°
d 140° **e** 630° **f** 252°

3 a 45.84° **b** 93.97° **c** 143.24°
d 226.89° **e** 239.50° **f** 340.91°

4 a 0.65 **b** 1.29 **c** 2.01
d 2.13 **e** 5.93 **f** 2.31

Exercise 6B

1 a 0 **b** 0 **c** −1 **d** −1 **e** 0
f 0 **g** 0 **h** 0 **i** −1 **j** 0
k −1 **l** 0 **m** 0 **n** 1

2 a 0.99 **b** 0.52 **c** −0.87 **d** 0.92
e −0.67 **f** −0.23 **g** −0.99 **h** 0.44
i −34.23 **j** −2.57 **k** 0.95 **l** 0.75

3 a $\frac{1}{\sqrt{2}}$ **b** $-\frac{1}{2}$ **c** $-\frac{\sqrt{3}}{2}$ **d** $\frac{1}{2}$
e $-\frac{1}{2}$ **f** $-\frac{1}{\sqrt{2}}$ **g** $-\frac{1}{\sqrt{2}}$ **h** $\frac{1}{2}$
i $\frac{1}{2}$ **j** $\frac{\sqrt{3}}{2}$ **k** $\frac{1}{2}$ **l** $\frac{\sqrt{3}}{2}$
m $\frac{1}{\sqrt{2}}$ **n** $-\frac{1}{2}$ **o** $\frac{1}{\sqrt{2}}$ **p** $-\frac{1}{2}$
q $-\sqrt{3}$ **r** $-\sqrt{3}$ **s** $\frac{1}{\sqrt{3}}$ **t** $\sqrt{3}$
u −1 **v** $-\frac{1}{\sqrt{3}}$

4 a 0.52 **b** −0.68 **c** 0.52 **d** 0.4
e −0.52 **f** 0.68 **g** −0.4 **h** −0.68
i −0.52 **j** 0.68 **k** −0.4

5 a 0.4 **b** −0.7 **c** 0.4 **d** 1.2
e −0.4 **f** 0.7 **g** −1.2 **h** −0.7
i −0.4 **j** 0.7 **k** −1.2

6 a $\frac{1}{2},\ -\frac{\sqrt{3}}{2},\ -\frac{1}{\sqrt{3}}$ **b** $-\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ 1$
c $\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 1$ **d** $-\frac{\sqrt{3}}{2},\ -\frac{1}{2},\ \sqrt{3}$
e $\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 1$ **f** $-\frac{1}{2},\ \frac{\sqrt{3}}{2},\ -\frac{1}{\sqrt{3}}$

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Exercise 6C

1 **a** 0.6 **b** 0.6 **c** -0.7 **d** 0.3
e -0.3 **f** $\frac{10}{7}$ **g** -0.3 **h** 0.6
i -0.6 **j** -0.3 **k** $\frac{10}{7}$ **l** 0.3

2 **a** $-\frac{4}{5}, -\frac{4}{3}$ **b** $-\frac{12}{13}, -\frac{5}{12}$
c $-\frac{2\sqrt{6}}{5}, -2\sqrt{6}$ **d** $-\frac{5}{13}, \frac{12}{5}$
e $-\frac{3}{5}, -\frac{3}{4}$ **f** $-\frac{12}{13}, \frac{5}{12}$ **g** $-\frac{3}{5}, -\frac{3}{4}$

Exercise 6D

1 **a** $2\pi, 3$ **b** $\frac{2\pi}{3}, 5$ **c** $\pi, \frac{1}{2}$ **d** $6\pi, 2$
e $\frac{\pi}{2}, 3$ **f** $2\pi, \frac{1}{2}$ **g** $4\pi, 3$ **h** $3\pi, 2$

2 **a** Dilation of factor 4 from the x-axis, dilation of factor $\frac{1}{3}$ from the y-axis;
Amplitude = 4; Period = $\frac{2\pi}{3}$

b Dilation of factor 5 from the x-axis, dilation of factor 3 from the y-axis;
Amplitude = 5; Period = 6π

c Dilation of factor 6 from the x-axis, dilation of factor 2 from the y-axis;
Amplitude = 6; Period = 4π

d Dilation of factor 4 from the x-axis, dilation of factor $\frac{1}{5}$ from the y-axis;
Amplitude = 4; Period = $\frac{2\pi}{5}$

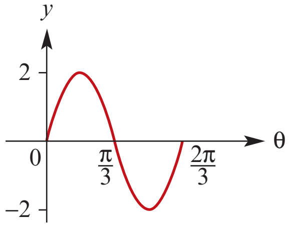
3 **a** Dilation of factor 2 from the x-axis, dilation of factor $\frac{1}{3}$ from the y-axis;
Amplitude = 2; Period = $\frac{2\pi}{3}$

b Dilation of factor 3 from the x-axis, dilation of factor 4 from the y-axis;
Amplitude = 3; Period = 8π

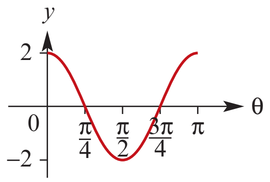
c Dilation of factor 6 from the x-axis, dilation of factor 5 from the y-axis;
Amplitude = 6; Period = 10π

d Dilation of factor 3 from the x-axis, dilation of factor $\frac{1}{7}$ from the y-axis;
Amplitude = 3; Period = $\frac{2\pi}{7}$

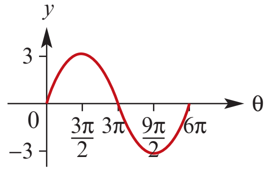
4 **a** Amplitude = 2
Period = $\frac{2\pi}{3}$



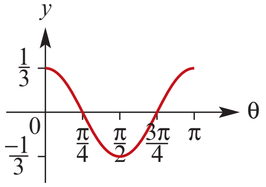
b Amplitude = 2
Period = π



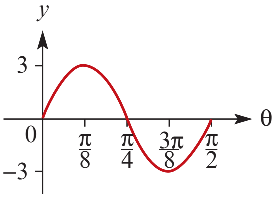
c Amplitude = 3
Period = 6π



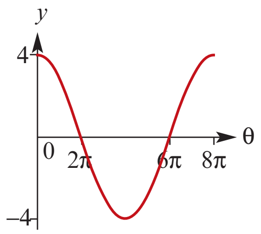
d Period = π
Amplitude = $\frac{1}{3}$



e Amplitude = 3
Period = $\frac{\pi}{2}$



f Amplitude = 4
Period = 8π



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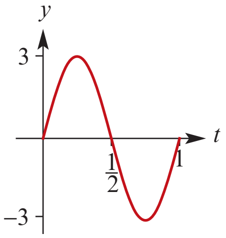

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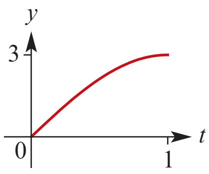

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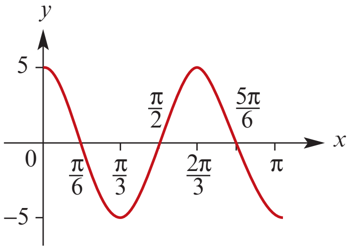

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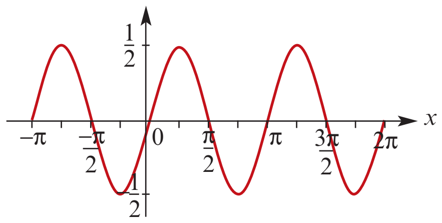
8



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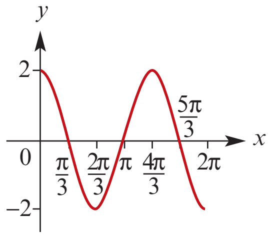

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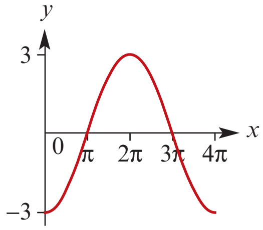


11 $y = 2\sin\left(\frac{x}{3}\right)$

12 $y = \frac{1}{2}\cos\left(\frac{x}{3}\right)$

13 $y = \frac{1}{2}\sin\left(\frac{x}{2}\right)$

Exercise 6E

1 **a** $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}$ **b** $\frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{23\pi}{6}$

c $\frac{4\pi}{3}, \frac{5\pi}{3}, \frac{10\pi}{3}, \frac{11\pi}{3}$ **d** $\frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}$

e $\frac{\pi}{2}, \frac{5\pi}{2}$ **f** $\pi, 3\pi$

2 **a** $-\frac{5\pi}{6}, -\frac{\pi}{6}$ **b** $-\frac{\pi}{6}, \frac{\pi}{6}$ **c** $-\frac{5\pi}{6}, \frac{5\pi}{6}$

3 **a** $\frac{\pi}{4}, \frac{3\pi}{4}$ **b** $\frac{3\pi}{4}, \frac{5\pi}{4}$ **c** $\frac{5\pi}{6}, \frac{7\pi}{6}$

d $\frac{7\pi}{6}, \frac{11\pi}{6}$ **e** $\frac{\pi}{4}, \frac{7\pi}{4}$ **f** $\frac{2\pi}{3}, \frac{4\pi}{3}$

4 **a** 0.643501, 2.49809 **b** 0.643501, 5.63968

c 3.60836, 5.81642 **d** 1.77215, 4.51103

5 **a** 17.46°, 162.54° **b** 66.42°, 293.58°

c 233.13°, 306.87° **d** 120°, 240°

6 **a** 60°, 300° **b** 60°, 120° **c** 225°, 315°

d 120°, 240° **e** 60°, 120° **f** 150°, 210°

7 **a** $\frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$

b $\frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}$ **c** $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$

d $\frac{5\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{23\pi}{12}$

e $\frac{5\pi}{12}, \frac{7\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}$ **f** $\frac{5\pi}{8}, \frac{7\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$

8 **a** $\frac{5\pi}{18}, \frac{7\pi}{18}, \frac{17\pi}{18}, \frac{19\pi}{18}, \frac{29\pi}{18}, \frac{31\pi}{18}$

b $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$

c $\frac{\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{17\pi}{12}, \frac{23\pi}{12}$

d $\frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}, \frac{25\pi}{18}, \frac{29\pi}{18}$

e $\frac{\pi}{8}, \frac{3\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}$

f $\frac{5\pi}{18}, \frac{7\pi}{18}, \frac{17\pi}{18}, \frac{19\pi}{18}, \frac{29\pi}{18}, \frac{31\pi}{18}$

g $\frac{5\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{23\pi}{12}$

h $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

i $\frac{3\pi}{8}, \frac{5\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}$

9 **a** 2.03444, 2.67795, 5.17604, 5.81954

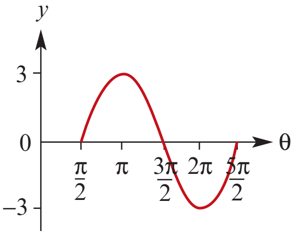
b 1.89255, 2.81984, 5.03414, 5.96143

c 0.57964, 2.56195, 3.72123, 5.70355

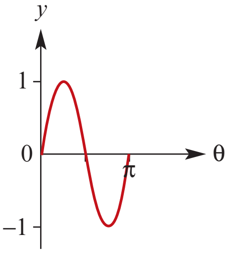
d 0.309098, 1.7853, 2.40349, 3.87969, 4.49789, 5.97409

Exercise 6F

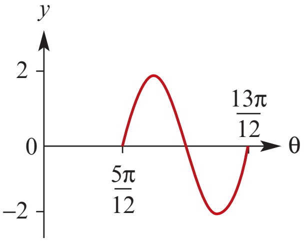
1 **a** Period = 2π; Amplitude = 3; $y = \pm 3$



b Period = π; Amplitude = 1; $y = \pm 1$



c Period = $\frac{2\pi}{3}$; Amplitude = 2; $y = \pm 2$



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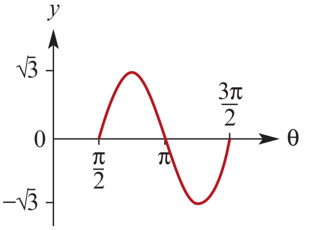
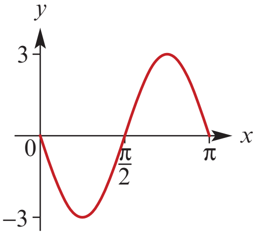
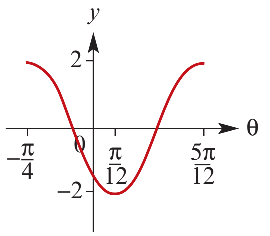
d Period = π; Amplitude = $\sqrt{3}$; $y = \pm\sqrt{3}$



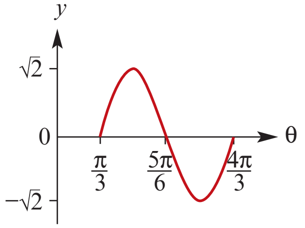
e Period = π; Amplitude = 3; $y = \pm 3$



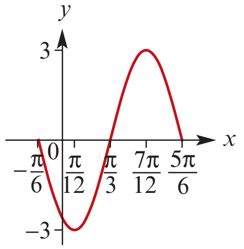
f Period = $\frac{2\pi}{3}$; Amplitude = 2; $y = \pm 2$



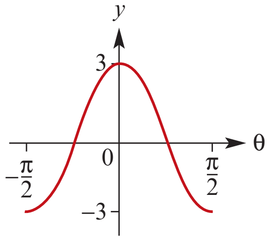
g Period = π; Amplitude = $\sqrt{2}$; $y = \pm\sqrt{2}$



h Period = π; Amplitude = 3; $y = \pm 3$

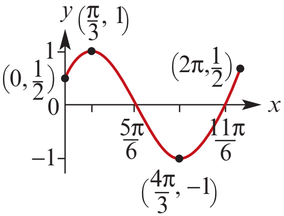


i Period = π; Amplitude = 3; $y = \pm 3$



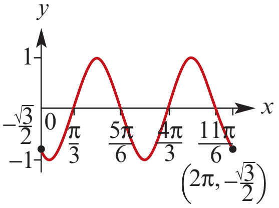
2 a $f(0) = \frac{1}{2}$, $f(2\pi) = \frac{1}{2}$

b



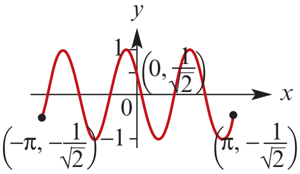
3 a $f(0) = -\frac{\sqrt{3}}{2}$, $f(2\pi) = -\frac{\sqrt{3}}{2}$

b



4 a $f(-\pi) = -\frac{1}{\sqrt{2}}$, $f(\pi) = -\frac{1}{\sqrt{2}}$

b



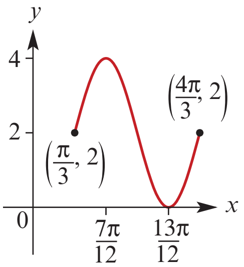
5 a $y = 3\sin\left(\frac{x}{2}\right)$ **b** $y = 3\sin(2x)$

c $y = 2\sin\left(\frac{x}{3}\right)$ **d** $y = \sin 2\left(x - \frac{\pi}{3}\right)$

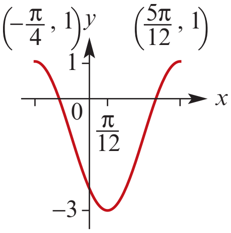
e $y = \sin\frac{1}{2}\left(x + \frac{\pi}{3}\right)$

Exercise 6G

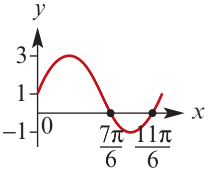
1 a



b



2 a



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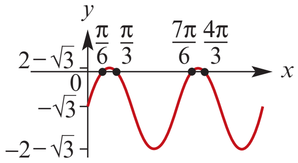
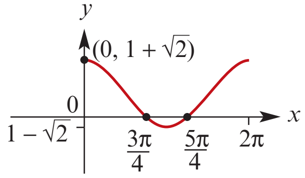
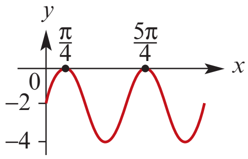
b



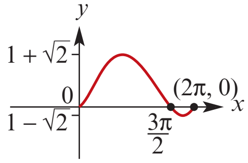
c



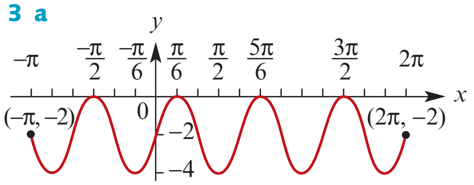
d



e



3 a



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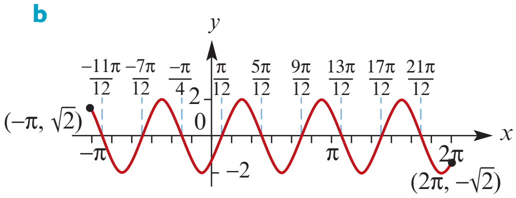

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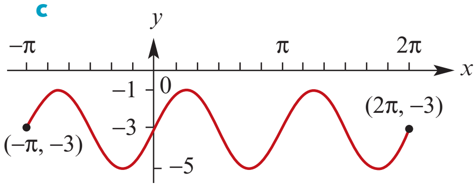
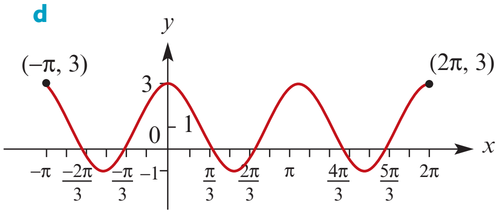
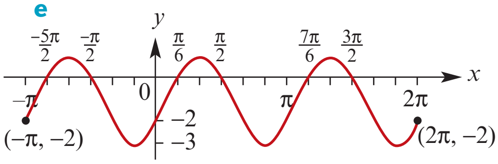


d



e



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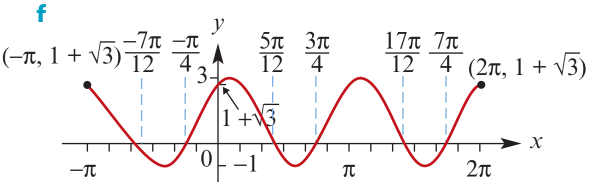
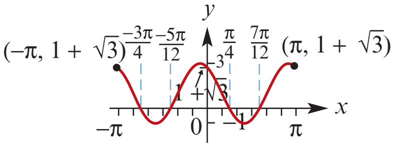


4 a



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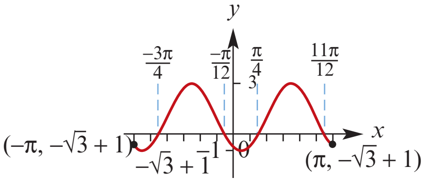

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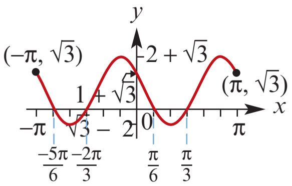
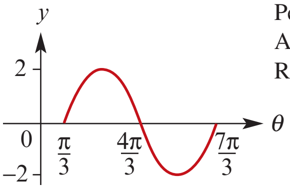
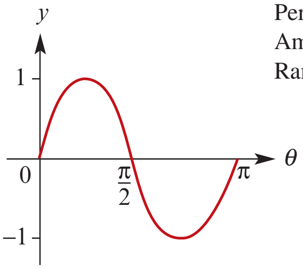


5 a



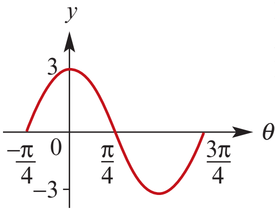
Period = 2π
Amplitude = 2
Range = $[-2, 2]$

b



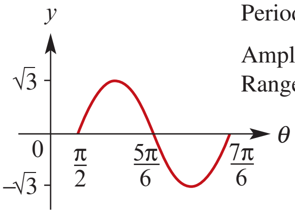
Period = π
Amplitude = 1
Range = $[-1, 1]$

c



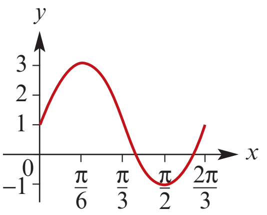
Period = π
Amplitude = 3
Range = $[-3, 3]$

d



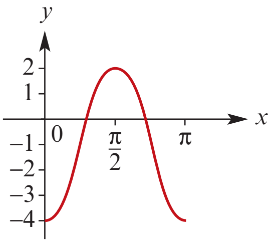
Period = $\frac{2\pi}{3}$
Amplitude = $\sqrt{3}$
Range = $[-\sqrt{3}, \sqrt{3}]$

e



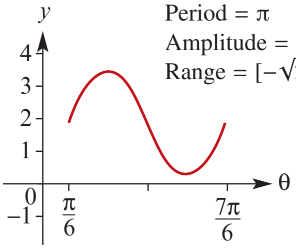
Period $= \frac{2\pi}{3}$
Amplitude $= 2$
Range $= [-1, 3]$

f



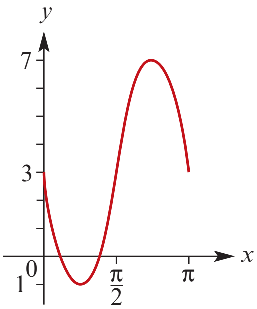
Period $= \pi$
Amplitude $= 3$
Range $= [-4, 2]$

g



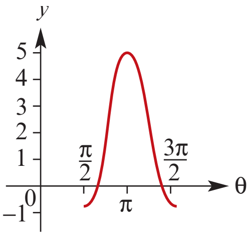
Period $= \pi$
Amplitude $= \sqrt{2}$
Range $= [-\sqrt{2} + 2, 2 + \sqrt{2}]$

h



Period $= \pi$
Amplitude $= 4$
Range $= [-1, 7]$

i



Period $= \pi$
Amplitude $= 3$
Range $= [-1, 5]$

6 a $y = \frac{1}{2}\cos\left(\frac{1}{3}\left(x - \frac{\pi}{4}\right)\right)$

b $y = 2\cos\left(x - \frac{\pi}{4}\right)$

c $y = -\frac{1}{3}\cos\left(x - \frac{\pi}{3}\right)$

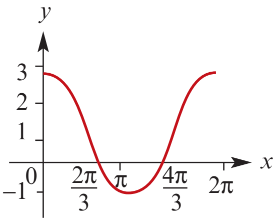
7 a
- Dilation of factor 3 from the x-axis
- Dilation of factor $\frac{1}{2}$ from the y-axis
- Reflection in the x-axis

b
- Dilation of factor 3 from the x-axis
- Dilation of factor $\frac{1}{2}$ from the y-axis
- Reflection in the x-axis
- Translation $\frac{\pi}{3}$ units to the right

c
- Dilation of factor 3 from the x-axis
- Dilation of factor $\frac{1}{2}$ from the y-axis
- Translation $\frac{\pi}{3}$ units to the right and 2 units up

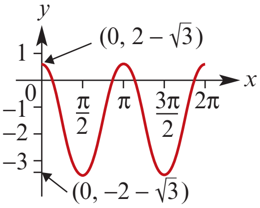
d
- Dilation of factor 2 from the x-axis
- Dilation of factor $\frac{1}{2}$ from the y-axis
- Reflection in the x-axis
- Translation $\frac{\pi}{3}$ units to the right and 5 units up

8 a



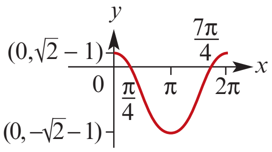
Intercepts: $\left(\frac{2\pi}{3}, 0\right), \left(\frac{4\pi}{3}, 0\right)$

b



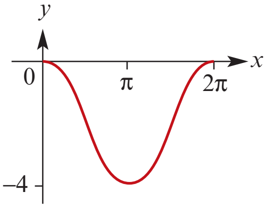
Intercepts: $\left(\frac{\pi}{12}, 0\right), \left(\frac{11\pi}{12}, 0\right), \left(\frac{13\pi}{12}, 0\right), \left(\frac{23\pi}{12}, 0\right)$

c



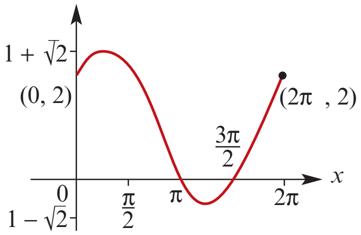
Intercepts: $\left(\frac{\pi}{4}, 0\right), \left(\frac{7\pi}{4}, 0\right)$

d



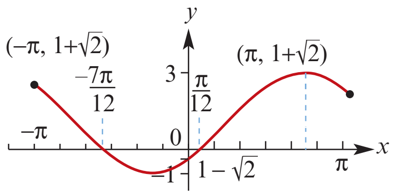
Intercepts: $(0, 0), (2\pi, 0)$

e

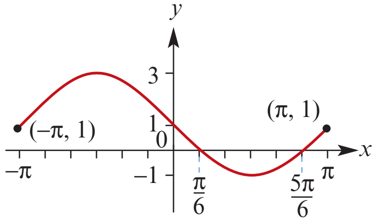


Intercepts: $(\pi, 0), \left(\frac{3\pi}{2}, 0\right)$

9 a



b



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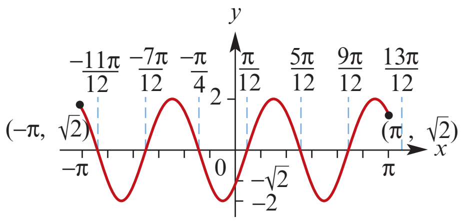

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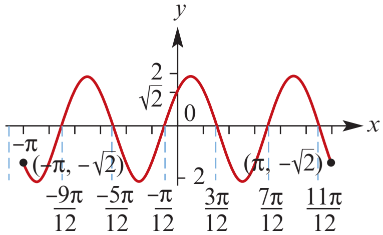

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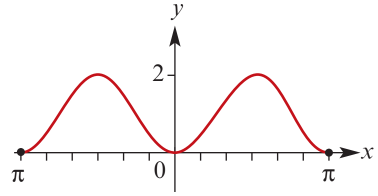
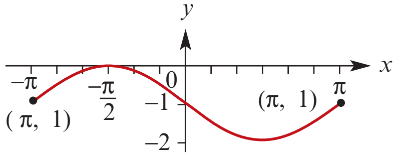
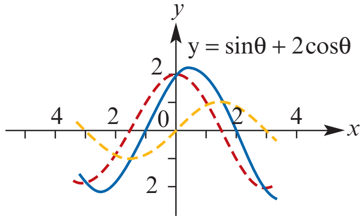


f

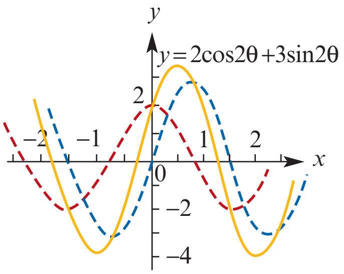


Exercise 6H

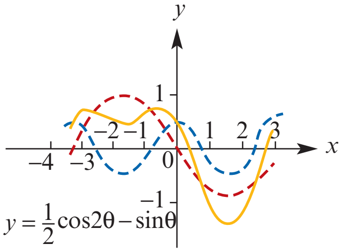
1 a



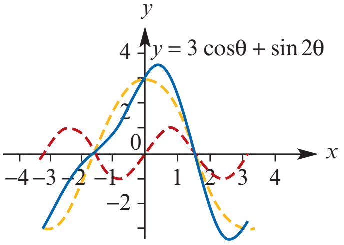
b



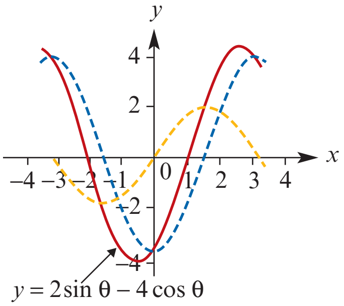
c



d



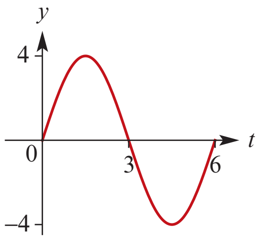
e



Exercise 6I

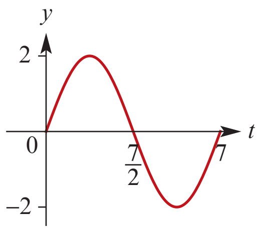
1 a $n = \frac{\pi}{3}$

$A = 4$



b $n = \frac{2\pi}{7}$

$A = 2$



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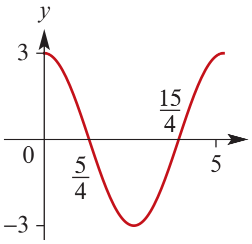
c $n = \frac{2\pi}{5}$

$A = 3$



2 $A = 3,\ n = \frac{\pi}{4}$

3 $A = -4,\ n = \frac{\pi}{6}$

4 $A = 0.5,\ \varepsilon = \frac{-\pi}{3}$

5 $A = 3,\ n = 3,\ b = 5$

6 $A = 4,\ n = \frac{\pi}{4},\ \varepsilon = \frac{-\pi}{2}$

(*Note:* ε can take infinitely many values)

7 $A = 2,\ n = \frac{\pi}{3},\ \varepsilon = \frac{-\pi}{6}$

(*Note*: ε can take infinitely many values)

8 $A = 4,\ n = \frac{\pi}{4},\ d = 2,\ \varepsilon = \frac{-\pi}{2}$

(*Note*: ε can take infinitely many values)

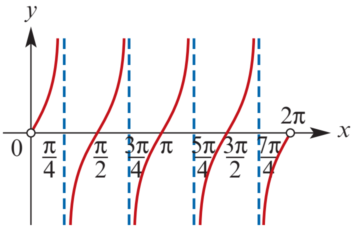
9 $A = 2,\ n = \frac{\pi}{3},\ d = 2,\ \varepsilon = \frac{-\pi}{6}$

(*Note:* ε can take infinitely many values)

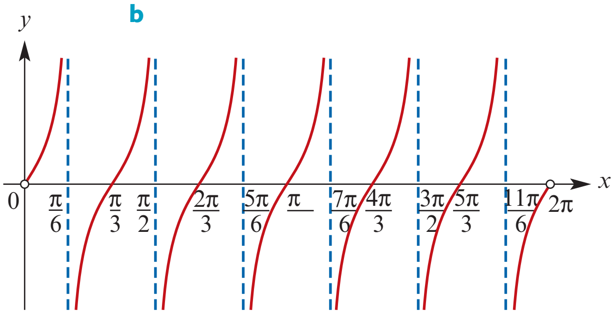
Exercise 6J

1 a $\frac{\pi}{3}$ **b** 2π **c** $\frac{2\pi}{3}$ **d** 1 **e** 2

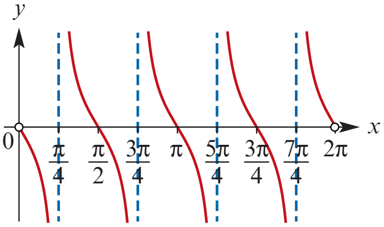
2 a



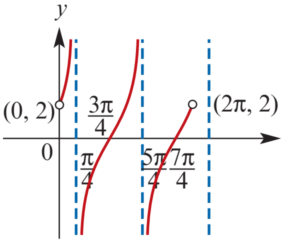
b



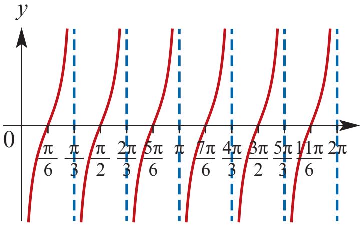
c



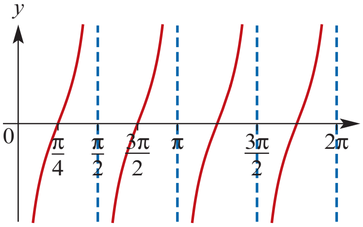
3 a



b



c



4 a $\frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$ **b** $-\frac{5\pi}{8}, -\frac{\pi}{8}, \frac{3\pi}{8}, \frac{7\pi}{8}$

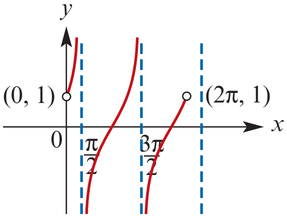
c $-\frac{2\pi}{3}, -\frac{\pi}{6}, \frac{\pi}{3}, \frac{5\pi}{6}$ **d** $-\frac{5\pi}{6}, -\frac{\pi}{3}, \frac{\pi}{6}, \frac{2\pi}{3}$

e $-\frac{11\pi}{12}, -\frac{5\pi}{12}, \frac{\pi}{12}, \frac{7\pi}{12}$

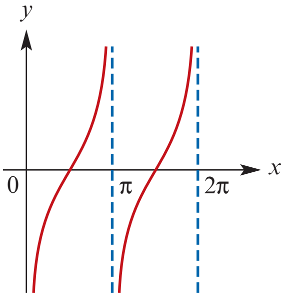
5 $\frac{11\pi}{24}, \frac{23\pi}{24}, \frac{35\pi}{24}, \frac{47\pi}{24}$

6 $\frac{7\pi}{12}, \frac{19\pi}{12}$

7 a



b



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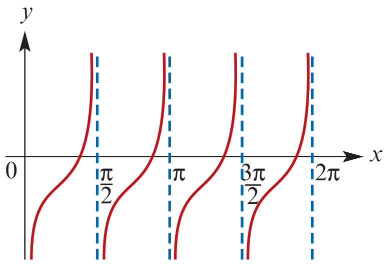
c



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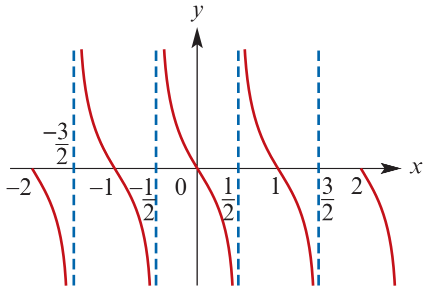

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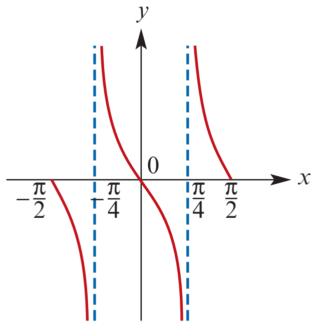
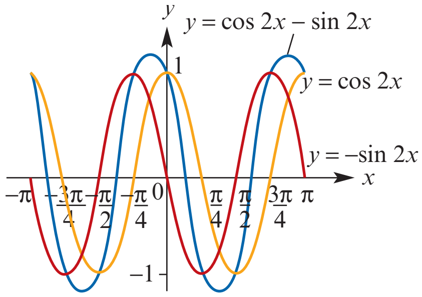


10 a c



b $\left(\frac{-5\pi}{8}, \frac{-1}{\sqrt{2}}\right), \left(\frac{-\pi}{8}, \frac{1}{\sqrt{2}}\right), \left(\frac{3\pi}{8}, \frac{-1}{\sqrt{2}}\right), \left(\frac{7\pi}{8}, \frac{1}{\sqrt{2}}\right)$

11 a $\frac{\pi}{6}, \frac{7\pi}{6}$

b $\frac{\pi}{16}, \frac{5\pi}{16}, \frac{9\pi}{16}, \frac{13\pi}{16}, \frac{17\pi}{16}, \frac{21\pi}{16}, \frac{25\pi}{16}, \frac{29\pi}{16}$

c $\frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{19\pi}{12}$

d $\frac{5\pi}{12}, \frac{11\pi}{12}, \frac{17\pi}{12}, \frac{23\pi}{12}$

e $\frac{\pi}{4}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{5\pi}{4}, \frac{19\pi}{12}, \frac{23\pi}{12}$

f 0.4636, 3.6052

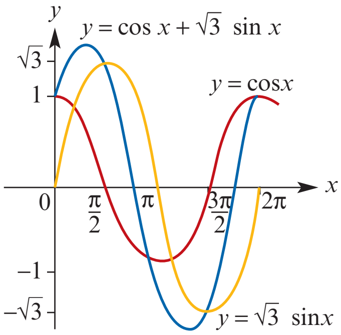
g 1.1071, 4.2487

h $\frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$

i $\frac{\pi}{18}, \frac{7\pi}{18}, \frac{13\pi}{18}, \frac{19\pi}{18}, \frac{25\pi}{18}, \frac{31\pi}{18}$

j $\frac{\pi}{9}, \frac{4\pi}{9}, \frac{7\pi}{9}, \frac{10\pi}{9}, \frac{13\pi}{9}, \frac{16\pi}{9}$

12 a c



b $\left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right), \left(\frac{7\pi}{6}, -\frac{\sqrt{3}}{2}\right)$

13 a $\frac{7\pi}{24}, \frac{19\pi}{24}, \frac{31\pi}{24}, \frac{43\pi}{24}$

b $\frac{5\pi}{12}, \frac{11\pi}{12}, \frac{17\pi}{12}, \frac{23\pi}{12}$

c $\frac{11\pi}{36}, \frac{23\pi}{36}, \frac{35\pi}{36}, \frac{47\pi}{36}, \frac{59\pi}{36}, \frac{71\pi}{36}$

14 $A = 5,\ n = 3$

15 $A = 6,\ n = \frac{\pi}{2}$

Exercise 6K

1 a i 2π **ii** 4π **iii** -4π

b i $\frac{4\pi}{3}, \frac{8\pi}{3}$ **ii** $\frac{14\pi}{3}, \frac{10\pi}{3}$ **iii** $\frac{-14\pi}{3}, \frac{-10\pi}{3}$

2 a $2n\pi \pm \frac{\pi}{6}, n \in \mathbb{Z}$

b $\frac{2n\pi}{3} + \frac{\pi}{9}$ or $\frac{2n\pi}{3} + \frac{2\pi}{9}, n \in \mathbb{Z}$

c $n\pi + \frac{\pi}{3}, n \in \mathbb{Z}$

3 a $\frac{\pi}{6}, \frac{5\pi}{6}$ **b** $\frac{\pi}{12}, \frac{11\pi}{12}$ **c** $\frac{\pi}{3}, \frac{5\pi}{6}$

4 $\frac{-11\pi}{6}, \frac{-7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$

5 $\frac{-\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}$

6 a $x = n\pi - \frac{\pi}{6}$ or $x = n\pi - \frac{\pi}{2}, n \in \mathbb{Z}$

b $x = \frac{n\pi}{2} - \frac{\pi}{12}, n \in \mathbb{Z}$

c $x = 2n\pi + \frac{5\pi}{6}$ or $x = 2n\pi - \frac{\pi}{2}, n \in \mathbb{Z}$

7 $x = \frac{(4n-1)\pi}{4}$ or $x = n\pi, n \in \mathbb{Z}$;
$\left\{\frac{-5\pi}{4}, -\pi, \frac{\pi}{4}, 0, \frac{3\pi}{4}, \pi, \frac{7\pi}{4}\right\}$

8 $x = \frac{n\pi}{3}, n \in \mathbb{Z}; \left\{-\pi, \frac{-2\pi}{3}, \frac{-\pi}{3}, 0\right\}$

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9 $x = \dfrac{6n-1}{12}$ or $x = \dfrac{3n+2}{6}$, $n \in \mathbb{Z}$;
$\left\{\dfrac{-2}{3}, \dfrac{-7}{12}, \dfrac{-1}{6}, \dfrac{-1}{12}, \dfrac{1}{3}, \dfrac{5}{12}, \dfrac{5}{6}, \dfrac{11}{12}\right\}$

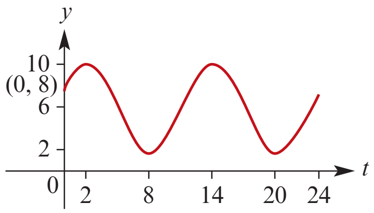
Exercise 6L

1 a i Amplitude = $1\frac{1}{2}$ **ii** Period = 12

iii $d(t) = 3.5 - 1.5\cos\left(\dfrac{\pi}{6}t\right)$ **iv** 1.5 m

b $[0, 3) \cup (9, 15) \cup (21, 24]$

2 a

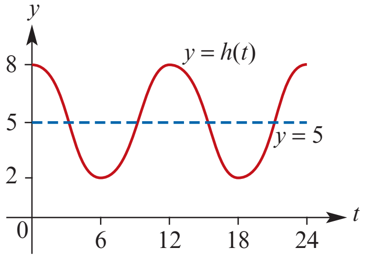


b 2:00 **c** 8:00, 20:00

3 a $A = 3$, $n = \dfrac{\pi}{6}$, $b = 5$, $\varepsilon = \dfrac{\pi}{2}$

b 2:21 a.m., 9:39 a.m., 2:21 p.m., 9:39 p.m.

c



4 a 5

b 1

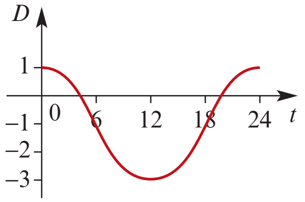
c $t = 0.524$ s, 2.618 s, 4.712 s

d $t = 0$ s, 1.047 s, 2.094 s

e Particle oscillates about the point $x = 3$ from $x = 1$ to $x = 5$

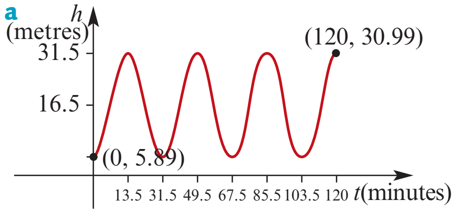
5 a 19.5°C **b** $D = -1 + 2\cos\left(\dfrac{\pi t}{12}\right)$

c



d $\{t : 4 < t < 20\}$

6 a



b 5.89 m **c** 27.51 s **d** 6 times

e 20 times **f** 4.21 m **g** 13.9 m

Chapter 6 review

Technology-free questions

1 a $\dfrac{\pi}{6}, \dfrac{5\pi}{6}$ **b** $\dfrac{-2\pi}{3}, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$

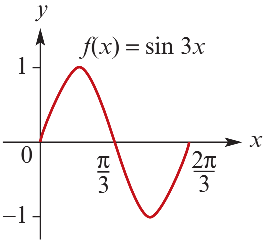
c $\dfrac{-\pi}{6}, \dfrac{\pi}{6}, \dfrac{11\pi}{6}$ **d** $\dfrac{-3\pi}{4}, \dfrac{-\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

e $\dfrac{-\pi}{6}, \dfrac{-5\pi}{6}, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$ **f** $\dfrac{-\pi}{4}, \dfrac{3\pi}{4}, \dfrac{7\pi}{4}$

g $\dfrac{-3\pi}{8}, \dfrac{-5\pi}{8}, \dfrac{3\pi}{8}, \dfrac{5\pi}{8}, \dfrac{11\pi}{8}, \dfrac{13\pi}{8}$

h $\dfrac{-7\pi}{18}, \dfrac{-11\pi}{18}, \dfrac{\pi}{18}, \dfrac{5\pi}{18}, \dfrac{13\pi}{18}, \dfrac{17\pi}{18}, \dfrac{25\pi}{18}, \dfrac{29\pi}{18}$

2 a



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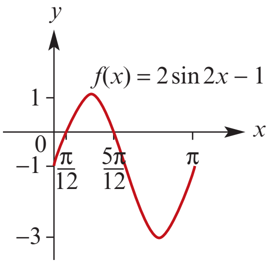

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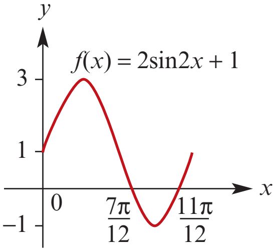

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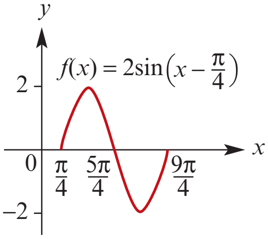

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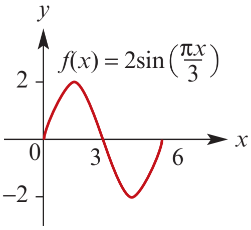

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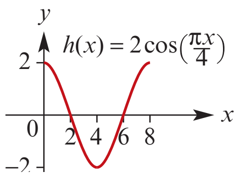
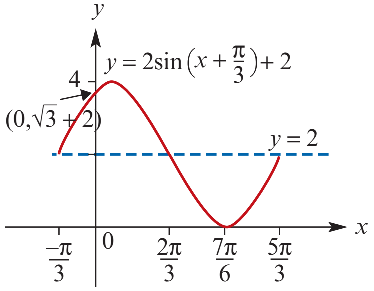
f



3 a 30, 150 **b** 45, 135, 225, 315
c 240, 300 **d** 90, 120, 270, 300
e 120, 240

4 a



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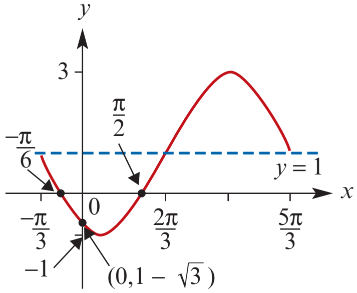
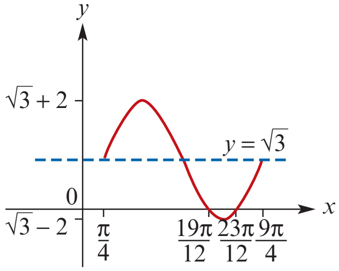


c



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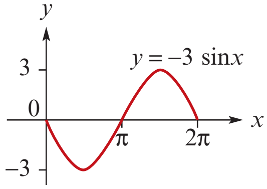

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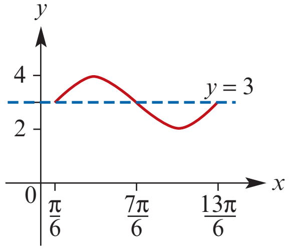

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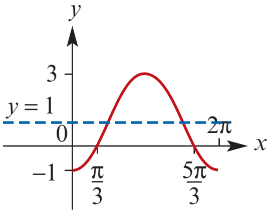
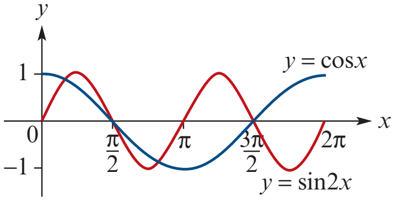
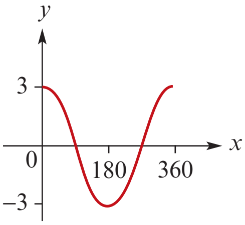


5

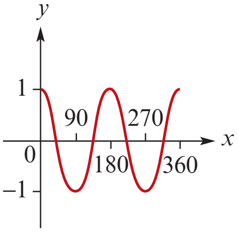


a 4 solutions **b** 4 solutions **c** 2 solutions

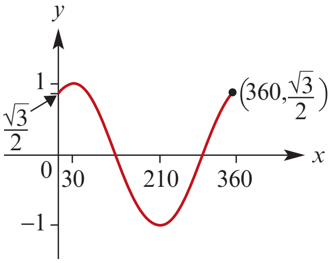
6 a



b



c



7 a $\frac{-2\pi}{3}, \frac{\pi}{3}$ **b** $\frac{-\pi}{4}, \frac{3\pi}{4}$
c $\frac{-5\pi}{8}, \frac{-\pi}{8}, \frac{3\pi}{8}, \frac{7\pi}{8}$ **d** $\frac{-2\pi}{3}, \frac{-\pi}{6}, \frac{\pi}{3}, \frac{5\pi}{3}$

8 $-\frac{2\pi}{3}, \frac{\pi}{3}$

9 a $\frac{1}{\sqrt{3}}$ **b** $-\frac{5\pi}{6}$

10 a $n\pi + \frac{\pi}{4}, n \in \mathbb{Z}$ **b** $\frac{2n\pi}{3}, n \in \mathbb{Z}$
c $-\frac{\pi}{4} + n\pi, n \in \mathbb{Z}$

Multiple-choice questions

1 C **2** A **3** E **4** D **5** A **6** C
7 C **8** B **9** C **10** E **11** D **12** B

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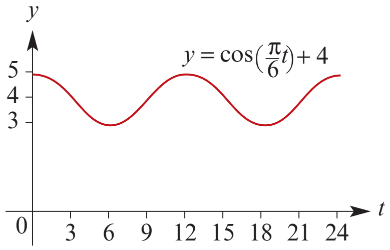
Extended-response questions

1 a



b 9:00, 15:00 **c** 8:00, 16:00

2 a Maximum = 210 cm; Minimum = 150 cm; Mean = 180 cm

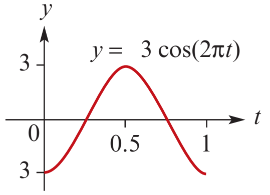
b $A = 30,\ n = \frac{\pi}{6},\ \varepsilon = \frac{-\pi}{2},\ b = 180$

c i 165 cm **ii** $180 - 15\sqrt{3} \approx 154$ cm

d ≈ 4:24, ≈ 7:36

3 a $a = -3,\ n = 2\pi$

b

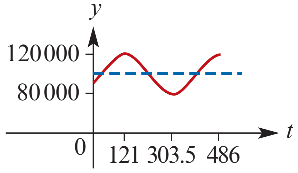


c i $t = \frac{1}{3}$ second **ii** $t = \frac{1}{6}$ second

d $t = 0.196$ seconds

4 a $a = 20\,000,\ b = 100\,000,\ n = \frac{2\pi}{365},\ \varepsilon \approx 5.77$

b



c i $t = 242.7,\ t = 364.3$

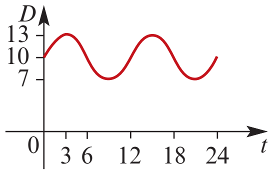
ii $t = 60.2,\ t = 181.8$

d ≈ 117 219 m^3/day

5 a i 1.83×10^{-3} hours **ii** 11.79 hours

b 25 April ($t = 3.856$), 14 August ($t = 7.477$)

6 a



b $\{ t : D(t) \geq 8.5 \} = [0, 7] \cup [11, 19] \cup [23, 24]$

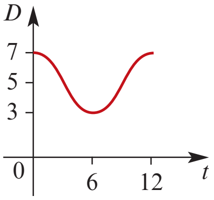
c 12.898 m

7 a $p = 5$

$q = 2$

$r = 30$

b



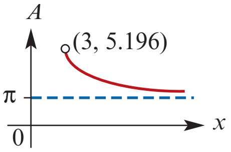
c A ship can enter 2 hours after low tide

8 a i $25\sqrt{3}$ **ii** 30

b 2.27, 0.53 **d** $b = 8$

e $\theta = 0.927$ or 1.837 **f** $a = 4\sqrt{3}$

9 c



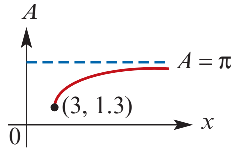
d i 2.055 **ii** 0.858

iii 0.0738 **iv** 0.0041

e $nr\tan\left(\frac{\pi}{n}\right)$

f i $n\sin\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{n}\right)$

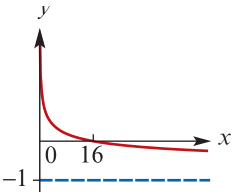
ii



Chapter 7

Exercise 7A

1 a

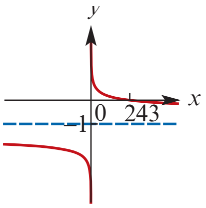


Domain = $\mathbb{R}^+$

Range = $(-1, \infty)$

Neither

b



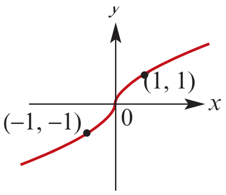
Domain = $\mathbb{R} \setminus \{0\}$

Range = $\mathbb{R} \setminus \{-1\}$

Neither

2 a 4 **b** 4 **c** 8 **d** −8 **e** −32 **f** 81

3 a

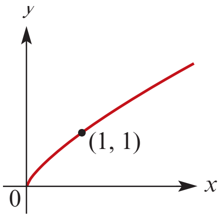


Domain = $\mathbb{R}$

Range = $\mathbb{R}$

Odd

b



Domain = $\mathbb{R}^+ \cup \{0\}$

Range = $\mathbb{R}^+ \cup \{0\}$

Neither

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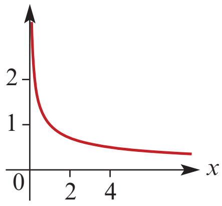
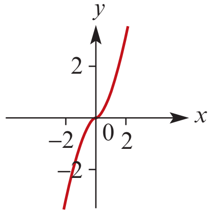
4 a i Domain = $\mathbb{R}^+$; Range = $\mathbb{R}^+$;
Asymptotes: $x = 0$, $y = 0$

ii



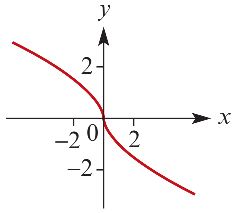
b i Domain = $\mathbb{R}$; Range = $\mathbb{R}$

ii



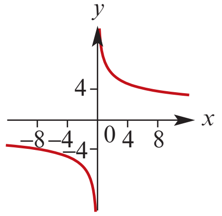
c i Domain = $\mathbb{R}$; Range = $\mathbb{R}$

ii



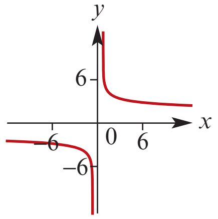
d i Domain = $\mathbb{R} \setminus \{0\}$; Range = $\mathbb{R} \setminus \{0\}$;
Asymptotes: $x = 0$, $y = 0$

ii



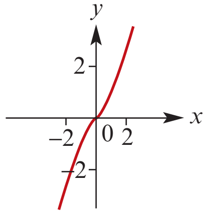
e i Domain = $\mathbb{R} \setminus \{0\}$; Range = $\mathbb{R} \setminus \{0\}$;
Asymptotes: $x = 0$, $y = 0$

ii



f i Domain = $\mathbb{R}$; Range = $\mathbb{R}$

ii



5 a $(0, 1)$ **b** $(0, 1)$

6 a Odd **b** Even **c** Odd
d Odd **e** Even **f** Odd

Exercise 7B

1 a $h(x) = f \circ g(x),\ f(x) = e^x,\ g(x) = x^3$
b $h(x) = f \circ g(x),\ f(x) = \sin x,\ g(x) = 2x^2$
c $h(x) = f \circ g(x),\ f(x) = x^n,\ g(x) = x^2 - 2x$
d $h(x) = f \circ g(x),\ f(x) = \cos x,\ g(x) = x^2$
e $h(x) = f \circ g(x),\ f(x) = x^2,\ g(x) = \cos x$
f $h(x) = f \circ g(x),\ f(x) = x^4,\ g(x) = x^2 - 1$
g $h(x) = f \circ g(x),\ f(x) = x^2,\ g(x) = \cos(2x)$
h $h(x) = f \circ g(x),\ f(x) = x^3 - 2x,$
$g(x) = x^2 - 2x$

2 a $f^{-1}\colon (0, \infty) \to \mathbb{R},\ f^{-1}(x) = \dfrac{1}{3}\log_e\left(\dfrac{x}{4}\right)$

b $g^{-1}\colon \mathbb{R} \setminus \{0\} \to \mathbb{R},\ g^{-1}(x) = \dfrac{8}{x^3}$

c $f \circ g\colon \mathbb{R} \setminus \{0\} \to \mathbb{R},\ f \circ g(x) = 4e^{\frac{6}{\sqrt[3]{x}}}$

d $g \circ f\colon \mathbb{R} \to \mathbb{R},\ g \circ f(x) = \dfrac{2}{\sqrt[3]{4e^{3x}}}$

e $(f \circ g)^{-1}\colon \mathbb{R}^+ \to \mathbb{R},\ (f \circ g)^{-1}(x) = \left(\dfrac{6}{\log_e(\frac{x}{4})}\right)^3$

f $(g \circ f)^{-1}\colon \mathbb{R}^+ \to \mathbb{R},\ (g \circ f)^{-1}(x) = \dfrac{1}{3}\log_e\left(\dfrac{2}{x^3}\right)$

3 a $f^{-1}\colon \mathbb{R}^+ \cup \{0\} \to \mathbb{R},\ f^{-1}(x) = x^{\frac{5}{2}}$
Both f and f^{-1} are strictly increasing

b $f^{-1}\colon \mathbb{R}^+ \cup \{0\} \to \mathbb{R},\ f^{-1}(x) = -x^{\frac{5}{2}}$
Both f and f^{-1} are strictly decreasing

c $f^{-1}\colon \mathbb{R}^+ \to \mathbb{R},\ f^{-1}(x) = x^{\frac{2}{5}}$
Both f and f^{-1} are strictly increasing

4 a i $f \circ g(x) = 3\sin(2x^2)$,
$g \circ f(x) = 9\sin^2(2x)$
ii $\operatorname{ran}(f \circ g) = [-3, 3]$, $\operatorname{dom}(f \circ g) = \mathbb{R}$,
$\operatorname{ran}(g \circ f) = [0, 9]$, $\operatorname{dom}(g \circ f) = \mathbb{R}$

b i $f \circ g(x) = -2\cos(2x^2)$,
$g \circ f(x) = 4\cos^2(2x)$
ii $\operatorname{ran}(f \circ g) = [-2, 2]$, $\operatorname{dom}(f \circ g) = \mathbb{R}$,
$\operatorname{ran}(g \circ f) = [0, 4]$, $\operatorname{dom}(g \circ f) = \mathbb{R}$

c i $f \circ g(x) = e^{x^2}$, $g \circ f(x) = e^{2x}$
ii $\operatorname{ran}(f \circ g) = (1, \infty)$, $\operatorname{dom}(f \circ g) = \mathbb{R}$,
$\operatorname{ran}(g \circ f) = (0, \infty)$, $\operatorname{dom}(g \circ f) = \mathbb{R}$

d i $f \circ g(x) = e^{2x^2} - 1$, $g \circ f(x) = (e^{2x} - 1)^2$
ii $\operatorname{ran}(f \circ g) = [0, \infty)$, $\operatorname{dom}(f \circ g) = \mathbb{R}$,
$\operatorname{ran}(g \circ f) = [0, \infty)$, $\operatorname{dom}(g \circ f) = \mathbb{R}$

e i $f \circ g(x) = -2e^{x^2} - 1$, $g \circ f(x) = (2e^x + 1)^2$
ii $\operatorname{ran}(f \circ g) = (-\infty, -3]$, $\operatorname{dom}(f \circ g) = \mathbb{R}$,
$\operatorname{ran}(g \circ f) = (1, \infty)$, $\operatorname{dom}(g \circ f) = \mathbb{R}$

f i $f \circ g(x) = \log_e(2x^2)$,
$g \circ f(x) = (\log_e(2x))^2$
ii $\operatorname{ran}(f \circ g) = \mathbb{R}$, $\operatorname{dom}(f \circ g) = \mathbb{R} \setminus \{0\}$,
$\operatorname{ran}(g \circ f) = [0, \infty)$, $\operatorname{dom}(g \circ f) = \mathbb{R}^+$

g i $f \circ g(x) = \log_e(x^2 - 1)$,
$g \circ f(x) = (\log_e(x - 1))^2$

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ii $\operatorname{ran}(f\circ g)=\mathbb{R}$, $\operatorname{dom}(f\circ g)=\mathbb{R}\setminus[-1,1]$, $\operatorname{ran}(g\circ f)=[0,\infty)$, $\operatorname{dom}(g\circ f)=(1,\infty)$

h **i** $f\circ g(x)=-\log_e(x^2)$, $g\circ f(x)=(\log_e x)^2$

ii $\operatorname{ran}(f\circ g)=\mathbb{R}$, $\operatorname{dom}(f\circ g)=\mathbb{R}\setminus\{0\}$, $\operatorname{ran}(g\circ f)=[0,\infty)$, $\operatorname{dom}(g\circ f)=\mathbb{R}^+$

5 **a** $g\circ f:\mathbb{R}\to\mathbb{R}$, $g\circ f(x)=\sin\left(2x-\dfrac{\pi}{3}\right)$

b Dilation of factor $\frac{1}{2}$ from the y-axis, then translation $\frac{\pi}{6}$ units to the right

6 **a** $g\circ f:(\frac{1}{3},\infty)\to\mathbb{R}$, $g\circ f(x)=\log_e(3x-1)$

b Dilation of factor $\frac{1}{3}$ from the y-axis, then translation $\frac{2}{3}$ units to the right

7 **a** $g(x)=3$, $g(x)=4$ **b** $g(x)=3x$, $g(x)=4x$

8 $g(x)=\log_e(2x-1)$

9 **a** $2e^{2x}$ **b** $\dfrac{1}{2}\log_e\left(\dfrac{x}{2}\right)$ **c** e^{x^2}

10 **a** $f^{-1}(x)=-\dfrac{1}{2}\log_e x$, $g^{-1}(x)=(x-1)^{\frac{1}{3}}$

b $f\circ g(x)=e^{-2(x^3+1)}$, $\operatorname{ran}(f\circ g)=\mathbb{R}^+$, $g\circ f(x)=e^{-6x}+1$, $\operatorname{ran}(g\circ f)=(1,\infty)$

11 **a** $f^{-1}(x)=\dfrac{1}{x}-1$ **b** $x=\dfrac{\sqrt{5}-1}{2}$

12 **a** $f^{-1}(x)=e^x-1$, $\operatorname{dom} f^{-1}=\mathbb{R}$, $g^{-1}(x)=\sqrt{x+1}-1$, $\operatorname{dom} g^{-1}=(-1,\infty)$

b $\log_e(x^2+2x+1)$

13 $f\circ g(x)=\log_e\left(\dfrac{1}{x}\right)$, $f(x)+f\circ g(x)=0$

14 x

15 **a** $f(g(x))=(x^2-10)(x^2-8)$, $g(f(x))=x^4-20x^3+148x^2-480x+572$

b $x=1$

16 $x=\pm\sqrt{6}$ or $x=\pm\sqrt{2}$

18 $a=\dfrac{1}{6}$, $b=-\dfrac{1}{2}$

20 $b=0$, $a=6$, $g(x)=e^{6x}$

21 **a** $f^{-1}:[1,\infty)\to\mathbb{R}$, $f^{-1}(x)=\log_e(x+\sqrt{x^2-1})$

b $g^{-1}:\mathbb{R}\to\mathbb{R}$, $g^{-1}(x)=\log_e(x+\sqrt{x^2+1})$

c Yes **d** Yes

22 **a** $x>y \Rightarrow f(x)>f(y) \Rightarrow g(f(x))>g(f(y))$

b $x>y \Rightarrow f(x)<f(y) \Rightarrow g(f(x))>g(f(y))$

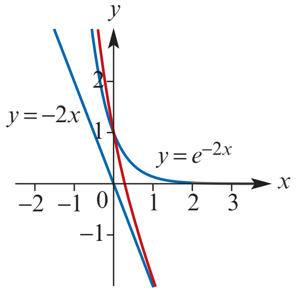
c $g\circ f$ is strictly decreasing

Exercise 7C

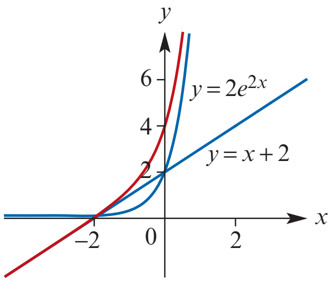
1 **a** **i** $e^{-2x}-2x$ **ii** $-2xe^{-2x}$

b **i** $e^{-1}+1$ **ii** e^{-1}

2



3



4 **a** **i** $\sin\left(\dfrac{\pi x}{2}\right)-2x$ **ii** $-2x\sin\left(\dfrac{\pi x}{2}\right)$

b **i** -1 **ii** -2

5 **a** **i** $\cos\left(\dfrac{\pi x}{2}\right)+e^x$ **ii** $e^x\cos\left(\dfrac{\pi x}{2}\right)$

b **i** 2 **ii** 1

6 $f(x)=\dfrac{f(x)+f(-x)}{2}+\dfrac{f(x)-f(-x)}{2}$

Exercise 7D

1 **a** $f(x-y)=2(x-y)=2x-2y=f(x)-f(y)$

b $f(x-y)=x-y-3\neq f(x)-f(y)$

2 $f(x-y)=f(x)-f(y)$

3 $f(x+y)=f(x)+f(y)-3$; $a=-3$

4 $f(x)+f(y)=\dfrac{3}{x}+\dfrac{3}{y}=\dfrac{3(x+y)}{xy}=(x+y)f(xy)$

5 $g(x)=0$ or $g(x)=1$

6 $g(x)=\pm1$

7 $f(2+3)=5^3=125\neq f(2)+f(3)$

8 $f\left(\dfrac{\pi}{2}+\dfrac{\pi}{2}\right)=\sin\pi=0\neq2=f\left(\dfrac{\pi}{2}\right)+f\left(\dfrac{\pi}{2}\right)$

9 $f(x)+f(y)=\dfrac{1}{x^2}+\dfrac{1}{y^2}=\dfrac{x^2+y^2}{x^2y^2}=(x^2+y^2)f(xy)$

10 **a** $h(1+2)=3^2=9$, $h(1)+h(2)=1+4=5$

11 $g(x+y)=2^{3x+3y}=2^{3x}\times2^{3y}=g(x)\times g(y)$

12 $f(xy)=(xy)^n=x^ny^n=f(x)f(y)$

$f\left(\dfrac{x}{y}\right)=\left(\dfrac{x}{y}\right)^n=\dfrac{x^n}{y^n}=\dfrac{f(x)}{f(y)}$

13 Let $x=2$ and $y=3$. Then $f(xy)=f(6)=6a$ and $f(x)f(y)=2a\cdot3a=6a^2$. But $6a=6a^2$ implies $a=0$ or $a=1$.

Exercise 7E

1 **a** $\dfrac{4}{m}$ **b** $m\geq4$ or $m<0$

c $f^{-1}:\mathbb{R}\to\mathbb{R}$, $f^{-1}(x)=\dfrac{x+4}{m}$

d $\left(\dfrac{4}{m-1},\dfrac{4}{m-1}\right)$, where $m\in\mathbb{R}\setminus\{0,1\}$

e $y=-\dfrac{1}{m}x-4$

2 **a** $\dfrac{c}{2}$ **b** $c\leq2$

c $f^{-1}:\mathbb{R}\to\mathbb{R}$, $f^{-1}(x)=\dfrac{c-x}{2}$

d $\left(\dfrac{c}{3},\dfrac{c}{3}\right)$ **e** $y=\dfrac{1}{2}x+c$

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3 **a** 0 and b
b $\left(\frac{b}{2}, \frac{-b^2}{4}\right)$
c **i** $(0, 0)$ and $(b-1, 1-b)$
ii $b = 1$ **iii** $b \in \mathbb{R} \setminus \{1\}$

4 $a = 5 - c$ and $b = -1$, where $y = ax^2 + bx + c$

5 **a** $-1 \pm 2\sqrt{2}$ **b** $\pm 2\sqrt{2}$ **c** $a = -8, b = 16$

6 **a** $(-\infty, 2a]$
b $\left(\frac{-1+\sqrt{1+8a}}{2}, \frac{-1+\sqrt{1+8a}}{2}\right)$
c $a = 1$ **d** $a = 3$ **e** $a = \frac{c^2+c}{2}$

7 **a** $(0, 0)$ and $(a, 0)$ **b** $(0, 0)$ **c** $\frac{a^4}{16}$
d $a = 3$ or $a = -5$

8 **a** $\frac{1}{b}\log_e\left(\frac{c}{a}\right)$ **b** $e^{\frac{b}{c}} - a$
c $\frac{a+1}{c}$ **d** $\frac{\log_e(c) - b}{a}$

9 **a** $x = a$ **b** $(a+1, 0)$
c $(a + e^{\frac{1}{c}}, 1)$ **d** $c = \frac{1}{\log_e(2-a)}$

10 **a** $y = -b$
b $(\log_e(b) + 1, 0)$
c **i** $b = \frac{1}{e}$ **ii** $0 < b < \frac{1}{e}$

11 $a = \frac{3d+4}{6}$, $b = 2 - d$ and $c = \frac{-3d-28}{6}$, where $y = ax^3 + bx^2 + cx + d$

12 **a** $c = 28 - 8\sqrt{6}$ or $c = 28 + 8\sqrt{6}$
b $c \in (-\infty, 8) \cup (8, 28 - 8\sqrt{6}) \cup (28 + 8\sqrt{6}, \infty)$

13 $a = \frac{5d-9}{30}$, $b = \frac{41-10d}{30}$ and $c = \frac{-25d-2}{30}$, where $y = ax^3 + bx^2 + cx + d$

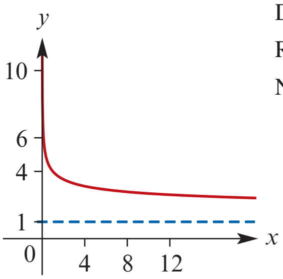
14 **a** $x = \frac{a-x'}{4}$ and $y = \frac{y'-2}{k}$
b $y = \frac{4k}{3-x} + 2$ **c** $k = \frac{-3}{2}$

15 **a** $x = \frac{a-x'}{4}$ and $y = \frac{y'+2}{2}$
b $y = 2 \times 2^{\frac{a-x}{4}} - 2$ **c** $a = 0$

Chapter 7 review

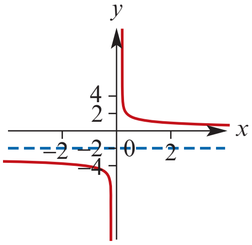
Technology-free questions

1 **a**



Domain = $\mathbb{R}^+$
Range = $(1, \infty)$
Neither

b



Domain = $\mathbb{R} \setminus \{0\}$
Range = $\mathbb{R} \setminus \{-2\}$
Neither

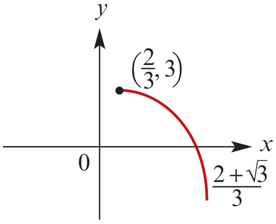
2 **a** 9 **b** 9 **c** 27
d 9 **e** −243 **f** 625

3 **a** **i** $f \circ g(x) = 3\cos(2x^2)$, $g \circ f(x) = 9\cos^2(2x)$
ii $\text{dom}(f \circ g) = \mathbb{R}$, $\text{ran}(f \circ g) = [-3, 3]$, $\text{dom}(g \circ f) = \mathbb{R}$, $\text{ran}(g \circ f) = [0, 9]$
b **i** $f \circ g(x) = \log_e(3x^2)$, $g \circ f(x) = (\log_e(3x))^2$
ii $\text{dom}(f \circ g) = \mathbb{R} \setminus \{0\}$, $\text{ran}(f \circ g) = \mathbb{R}$, $\text{dom}(g \circ f) = \mathbb{R}^+$, $\text{ran}(g \circ f) = [0, \infty)$
c **i** $f \circ g(x) = \log_e(2 - x^2)$, $g \circ f(x) = (\log_e(2-x))^2$
ii $\text{dom}(f \circ g) = (-\sqrt{2}, \sqrt{2})$, $\text{ran}(f \circ g) = (-\infty, \log_e 2)$, $\text{dom}(g \circ f) = (-\infty, 2)$, $\text{ran}(g \circ f) = [0, \infty)$
d **i** $f \circ g(x) = -\log_e(2x^2)$, $g \circ f(x) = (\log_e(2x))^2$
ii $\text{dom}(f \circ g) = \mathbb{R} \setminus \{0\}$, $\text{ran}(f \circ g) = \mathbb{R}$, $\text{dom}(g \circ f) = (0, \infty)$, $\text{ran}(g \circ f) = [0, \infty)$

4 **a** $h(x) = f \circ g(x)$, $g(x) = x^2$, $f(x) = \cos x$ (*Note:* answer not unique)
b $h(x) = f \circ g(x)$, $g(x) = x^2 - x$, $f(x) = x^n$ (*Note:* answer not unique)
c $h(x) = f \circ g(x)$, $g(x) = \sin x$, $f(x) = \log_e x$ (*Note:* answer not unique)
d $h(x) = f \circ g(x)$, $g(x) = \sin(2x)$, $f(x) = -2x^2$ (*Note:* answer not unique)
e $h(x) = f \circ g(x)$, $g(x) = x^2 - 3x$, $f(x) = x^4 - 2x^2$ (*Note:* answer not unique)

5 **a** **i** $(f+g)(x) = 2\cos\left(\frac{\pi x}{2}\right) + e^{-x}$
ii $(fg)(x) = 2e^{-x}\cos\left(\frac{\pi x}{2}\right)$
b **i** $(f+g)(0) = 3$ **ii** $(fg)(0) = 2$

6 **a** $\frac{2}{3}$
b $(-\infty, 3]$
c



d $f^{-1}(x) = \frac{2+\sqrt{3-x}}{3}$, ran $= [\frac{2}{3}, \infty)$, dom $= (-\infty, 3]$

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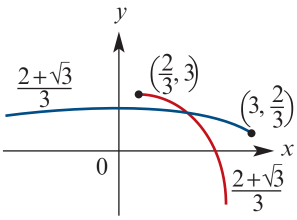
e



7 **a** $x = a$ **b** $(a+1, 0)$

c $(e+a, c)$ **d** $f^{-1}(x) = e^{\frac{x}{c}} + a$

e (a, ∞) **f** $c = \dfrac{1}{\log_e 2}, a = 0$

8 $a = \dfrac{1}{4}, b = \dfrac{3}{2}$

9 **a** $f^{-1}(x) = \left(\dfrac{x-1}{3}\right)^3$ **b** $f^{-1}(x) = \left(\dfrac{x+2}{4}\right)^3$

c $f^{-1}(x) = \dfrac{1}{3}\left((x-4)^{\frac{1}{3}} + 2\right)$

d $f^{-1}(x) = \left(\dfrac{3-x}{2}\right)^{\frac{1}{3}}$

Multiple-choice questions

1 B **2** E **3** E **4** D **5** E
6 E **7** B **8** C **9** D **10** C
11 B **12** D **13** E **14** B

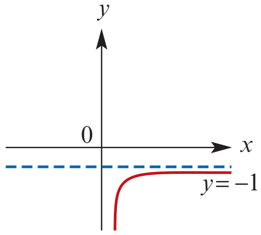
Extended-response questions

1 **a** $(0, 1), (-\infty, 0)$

b $f^{-1}(x) = -\log_e x, \ g^{-1}(x) = \dfrac{1}{x} + 1$

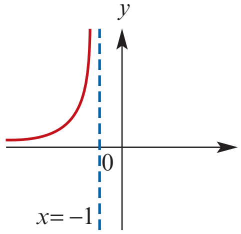
c **i** $g \circ f(x) = \dfrac{1}{e^{-x} - 1} = \dfrac{e^x}{1 - e^x}$

ii

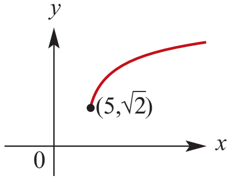


d **i** $(g \circ f)^{-1}(x) = \log_e\left(\dfrac{x}{x+1}\right)$

ii



2 **a** **i**

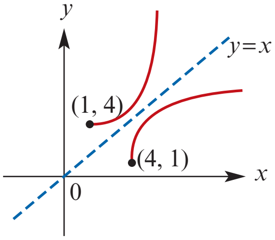


ii $[\sqrt{2}, \infty)$

iii $f^{-1}: [\sqrt{2}, \infty) \to \mathbb{R}, f^{-1}(x) = x^2 + 3$

b **i** $p = 3$ **ii** $h^{-1}(x) = x^2 + 3$

iii



3 **a** $(0, 1]$

b $(0, 1]$

c $\operatorname{ran} g \subseteq \operatorname{dom} f, \ f \circ g(x) = \sin\left(\dfrac{1}{x}\right)$

d Not defined as $\operatorname{ran} f \not\subseteq \operatorname{dom} g$

e $g^{-1}(x) = \dfrac{1}{x}$, $\operatorname{dom} g^{-1} = (0, 1]$, $\operatorname{ran} g^{-1} = [1, \infty)$

f $\operatorname{ran} f = \operatorname{dom} g^{-1}, \ g^{-1} \circ f(x) = \dfrac{1}{\sin x}$, $\operatorname{dom}(g^{-1} \circ f) = (0, \pi), \operatorname{ran}(g^{-1} \circ f) = [1, \infty)$

4 **a** $a = 2$ **b** $c = 2 - k \log_e(2)$

c $k = \dfrac{10}{\log_e\left(\frac{d+2}{2}\right)}$ **d** $k = 10$

Chapter 8

Technology-free questions

1 **a** Domain $= \mathbb{R} \setminus \{0\}$; Range $= \mathbb{R} \setminus \{2\}$

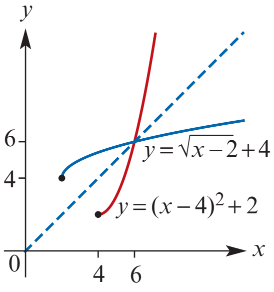
b Domain $= \left[\dfrac{2}{3}, \infty\right)$; Range $= (-\infty, 3]$

c Domain $= \mathbb{R} \setminus \{2\}$; Range $= (3, \infty)$

d Domain $= \mathbb{R} \setminus \{2\}$; Range $= \mathbb{R} \setminus \{4\}$

e Domain $= [2, \infty)$; Range $= [-5, \infty)$

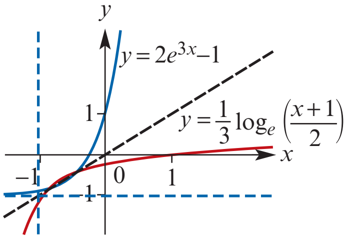
2 $f^{-1}: [4, \infty) \to \mathbb{R}, f^{-1}(x) = (x-4)^2 + 2$



3 $f^{-1}: \mathbb{R} \setminus \{1\} \to \mathbb{R}, f^{-1}(x) = \dfrac{x+2}{1-x}$

4 **a** $f^{-1}(x) = \dfrac{1}{3}\log_e\left(\dfrac{x+1}{2}\right)$, $\operatorname{dom} f^{-1} = (-1, \infty)$

b



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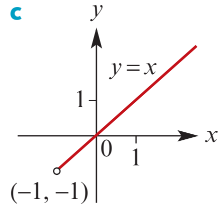
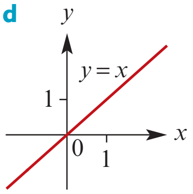
c



d



e $y = 2x$

5 1

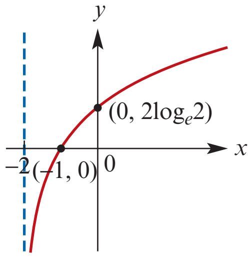
6 $x = \sqrt[3]{12a}$

7 $x = -9$

8 $x = \log_e\left(\frac{3}{2}\right)$

9 a $a = -1$ and $b = 2\log_e 2$

b



10 $x = 0$ or $x = 1$

11 $x = -\frac{7\pi}{9}$ or $x = \frac{\pi}{9}$ or $x = \frac{5\pi}{9}$

12 a Range = [2, 8]; Period = 6

b $x = \frac{\pi}{12}$ or $x = \frac{3\pi}{4}$

13 a $m \in \mathbb{R} \setminus \{-1, 2\}$ **b** $m = 2$ **c** $m = -1$

14 $a = -2$ and $b = 1$

15 a $m = \pm 2\sqrt{2}$ **b** $m > 2\sqrt{2}$ or $m < -2\sqrt{2}$

c $-2\sqrt{2} < m < 2\sqrt{2}$

16 a i $a = -3$ **ii** $a = 5$ or $a = 1$ **iii** $a = -3$

b $5y - 3x + 4 = 0$, $\tan^{-1}\left(\frac{3}{5}\right)$

17 a Odd

b $f^{-1}(x) = \sqrt[3]{\frac{x}{2}}$

c i 2 **ii** −1

iii $x = 0$ or $x = 2^{-\frac{1}{2}}$ or $x = -2^{-\frac{1}{2}}$

18 a 4 **b** $\sqrt{5}$ **c** $2 - 2a$ **d** $\sqrt{2a - 5}$

e $x = -8$ **f** $x = \frac{103}{2}$ **g** $x < 1$

19 a i $f \circ g(x) = 4x^2 + 8x - 3$

ii $g \circ f(x) = 16x^2 - 16x + 3$

iii $g \circ f^{-1}(x) = \frac{1}{16}(x^2 + 14x + 33)$

b Dilation of factor $\frac{1}{4}$ from the y-axis, then translation $\frac{3}{4}$ units to the right

c Translation 1 unit to the left and 1 unit down

20 $x = 2n\pi \pm \frac{\pi}{6}, n \in \mathbb{Z}$

21 $A = \frac{8}{5}$ and $k = \log_e\left(\frac{5}{2}\right)$

22 a $-2 \le x \le \frac{1}{2}$ or $x \ge 3$ **b** $x < 0$

Multiple-choice questions

1 D	**2** A	**3** B	**4** E	**5** E	**6** C
7 A	**8** A	**9** B	**10** C	**11** C	**12** B
13 A	**14** E	**15** D	**16** C	**17** E	**18** C
19 B	**20** D	**21** C	**22** C	**23** D	**24** D
25 E	**26** A	**27** D	**28** D	**29** A	**30** E
31 D	**32** E	**33** A	**34** A	**35** E	**36** C
37 D	**38** C	**39** E	**40** B	**41** B	**42** C
43 D	**44** B	**45** B	**46** E	**47** A	**48** C
49 D	**50** B	**51** C	**52** C	**53** A	**54** B
55 A	**56** B	**57** D	**58** D	**59** D	**60** A
61 A	**62** D	**63** B	**64** D	**65** B	**66** E
67 C	**68** C	**69** B	**70** C		

Extended-response questions

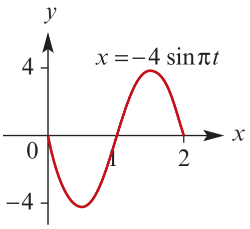
1 a $a = -0.09$, $b = 9$ **b** $DE = 2.79$ m

c Length = $2\sqrt{30} \approx 10.95$ m

2 a $a = -3$ **b** $x = -1$, $x = -\frac{1}{2}$, $x = 2$

c ii $b = \frac{7}{2}$, $c = \frac{3}{2}$

3 a



b i $x = 0$ **ii** $x = -4$ **iii** $x = 0$

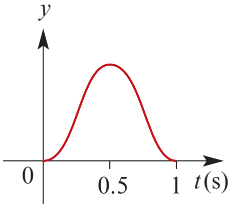
c $t = \frac{7}{6}$

d Period = $\frac{2\pi}{\pi}$ = 2 seconds

4 a i 0 **ii** 2.5 **iii** 0

b 1 second

c



d $t = 0.35$ seconds

5 a $k = 0.0292$ **b** 150×10^6

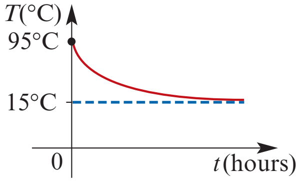
c 6.4494×10^8 **d** 23.417 years

6 a $A = 80$, $k = 0.3466$

b 17.5°C

c 6 hours 18 minutes and 14 seconds after 2:00 p.m., i.e. 8:18:14

d



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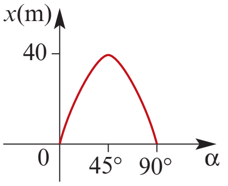
7 **a** 62.5 metres **b**



c 24.3° or 65.7°

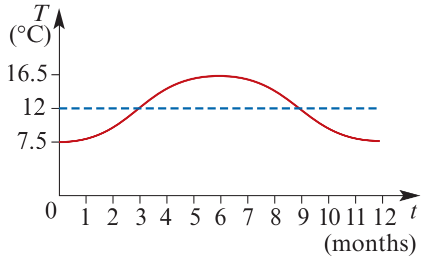
8 **a** Area = $0.02(0.92)^{\frac{x}{10}}$

b 0.0199 mm^2 **c** Load = $0.02(0.92)^{10-2.9x}$

d $x < 2.59$ m

9 **a** **i** 12 units **ii** $OQ = h - k$, $OR = h + k$

b



c $h = 12$, $k = 4.5$

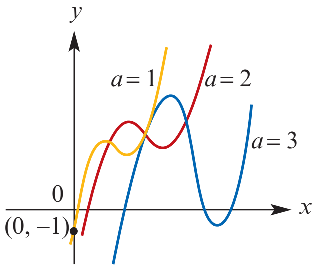
10 **a** Carriage A: $(0.83)^n I$

Carriage B: $0.66(0.89)^n I$

b 6 stations

11 **a** **i** $\left(3 + \frac{1}{\sqrt{a}}, 0\right)$, $\left(3 - \frac{1}{\sqrt{a}}, 0\right)$ **ii** $\frac{1}{\sqrt{a}}$

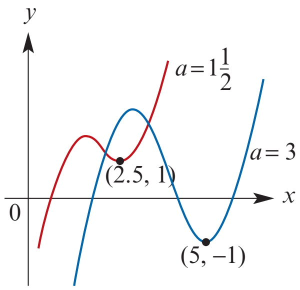
b **i**



ii $a = \frac{3\sqrt{3}}{2}$ **iii** $a > \frac{3\sqrt{3}}{2}$ **iv** $a = 3$

v $a = 1\frac{1}{2}$

vi



c **i** $\left(a, \frac{-4}{27}a^3 + a\right)$ **ii** $\frac{4}{27}a^3$, $\frac{2a}{3}$

iii $A = \frac{4a^4}{81}$ **iv** $a = 3$ **v** $3\sqrt[4]{375}$

12 **a** $D = 0.05t^2 - 0.25t + 1.8$

b \$3 000 000

13 $a = \frac{-7}{48}$, $b = \frac{23}{24}$, $c = 7.5$

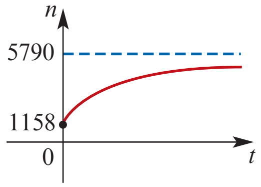
Rainfall at noon was $\frac{35}{6}$ mm per hour

Rainfall greatest at $\frac{23}{7}$ hours after 4 a.m.

(approx 7:17 a.m.)

14 **a** **i** $n = 5790$ **ii** 1158

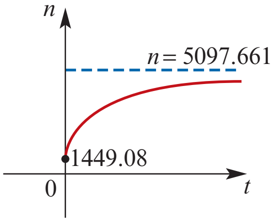
iii



iv $t = \frac{-100}{3}\log_e\left(\frac{179}{1600}\right)$

b **i** $a = 2.518$, $b = 0.049$, $c = 5097.661$

ii



Chapter 9

Exercise 9A

1 −1

2 −1

3 **a** $h + 9$ **b** 9

4 **a** $x + 1$ **b** $2x^3 + 1$ **c** 40 **d** 0

e 5 **f** 1 **g** $2x + 1$ **h** $3x$

i $3x^3 + x$ **j** $6x$

5 **a** $2 + 3h + h^2$ **b** 2

6 $2x + h$, $2x$

7 $h + 6$, 6

8 **a** $10x$ **b** 3 **c** 0 **d** $6x + 4$

e $15x^2$ **f** $10x - 6$

Exercise 9B

1 **a** $5x^4$ **b** $28x^6$ **c** 6 **d** $10x - 4$

e $12x^2 + 12x + 2$ **f** $20x^3 + 9x^2$

g $-4x + 4$ **h** $18x^2 - 4x + 4$

2 **a** −4 **b** −8 **c** −2 **d** −4

3 **a** −4 **b** −36

4 **a** $3t^2$ **b** $3t^2 - 2t$ **c** $x^3 + 9x^2$

5 **a** −2 **b** 0 **c** $15x^2 - 6x + 2$

d $\frac{6x^2 - 8}{5}$ **e** $4x - 5$ **f** $12x - 12$

g $50x^4$ **h** $27x^2 + 3$

6 **a** $4x - 15x^2$ **b** $-4z - 6$ **c** $18z^2 - 8z$

d $-2 - 15x^2$ **e** $-4z - 6$ **f** $-3z^2 - 8z$

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7 a $(-\tfrac{1}{2}, 3\tfrac{1}{2})$ **b** $(2, 32), (-2, -32)$
c $(2, 6)$ **d** $(0, 0), (2, -4)$

8 a $(1, 7)$ **b** $\left(\dfrac{5}{4}, \dfrac{59}{8}\right)$

9 a $x = 1$ **b** $x = 0$ **c** $x = \dfrac{1+\sqrt{3}}{2}$
d $x = \dfrac{3+\sqrt{3}}{6}$ **e** $x = \dfrac{1-\sqrt{3}}{2}$

10 a 78.69° **b** 0° **c** 45° **d** 135°
e 63.43° **f** 116.57°

11 a $8x - 4$ **b** $2x + 2$ **c** $6x^2 - 12x + 18$
d $x^2 - 2x + 1$

12 a $(3, 16)$, gradient $= 8$
b $(0, -1)$, gradient $= -1$
c $(-1, 6)$, gradient $= -8$
d $(4, 594)$, gradient $= 393$
e $(1, -28)$, gradient $= -92$
f $(2\tfrac{1}{2}, 0)$, gradient $= 0$

13 a $x = 1$ **b** $x = 1$ **c** $x > 1$ **d** $x < 1$
e $x = 2\tfrac{2}{3}$ **f** $x = 4$ or $x = -2$

14 a $(-\infty, -1) \cup (1, \infty)$ **b** $(-1, 1)$ **c** $\{1, -1\}$

15 a $(-1, 0.5) \cup (2, \infty)$ **b** $(-\infty, -1) \cup (0.5, 2)$
c $\{-1, 0.5, 2\}$

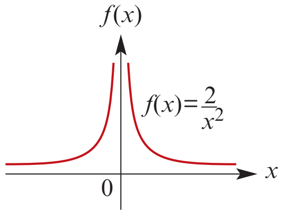
16 a $(-\tfrac{1}{4}, \infty) \setminus \{2\}$ **b** $(-\infty, -\tfrac{1}{4})$ **c** $\{-\tfrac{1}{4}, 2\}$

17 a $(2, -12)$ **b** $(3, -11)$ **c** $\left(\dfrac{5}{4}, -\dfrac{183}{16}\right)$

21 a $(-\infty, -1]$ **b** $[2, \infty)$ **c** $(-\infty, 0]$
d $[\tfrac{3}{2}, \infty)$

Exercise 9C

1 a



b $\dfrac{-2(2+h)}{(1+h)^2}$ **c** -4

2 a $-\dfrac{1}{(x-3)^2}$ **b** $\dfrac{-1}{(x+2)^2}$

3 $-4x^{-5}$

4 a $-6x^{-3} - 5x^{-2}$ **b** $12x - \dfrac{15}{x^4}$
c $\dfrac{15}{x^4} - \dfrac{8}{x^3}$ **d** $-18x^{-4} - 6x^{-3}$ **e** $-\dfrac{2}{x^2}$

5 a $\dfrac{4}{z^2}$ **b** $\dfrac{-18-2z}{z^4}$ **c** $3z^{-4}$
d $\dfrac{-2z^3 + z^2 - 4}{z^2}$ **e** $\dfrac{6-12z}{z^4}$
f $-6x - \dfrac{6}{x^2}$

6 a $11\tfrac{3}{4}$ **b** $\dfrac{1}{32}$ **c** -1 **d** 5

7 $f'(x) = 10x^{-6} > 0$ for all $x \in \mathbb{R} \setminus \{0\}$

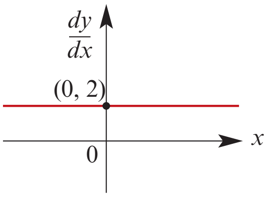
8 $\pm\dfrac{1}{2}$ **9** $a = -1,\ b = 4$

10 $\dfrac{1}{2}$ **11** $a = -9,\ b = 1$

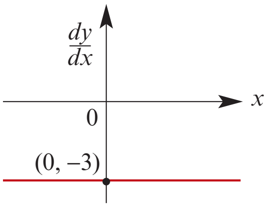
12 $k = 0$ or $k = \dfrac{3}{2}$

Exercise 9D

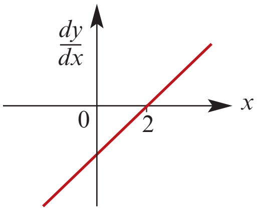
1 a



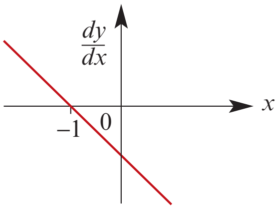
b



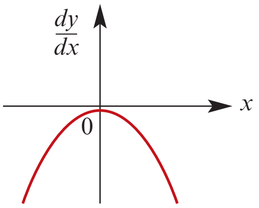
c



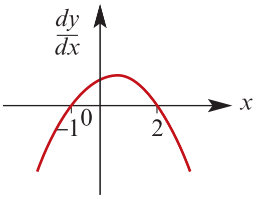
d



e



f



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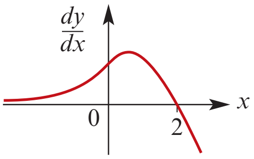

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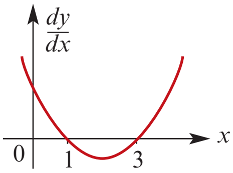

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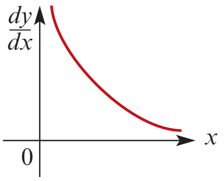

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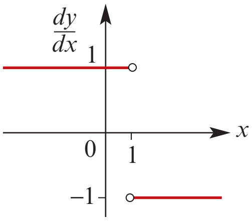

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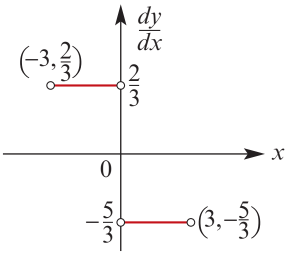

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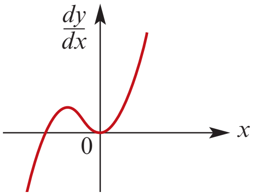

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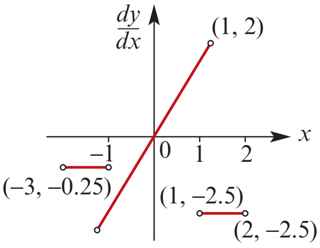

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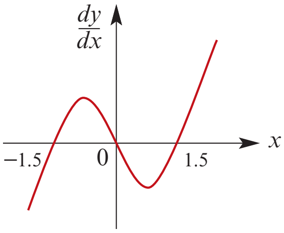

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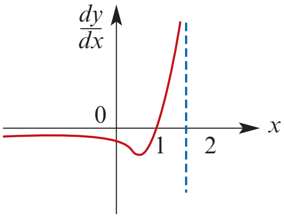

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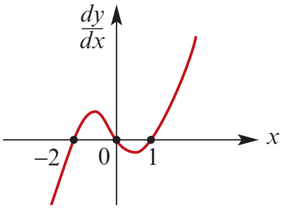

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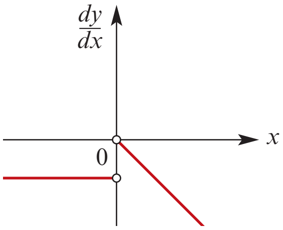

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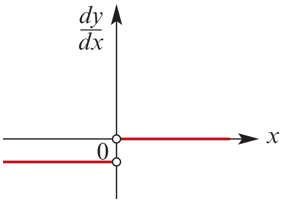
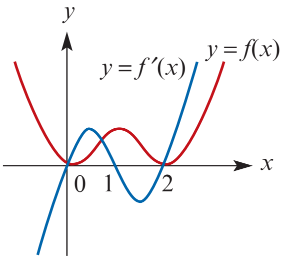


3 a D **b** F **c** B **d** C **e** A **f** E

4 a b



c i 0 **ii** 0 **iii** 0 **iv** 96

d i 1 **ii** 0.423

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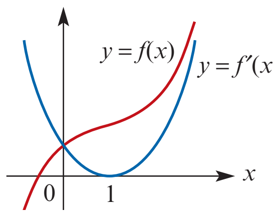
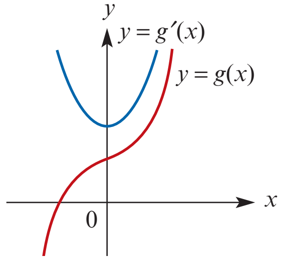
5



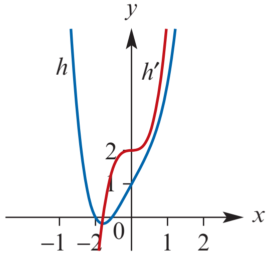
Gradient is 0 at $(1, \frac{4}{3})$;
Gradient is positive for $\mathbb{R} \setminus \{1\}$

6



Gradient is always positive;
Minimum gradient where $x = 0$

7 a



b i $x = -1.4945$ or $x = 0.7976$
ii $x = 0.6300$

Exercise 9E

1 a $8x(x^2+1)^3$ **b** $20x(2x^2-3)^4$
c $24(6x+1)^3$ **d** $an(ax+b)^{n-1}$
e $2anx(ax^2+b)^{n-1}$ **f** $\dfrac{6x}{(1-x^2)^4}$
g $-3\left(x^2-\dfrac{1}{x^2}\right)^{-4}\left(2x+\dfrac{2}{x^3}\right)$ **h** $(1-x)^{-2}$

2 a $6(x+1)^5$ **b** $4x^3(3x+1)(x+1)^7$
c $4\left(6x^3+\dfrac{2}{x}\right)^3\left(18x^2-\dfrac{2}{x^2}\right)$ **d** $-4(x+1)^{-5}$

3 -10 **4** $-\dfrac{1}{2}$ and $\dfrac{1}{2}$ **5** $2x\sqrt{3x^2+1}$

6 a $n[f(x)]^{n-1}f'(x)$ **b** $\dfrac{-f'(x)}{[f(x)]^2}$

Exercise 9F

1 $x^{-\frac{1}{2}}$

2 a $\dfrac{x^{-\frac{4}{5}}}{5}$ **b** $\dfrac{5}{2}x^{\frac{3}{2}}$
c $\dfrac{5}{2}x^{\frac{3}{2}}-\dfrac{3}{2}x^{\frac{1}{2}}$ **d** $\dfrac{3}{2}x^{-\frac{1}{2}}-\dfrac{20}{3}x^{\frac{2}{3}}$
e $-\dfrac{6}{7}x^{-\frac{13}{7}}$ **f** $-\dfrac{1}{4}x^{-\frac{5}{4}}+2x^{-\frac{1}{2}}$

3 a $\dfrac{1}{27}$ **b** $\dfrac{1}{12}$ **c** $\dfrac{2}{9}$ **d** $\dfrac{5}{2}$

4 a $\dfrac{1}{\sqrt{2x+1}}$ **b** $\dfrac{-3}{2\sqrt{4-3x}}$
c $\dfrac{x}{\sqrt{x^2+2}}$ **d** $\dfrac{-1}{\sqrt[3]{(4-3x)^2}}$
e $\dfrac{3}{2}\sqrt{x}-\dfrac{1}{\sqrt{x^3}}$ **f** $3\sqrt{x}\left(\dfrac{5x+6}{2}\right)$

7 a $\dfrac{x}{\sqrt{x^2+2}}$ **b** $\dfrac{2x-5}{3\sqrt[3]{(x^2-5x)^2}}$
c $\dfrac{2x+2}{5\sqrt[5]{(x^2+2x)^4}}$

Exercise 9G

1 a $5e^{5x}$ **b** $-21e^{-3x}$
c $-12e^{-4x}+e^x-2x$ **d** e^x-e^{-x}
e $e^{-2x}(e^x-1)$ **f** $2e^{2x}-2e^{-2x}$

2 a $-6x^2e^{-2x^3}$ **b** $2xe^{x^2}+3$
c $(2x-4)e^{x^2-4x}+3$ **d** $(2x-2)e^{x^2-2x+3}-1$
e $-\dfrac{1}{x^2}e^{\frac{1}{x}}$ **f** $\dfrac{1}{2}x^{-\frac{1}{2}}e^{x^{\frac{1}{2}}}$

3 a $\dfrac{9}{2}$ **b** $\dfrac{1}{2}e^{\frac{1}{2}}+4$

4 a 5 **b** $5e^4+2$

5 a $2f'(x)e^{2f(x)}$ **b** $2e^{2x}f'(e^{2x})$

6 a $8e^{2x}(e^{2x}-1)^3$ **b** $\dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$
c $\dfrac{e^x}{2\sqrt{e^x-1}}$ **d** $\dfrac{2}{3}e^{x^{\frac{2}{3}}}x^{-\frac{1}{3}}$
e $(2x-3)e^{(x-1)(x-2)}$ **f** e^{e^x+x}

Exercise 9H

1 a $\dfrac{2}{x}$ **b** $\dfrac{2}{x}$ **c** $2x+\dfrac{3}{x}$
d $\dfrac{3x-1}{x^2}$ **e** $\dfrac{3+x}{x}$ **f** $\dfrac{1}{x+1}$
g $\dfrac{1}{x+2}$ **h** $\dfrac{3}{3x-1}$ **i** $\dfrac{6}{6x-1}$

2 a $\dfrac{3}{x}$ **b** $\dfrac{3(\log_e x)^2}{x}$ **c** $\dfrac{2x+1}{x^2+x-1}$
d $\dfrac{3x^2+2x}{x^3+x^2}$ **e** $\dfrac{4}{2x+3}$ **f** $\dfrac{4}{2x-3}$

3 a $\dfrac{2x}{x^2+1}$ **b** 1

4 a $(e, 1)$, $m = \dfrac{1}{e}$
b $(e, \log_e(e^2+1))$, $m = \dfrac{2e}{e^2+1}$
c $(-e, 1)$, $m = -\dfrac{1}{e}$ **d** $(1, 1)$, $m = 2$
e $(1, 0)$, $m = 0$ **f** $(\frac{3}{2}, \log_e 2)$, $m = 1$

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5 $\frac{1}{2}$ **6** $\frac{1+2x}{1+x+x^2}$ **7** $\frac{3}{5}$ **8** 2

Exercise 9I

1 a $5\cos(5x)$ **b** $-5\sin(5x)$
c $5\sec^2(5x)$ **d** $2\sin x\cos x$
e $3\sec^2(3x+1)$ **f** $-2x\sin(x^2+1)$
g $2\sin\left(x-\frac{\pi}{4}\right)\cos\left(x-\frac{\pi}{4}\right)$
h $-2\cos\left(x-\frac{\pi}{3}\right)\sin\left(x-\frac{\pi}{3}\right)$
i $6\sin^2\left(2x+\frac{\pi}{6}\right)\cos\left(2x+\frac{\pi}{6}\right)$
j $6\sin^2\left(2x+\frac{\pi}{4}\right)\cos\left(2x+\frac{\pi}{4}\right)$

2 a $\frac{1}{\sqrt{2}}, \sqrt{2}$ **b** $1,\ 0$ **c** $2,\ 0$
d $0,\ 0$ **e** $1,\ 0$ **f** $1,\ 4$

3 a $-5\sin(x)-6\cos(3x)$
b $-\sin x+\cos x$ **c** $\cos x+\sec^2 x$
d $2\tan x\sec^2 x$

4 a $-\frac{\pi}{90}\sin x°$ **b** $\frac{\pi}{60}\cos x°$
c $\frac{\pi}{60}\sec^2(3x)°$

5 a $\tan x$ **b** $\frac{-1}{\sin x\cos x}$

6 a $2\cos(x)\,e^{2\sin x}$ **b** $-2\sin(2x)\,e^{\cos(2x)}$

Exercise 9J

1 a $20x^4+36x^2+4x$ **b** $9x^{\frac{1}{2}}+\frac{3}{2}x^{-\frac{1}{2}}$
c $3(2x-1)^2(8x-1)$ **d** $8(2x^2+1)(6x^2+1)$
e $5(3x+1)^{\frac{1}{2}}(3x+4)$ **f** $\frac{5x^2-8x+1}{\sqrt{2x-4}}$
g $x^2(3x^2+4x+3)(3x^2+2x+1)^{-2}$
h $2x^3(5x^2-2)(2x^2-1)^{-\frac{1}{2}}$
i $2x\sqrt[3]{x^2+2x}+\frac{2x^2(x+1)}{3\sqrt[3]{(x^2+2x)^2}}$
j $\frac{4(5x^2-4)^2(5x^2+2)}{x^3}$
k $\frac{3(x^6-16)}{x^4}$ **l** $\frac{2x^3(9x^2-8)}{5(x(x^2-1))^{4/5}}$

2 a $e^x(x^2+2x+1)$
b $e^{2x}(2x^3+3x^2+6x+5)$
c $2e^{4x+1}(x+1)(2x+3)$ **d** $\frac{-8x-7}{2e^{4x}\sqrt{x+1}}$

3 a $1+\log_e x$ **b** $2x+4x\log_e x$
c $e^x\log_e x+\frac{e^x}{x}$ **d** $1+\log_e(-x)$

4 a $\frac{2x^3(2-x)}{e^{2x}}$ **b** $2e^{2x+3}$
c $\frac{3}{2}(2e^{2x}+1)(e^{2x}+x)^{\frac{1}{2}}$ **d** $\frac{e^x(x-1)}{x^2}$
e $xe^{\frac{1}{2}x^2}$ **f** $-x^2e^{-x}$

5 a $e^x(f'(x)+f(x))$ **b** $\frac{e^x(f(x)-f'(x))}{[f(x)]^2}$
c $f'(x)e^{f(x)}$ **d** $2e^x f'(x)f(x)+[f(x)]^2e^x$

6 a $3x^2\cos(x)-x^3\sin(x)$
b $2x\cos x-(1+x^2)\sin x$
c $-e^{-x}\sin x+e^{-x}\cos x$
d $6\cos x-6x\sin x$
e $3\cos(3x)\cos(4x)-4\sin(4x)\sin(3x)$
f $2\sin(2x)+2\tan(2x)\sec(2x)$
g $12\sin x+12x\cos x$
h $2xe^{\sin x}+x^2\cos xe^{\sin x}$
i $2x\cos^2 x-2x^2\cos x\sin x$
j $e^x\tan x+e^x\sec^2 x$

7 a $-e^{\pi}$ **b** 0

8 2

Exercise 9K

1 a $\frac{4}{(x+4)^2}$ **b** $\frac{4x}{(x^2+1)^2}$ **c** $\frac{x^{-\frac{1}{2}}-x^{\frac{1}{2}}}{2(1+x)^2}$
d $\frac{(x+2)^2(x-3)(x-1)}{(x^2+1)^2}$ **e** $\frac{2+2x-x^2}{(x^2+2)^2}$
f $\frac{-4x}{(x^2-1)^2}$ **g** $\frac{x^2+4x+1}{(x^2+x+1)^2}$
h $\frac{-2(4x^3+3x^2+1)}{(2x^3+2x)^2}$

2 a $81,\ 378$ **b** $0,\ 0$ **c** $0, 0$
d $\frac{1}{2},\ 0$ **e** $\frac{3}{2},\ -\frac{1}{2}$

3 a $\frac{2x^2+x+1}{\sqrt{x^2+1}}$ **b** $\frac{x(7x^3+3x+4)}{2\sqrt{x^3+1}}$
c $\frac{5}{(x+3)^2}$

4 a $\frac{3e^x-2e^{4x}}{(3+e^{3x})^2}$ **b** $-\left(\frac{(x+1)\sin(x)+\cos(x)}{(x+1)^2}\right)$
c $\frac{x-x\log_e(x)+1}{x(x+1)^2}$

5 a $\frac{1-\log_e x}{x^2}$ **b** $\frac{1+x^2-2x^2\log_e x}{x(1+x^2)^2}$

6 a $\frac{9e^{3x}}{(3+e^{3x})^2}$ **b** $\frac{-2e^x}{(e^x-1)^2}$ **c** $\frac{-8e^{2x}}{(e^{2x}-2)^2}$

7 a -2 **b** -6π **c** $-e^{\pi}$ **d** $-\frac{1}{\pi}$

Exercise 9L

1 a 17 **b** 3 **c** -4 **d** $\frac{1}{8}$
e 3 **f** 4 **g** 2 **h** $2\sqrt{3}$
i -2 **j** 12 **k** $\frac{11}{9}$ **l** $\frac{1}{4}$

2 a 3, 4 **b** 7

3 a Discontinuity at $x=0$, as $f(0)=0$, $\lim_{x\to 0^+} f(x)=0$, but $\lim_{x\to 0^-} f(x)=2$
b Discontinuity at $x=1$ as $f(1)=3$, $\lim_{x\to 1^+} f(x)=3$, but $\lim_{x\to 0^-} f(x)=-1$

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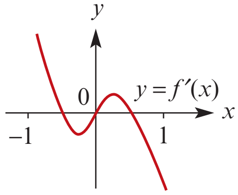
c Discontinuity at $x = 0$ as $f(0) = 1$, $\lim_{x\to 0^+} f(x) = 1$, but $\lim_{x\to 1^-} f(x) = 0$

4 $\mathbb{R}\setminus\{1\}$

Exercise 9M

1 a



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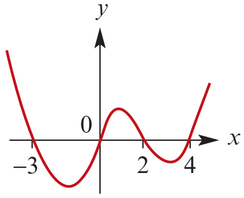

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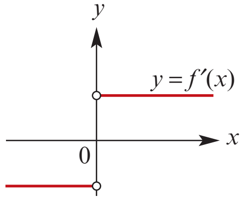

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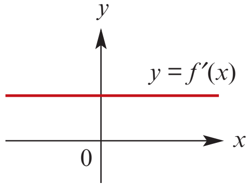
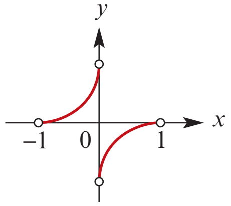


e



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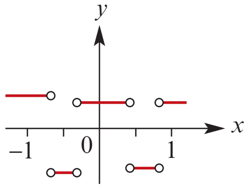
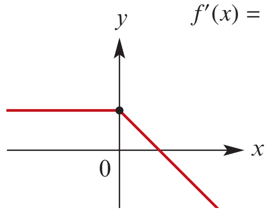


2 $f'(x) = \begin{cases} -2x + 3 & \text{if } x \ge 0 \\ 3 & \text{if } x < 0 \end{cases}$



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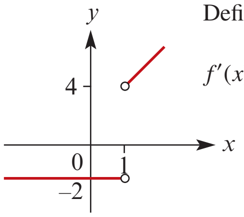
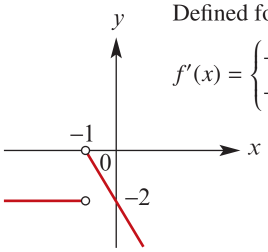


Defined for $\mathbb{R}\setminus\{1\}$

$f'(x) = \begin{cases} 2x + 2 & \text{if } x > 1 \\ -2 & \text{if } x < 1 \end{cases}$

4



Defined for $\mathbb{R}\setminus\{-1\}$

$f'(x) = \begin{cases} -2x - 2 & \text{if } x > -1 \\ -2 & \text{if } x < -1 \end{cases}$

5 a $\mathbb{R}\setminus\{1\}$, $f'(x) = \frac{1}{3}(x-1)^{-\frac{2}{3}}$

b $\mathbb{R}\setminus\{0\}$, $f'(x) = \frac{1}{5}(x)^{-\frac{4}{5}}$

c $\mathbb{R}\setminus\{0\}$, $f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$

d $\mathbb{R}\setminus\{-2\}$, $f'(x) = \frac{2}{5}(x+2)^{-\frac{3}{5}}$

Chapter 9 review

Technology-free questions

1 a 8 **b** −8

2 a $1 - \frac{x}{\sqrt{1-x^2}}$ **b** $\frac{-4x^2 - 2x + 12}{(x^2+3)^2}$

c $\frac{3}{2\sqrt{1+3x}}$ **d** $\frac{-2}{x^2} - \frac{1}{2}x^{-\frac{3}{2}}$

e $\frac{3x-15}{2\sqrt{x-3}}$ **f** $\frac{1+2x^2}{\sqrt{1+x^2}}$

g $\frac{4x}{(x^2+1)^2}$ **h** $\frac{-x^2+1}{(x^2+1)^2}$

i $\frac{10x}{3}(2+5x^2)^{-\frac{2}{3}}$ **j** $\frac{-2x^2-2x+4}{(x^2+2)^2}$

k $4x(3x^2+2)^{-\frac{1}{3}}$

3 a −6 **b** 1 **c** 5 **d** $\frac{1}{6}$

4 a $\frac{1}{x+2}$ **b** $3\cos(3x+2)$

c $-\frac{1}{2}\sin\left(\frac{x}{2}\right)$ **d** $(2x-2)e^{x^2-2x}$

e $\frac{1}{x-3}$ **f** $2\pi\cos(2\pi x)$

g $6\sin(3x+1)\cos(3x+1)$

h $\frac{1}{2x\sqrt{\log_e x}}$ **i** $\frac{2-2\log_e 2x}{x^2}$

j $2x\sin(2\pi x) + 2\pi x^2\cos(2\pi x)$

5 a $e^x\sin(2x) + 2e^x\cos(2x)$

b $4x\log_e x + 2x$ **c** $\frac{1-3\log_e x}{x^4}$

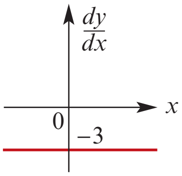
d $2\cos(2x)\cos(3x) - 3\sin(2x)\sin(3x)$

e $\dfrac{2}{\cos^2(2x)} = 2\sec^2(2x)$
f $-9\cos^2(3x+2)\sin(3x+2)$
g $2x\sin^2(3x) + 6x^2\cos(3x)\sin(3x)$

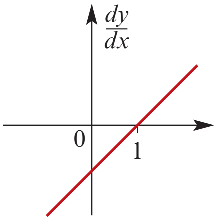
6 a $2e^2 \approx 14.78$ **b** 0
c $15e^3 + 2 \approx 303.28$ **d** 1

7 a ae^{ax} **b** ae^{ax+b} **c** $-be^{a-bx}$
d $abe^{ax} - abe^{bx}$ **e** $(a-b)e^{(a-b)x}$

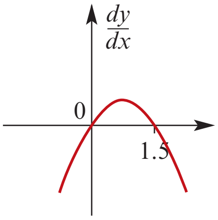
8 a



b



c



9 $2\left(4 - \dfrac{9}{x^2}\right)\left(4x + \dfrac{9}{x}\right),\ x = \pm\dfrac{3}{2}$

10 b $(\frac{3}{2}, \infty) \cap (-1, 4) = (\frac{3}{2}, 4)$

11 a $xf'(x) + f(x)$ **b** $\dfrac{-f'(x)}{[f(x)]^2}$
c $\dfrac{f(x) - xf'(x)}{[f(x)]^2}$ **d** $\dfrac{2xf(x) - 2x^2f'(x)}{[f(x)]^3}$

12 a $f \circ g(x) = 2\cos^3 x - 1$
b $g \circ f(x) = \cos(2x^3 - 1)$
c $g' \circ f(x) = -\sin(2x^3 - 1)$
d $(g \circ f)'(x) = -(6x^2)\sin(2x^3 - 1)$
e $\dfrac{3}{2}$ **f** $-\dfrac{3\sqrt{3}}{4}$

Multiple-choice questions

1 A	**2** B	**3** C	**4** A	**5** A
6 B	**7** C	**8** D	**9** E	**10** A
11 B	**12** C	**13** B		

Extended-response questions

1 a i -4 **ii** -6 **iii** -18 **iv** -18 **v** 6 **vi** $-\dfrac{1}{6}$
b $a = \dfrac{5}{2},\ b = 1,\ c = -\dfrac{7}{2},\ d = 6$

2 a i -1 and 3 **ii** $x > 3$ and $x < -1$
b $(3, 6)$ and $(7, 1)$ **c** $\left(\dfrac{1}{2}, 6\right)$ and $\left(\dfrac{5}{2}, 1\right)$
d $(2, 6)$ and $(10, 1)$ **e** $(2, 18)$ and $(10, 3)$

3 a $x = \alpha$ or $x = \beta$
b $(x-\beta)^{m-1}(x-\alpha)^{n-1}((m+n)x - \alpha m - \beta n)$
c $x = \alpha$ or $x = \beta$ or $x = \dfrac{\alpha m + \beta n}{m+n}$
d i $x > \dfrac{\alpha m + \beta n}{m+n},\ x \neq \beta$
ii $x < \alpha$ or $x > \dfrac{\alpha m + \beta n}{m+n}$

4 b $\dfrac{nx^{n-1}}{(x^n+1)^2}$ **d** $x = 0$ **e** $x > 0$

Chapter 10

Exercise 10A

1 $y = 4x - 5$

2 $y = -\dfrac{1}{3}x - 1$

3 $y = x - 2$ and $y = -x + 3$

4 $y = 18x + 1,\ y = -\dfrac{1}{18}x + 1$

5 $\left(\dfrac{3}{2}, -\dfrac{11}{4}\right),\ c = -\dfrac{29}{4}$

6 a i $y = 2x - 3$ **ii** $y = -\dfrac{1}{2}x - \dfrac{1}{2}$
b i $y = -3x - 1$ **ii** $y = \dfrac{1}{3}x - 1$
c i $y = -x - 2$ **ii** $y = x$
d i $y = 8x + 2$ **ii** $y = -\dfrac{1}{8}x - \dfrac{49}{8}$
e i $y = \dfrac{3}{2}x + 1$ **ii** $y = -\dfrac{2}{3}x + 1$
f i $y = \dfrac{1}{2}x + \dfrac{1}{2}$ **ii** $y = -2x + 3$
g i $y = \dfrac{2}{3}x + \dfrac{4}{3}$ **ii** $y = -\dfrac{3}{2}x + \dfrac{7}{2}$
h i $y = 4x - 16$ **ii** $y = -\dfrac{1}{4}x - \dfrac{15}{2}$
i i $y = -2$ **ii** $x = 2$
j i $y = 4x - 4$ **ii** $y = -\dfrac{1}{4}x + \dfrac{1}{4}$

7 $y = 56x - 160$

8 a $y = -1$ **b** $y = \dfrac{3}{2}x + \dfrac{1}{2}$
c $y = -2x - 1$ **d** $y = -4x + 5$

9 a $y = 2x$ **b** $y = -1$ **c** $y = 2x - \dfrac{\pi - 2}{2}$
d $y = 2x$ **e** $y = x$ **f** $y = -x + \dfrac{\pi - 2}{2}$

10 a $y = 2$ **b** $y = x$
c $y = 4e^2x - 3e^2$ **d** $y = \dfrac{e}{2}(x + 1)$
e $y = 3xe - 2e$ **f** $y = 4e^{-2}$

11 a $y = x - 1,\ y = -x + 1$
b $y = 2x - 1$ **c** $y = kx - 1$

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12 **a** $x = 0$ **b** $x = 0$ **c** $x = 4$
d $x = -5$ **e** $x = -\frac{1}{2}$ **f** $x = -5$

13 $\frac{\pi - 2}{2}$ **14** $a = 1$ **15** $a = e$

16 $a = 0$ **17** $a = 0$ or $a = \frac{3}{2}$

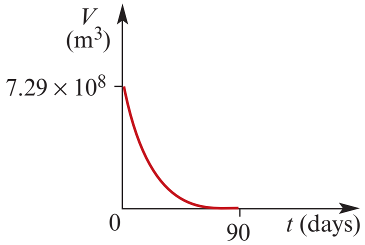
Exercise 10B

1 **a** 21 **b** $3h + 18$ **c** 18

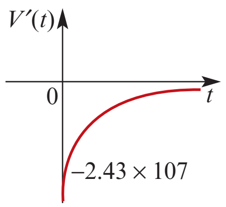
2 **a** $\frac{dV}{dt}$ **b** $\frac{dS}{dr}$ **c** $\frac{dV}{dx}$ **d** $\frac{dA}{dt}$ **e** $\frac{dV}{dh}$

3 Wanes by 0.006 units per day

4 **a** $-3 \times 10^3(90 - t)^2$ **b** 90 days
c 7.29×10^8 m^3 **d** 80 days
e

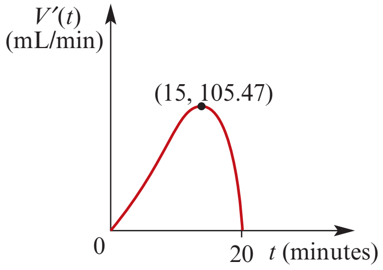


f



5 **a** $V'(t) = \frac{t^3}{160}(20 - t)$

b



c $t = 15$

6 **a** $t \approx 100$, $t \approx 250$, $t \approx 500$
b $\approx 430\,000$ m^3/day **c** $\approx 270\,000$ m^3/day
d $(100, 250) \cup (500, 600)$

7 **a** $\lambda = 0.1373$, $P_0 = 30$
b 9.625 hours
c **i** 4.120 units/hour **ii** 1.373 units/hour

8 **a** $-0.3(T - 15)$
b **i** -22.5°C/minute **ii** -13.5°C/minute
iii -4.5°C/minute

9 $\frac{dy}{dx} = 3 - 2\sin x$, gradient always positive

10 **a** 4.197 **b** -0.4

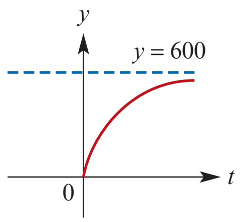
11 **a** $v = 6t^2 - 18t + 12$
b $t = 1, x = 5;\ t = 2, x = 4$
c -6 m/s^2, 6 m/s^2 **d** -1.5 m/s

12 **a** 8 cm from O **b** 2 cm/s
c 9 cm from O after 1 s **d** -2 cm/s^2

13 **a** $\frac{\sqrt{2}t}{\sqrt{t^2 + 1}}$ **b** $\frac{\sqrt{2}}{(t^2 + 1)^{\frac{3}{2}}}$
c 1 cm/s, $\frac{1}{2}$ cm/s^2

14 When $t = 0$, $v = 0.4$ m/s;
When $t = 1$, $v = 0.4e \approx 1.0874$ m/s;
When $t = 2$, $v = 0.4e^2 \approx 2.956$ m/s

15 **a**



b 3.33

16 **a** $-2y$ **b** ky

17 **a** 0.18 kg
b 3.47 hours
c **i** 6.93 hours **ii** 10.4 hours
d 0.2 m

Exercise 10C

1 **a** $(2, -16), (-2, 16)$ **b** $(1, -2)$
c $(0, 0), (1, 1)$ **d** $(4, 48)$
e $(0, 0), \left(\frac{2}{\sqrt{3}}, \frac{16}{3}\right), \left(\frac{-2}{\sqrt{3}}, \frac{16}{3}\right)$ **f** $\left(\frac{1}{3}, \frac{14}{3}\right)$
g $(3, 2)$ **h** $(0, -10), (2, 6)$

2 **a** $(0, 1)$ **b** $\left(\frac{1}{3e}, -\frac{1}{3e}\right)$
c $(0, 1), (-\pi, 1), \left(-\frac{\pi}{2}, -1\right), \left(\frac{\pi}{2}, -1\right), (\pi, 1)$
d $(-1, -e^{-1})$ **e** $(0, 0), (2, 4e^{-2})$
f $(e^{-1}, -2e^{-1})$

3 **a** $a = 6$ **b** $b = 3$

4 $b = -2,\ c = 1,\ d = 3$

5 $a = 2,\ b = -4,\ c = -1$

6 $a = \frac{2}{3},\ b = -2\frac{1}{2},\ c = -3,\ d = 7\frac{1}{2}$

7 **a** $a = 2$ and $b = 9$ **b** $(-1, -5)$

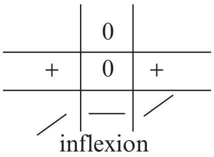
8 $x = \frac{1}{2}$ or $x = \frac{1 - 4n}{2n + 2}$

9 $x = \pm 1$ or $x = 0$

10 $(1, \frac{1}{2})$ or $(-1, -\frac{1}{2})$

Exercise 10D

1 **a** $x = 0$



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b $x = 2, x = -5$

	-5		2	
$+$	0	$-$	0	$+$
/	—	\	—	/
	max.		min.	

c $x = -1, x = \frac{1}{2}$

	-1		$\frac{1}{2}$	
$+$	0	$-$	0	$+$
/	—	\	—	/
	max.		min.	

d $x = -3, x = 4$

	-3		4	
$-$	0	$+$	0	$-$
\	—	/	—	\
	min.		max.	

e $x = -3, x = 4$

	-3		4	
$+$	0	$-$	0	$+$
/	—	\	—	/
	max.		min.	

f $x = 0, x = \frac{27}{5}$

	0		$\frac{27}{5}$	
$+$	0	$-$	0	$+$
/	—	\	—	/
	max.		min.	

g $x = 1, x = 3$

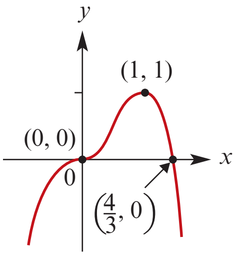
	1		3	
$+$	0	$-$	0	$+$
/	—	\	—	/
	max.		min.	

h $x = 1, x = 3$

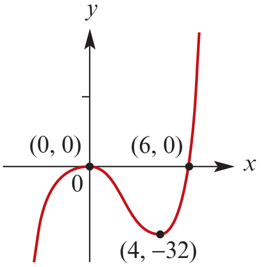
	1		3	
$-$	0	$+$	0	$-$
\	—	/	—	\
	min.		max.	

2 a $x = -2$ (max), $x = 2$ (min)
b $x = 0$ (min), $x = 2$ (max)
c $x = \frac{1}{3}$ (max), $x = 3$ (min)
d $x = 0$ (inflection)
e $x = -2$ (inflection), $x = 0$ (min)
f $x = -\frac{1}{\sqrt{3}}$ (max), $x = \frac{1}{\sqrt{3}}$ (min)

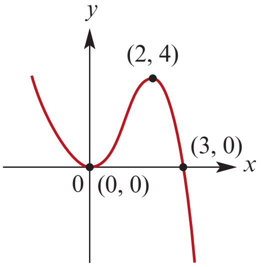
3 a i $(0, 0)$, $(\frac{4}{3}, 0)$
ii $(0, 0)$ inflection, $(1, 1)$ max



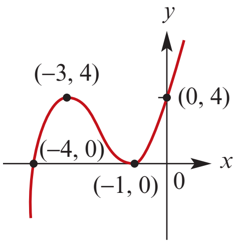
b i $(0, 0)$, $(6, 0)$
ii $(0, 0)$ max, $(4, -32)$ min



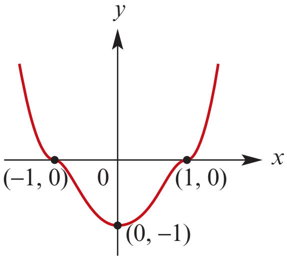
c i $(0, 0)$, $(3, 0)$
ii $(0, 0)$ min, $(2, 4)$ max



d i $(-4, 0)$, $(-1, 0)$, $(0, 4)$
ii $(-3, 4)$ max, $(-1, 0)$ min



e i $(-1, 0)$, $(0, -1)$, $(1, 0)$
ii $(-1, 0)$ infl, $(0, -1)$ min, $(1, 0)$ infl



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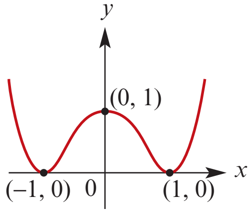
f **i** $(-1,0),\ (0,1),\ (1,0)$

ii $(-1,0)$ min, $(0,1)$ max, $(1,0)$ min

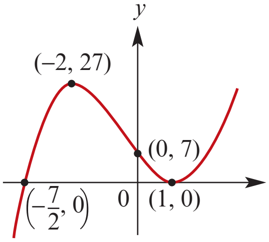


4 **a** $(-2,27)$ max, $(1,0)$ min

b $(1,0)$ is a turning point

c $\left(-\frac{7}{2},0\right),\ (0,7)$

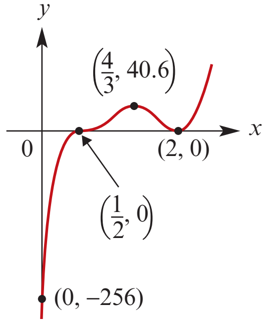
d



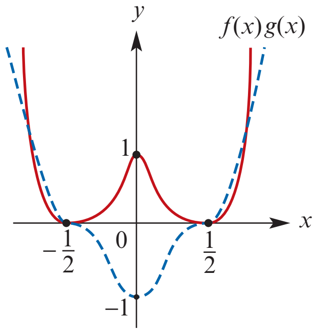
5 **b** $a=3,\ b=2,\ (0,2)$ min, $(-2,6)$ max

6 **a** $(0,-256),\ (\frac{1}{2},0),\ (2,0)$

b $\left(\frac{1}{2},0\right)$ inflection, $\left(\frac{4}{3},40.6\right)$ max, $(2,0)$ min



7 **a**



b **i** $\left(-\infty,-\frac{1}{\sqrt{2}}\right)\cup\left(-\frac{1}{2},\frac{1}{2}\right)\cup\left(\frac{1}{\sqrt{2}},\infty\right)$

ii $\left(\frac{-\sqrt{66}}{12},-\frac{1}{2}\right)\cup\left(-\frac{1}{2},0\right)\cup\left(\frac{\sqrt{66}}{12},\infty\right)$

8 **a** $(-2,0)$ max, $\left(\frac{4}{3},-18\frac{14}{27}\right)$ min

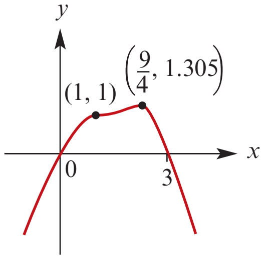
b No stationary points

9 **a** $(0,-1)$ stationary point of inflection, $(-1,-1)$ minimum

b $(0,-1)$ stationary point of inflection, $(-1.5,-2.6875)$ minimum

c No stationary points, gradient is always positive

10 **b** $x \le \frac{9}{4}$ **c**



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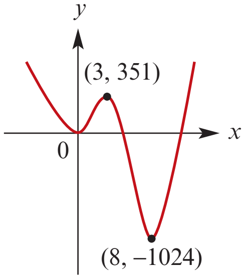


12 **a** $x=-1$ (infl), $x=1$ (min), $x=5$ (max)

b $x=0$ (max), $x=2$ (min)

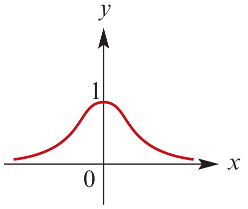
c $x=-4$ (min), $x=0$ (max)

d $x=-3$ (min), $x=2$ (infl)

13 **a** $(0,0)$ local max;
$(2\sqrt{2},-64)$ and $(-2\sqrt{2},-64)$ local min

b $(0,0)$ local max;
$\left(\pm 4\sqrt{\frac{m-1}{m}},-\frac{16^m(m-1)^{m-1}}{m^m}\right)$ local min

14



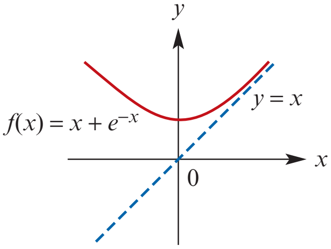
15 $\{x: -2<x<0\}$

16 $x<1$; Max value $=\frac{100}{e^4}\approx 1.83$

17 **a** Min value $=f(0)=0$

18 **a** $(0,1)$ min **b** $y=x$

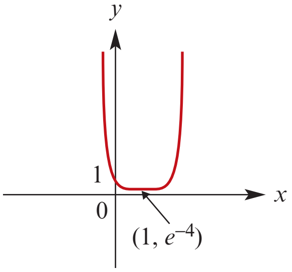
c



19 $p=1,\ q=-6,\ r=9$

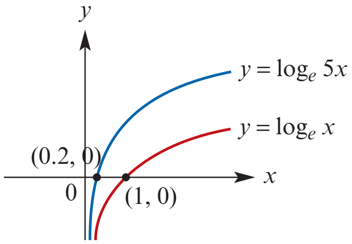
20 **a** $(8x-8)e^{4x2-8x}$ **b** $(1, e^{-4})$ min

c



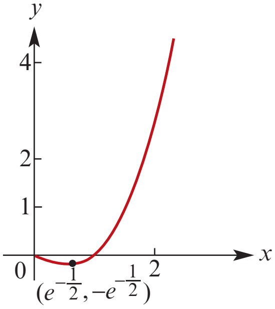
d $y = -\frac{1}{8}x + \frac{5}{4}$

21 Tangents are parallel for any given value of x.

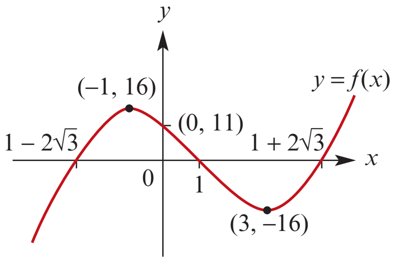


22 **a** $2x\log_e(x) + x$ **b** $x = 1$ **c** $x = e^{-\frac{1}{2}}$

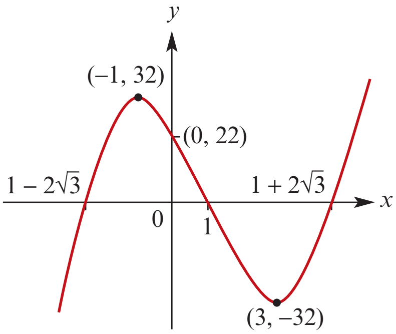
d



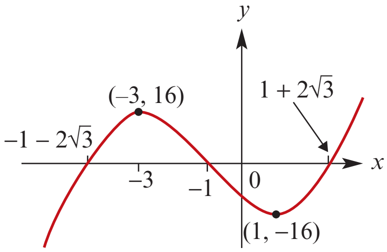
23 **a**



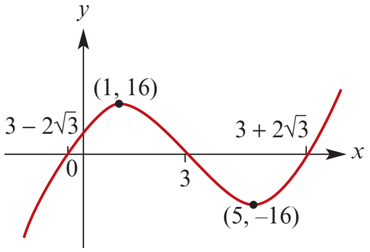
b Dilated by a factor of 2 from the x-axis:



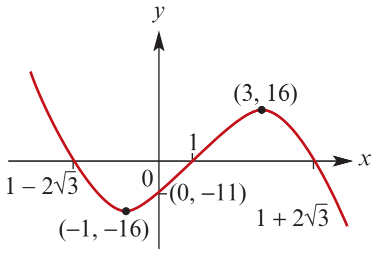
c Translated 2 units to the left:



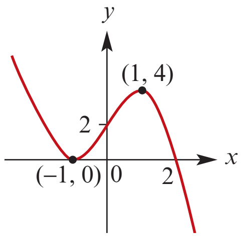
d Translated 2 units to the right:



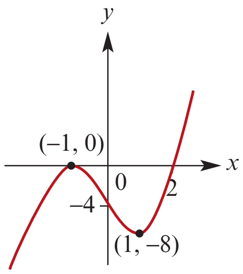
e Reflected in the x-axis:



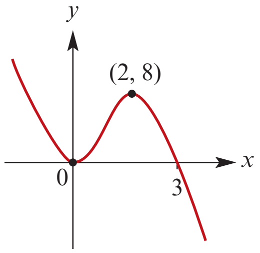
24 **a**



b



c



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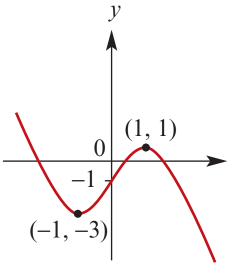
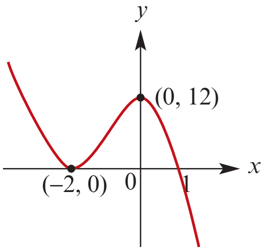
d



e



25 a $(a+\ell, 0)$, $(b+\ell, 0)$ **b** $(h+\ell, kp)$

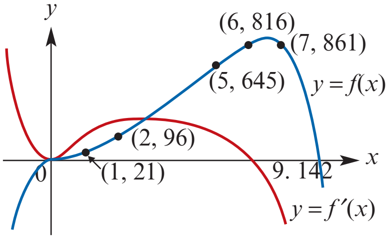
26 a Max $x = \frac{\pi}{3}, \frac{5\pi}{3}$; Min $x = 0, \pi, 2\pi$

b Max $x = \frac{\pi}{6}$; Min $x = \frac{5\pi}{6}$; Infl $x = \frac{3\pi}{2}$

c Max $x = \frac{\pi}{2}, \frac{3\pi}{2}$; Min $x = \frac{7\pi}{6}, \frac{11\pi}{6}$

d Max $x = \frac{\pi}{3}$; Infl $x = \pi$; Min $x = \frac{5\pi}{3}$

27 a b $y = -x^4 + 8x^3 + 10x^2 + 4x$



Local max at $(6.761, 867.07)$; No stationary point of inflection: $\frac{dy}{dx} = 4$ when $x = 0$

c -960

d $x = 4.317$ or $x = 8.404$

Exercise 10E

1 Absolute max = 2; Absolute min = −70

2 Absolute max = 15; Absolute min = −30

3 Absolute max = 0; Absolute min = −20.25

4 Absolute max = 2304; Absolute min = −8

5 b $\frac{dV}{dx} = 30x - 36x^2$

c Local max at $\left(\frac{5}{6}, \frac{125}{36}\right)$

d Absolute max value is 3.456 when $x = 0.8$

e Absolute max value is $\frac{125}{36}$ when $x = \frac{5}{6}$

6 a $25 \le y \le 28$

b Absolute max = 125; Absolute min = 56

7 a $\frac{1}{(x-4)^2} - \frac{1}{(x-1)^2}$ **b** $\left(\frac{5}{2}, \frac{4}{3}\right)$

c Absolute max = $\frac{3}{2}$; Absolute min = $\frac{4}{3}$

8 b $\frac{dA}{dx} = \frac{1}{4}(x-5)$ **c** $x = 5$

d $\frac{25}{4}$ m^2, but only one square is formed

9 Absolute max = 12.1; Absolute min = 4

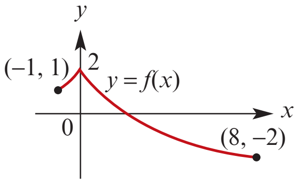
10 a $\frac{1}{(x-4)^2} - \frac{1}{(x+1)^2}$ **b** $\left(\frac{3}{2}, \frac{4}{5}\right)$

c Absolute max = $\frac{5}{4}$; Absolute min = $\frac{4}{5}$

11 Absolute max = $\frac{\sqrt{2}}{2}$; Absolute min = −1

12 Absolute max = 1; Absolute min = $\frac{\sqrt{2}}{2}$

13 Absolute max = 2; Absolute min = −2



14 Absolute max = $\frac{1}{e^2} + 2e^2$;

Absolute min = $2\sqrt{2}$

15 Absolute max = $2e^9$; Absolute min = 2

16 Absolute max = $-\log_e 10$;

Absolute min = $-\frac{10}{e}$

Exercise 10F

1 625 m^2

2 First = $\frac{4}{3}$; Second = $\frac{8}{3}$

3 Max value of P is 2500

4 Max area is 2 km × 1 km = 2 km^2

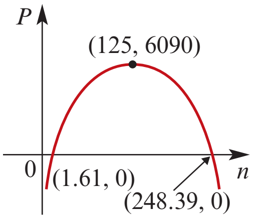
5 $p = \frac{3}{2}$, $q = \frac{8}{3}$

6 b $V = \frac{75x - x^3}{2}$ **c** 125 cm^3

7 a i $n = 125$

ii Maximum daily profit is \$6090

b



c $2 \le n \le 248$

d $n = 20$

8 12°C

9 8 mm for maximum; $\frac{4}{3}$ mm for minimum

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10 a $8\cos\theta$

b Area $= 16(1+\cos\theta)\sin\theta$;
Max area $= 12\sqrt{3}$ square units

11 $(1, 1)$

12 a $\dfrac{75}{\cos\theta}$ seconds

b $220 - 60\tan\theta$ seconds

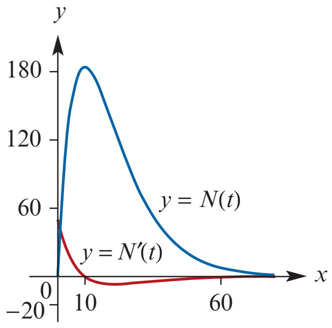
d $\dfrac{dT}{d\theta} = \dfrac{75\sin\theta - 60}{\cos^2\theta}$

e $\theta = \sin^{-1}\left(\dfrac{4}{5}\right) \approx 53.13°$

f Min time $T = 265$ seconds occurs when distance BP is 400 metres

13 Max population $\dfrac{500}{e}$ occurs when $t = 10$

14 a

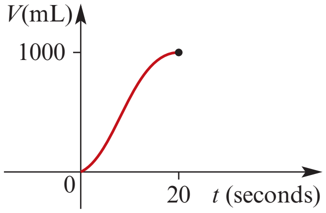


b Max rate of increase is 50, occurs at $t = 0$;
Max rate of decrease is $\dfrac{50}{e^2}$, occurs at $t = 20$

15 a i $V(0) = 0$ mL **ii** $V(20) = 1000$ mL

b $V'(t) = \dfrac{3}{4}(20t - t^2)$

c

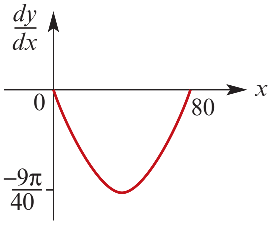


d Check the graph of $v'(t)$ on your calculator

e $t = 10$ s, 75 mL/s

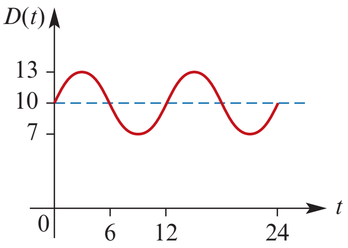
16 a $\dfrac{dy}{dx} = \dfrac{-9\pi}{40}\sin\left(\dfrac{\pi x}{80}\right)$

b



c $\left(40, \dfrac{-9\pi}{40}\right)$

17 a



b $\{t : D(t) \geq 8.5\} = [0, 7] \cup [11, 19] \cup [23, 24]$

c i 0 m/h **ii** $-\dfrac{\pi}{2}$ m/h **iii** $\dfrac{\pi}{2}$ m/h

d i $t = 0, 12, 24$ **ii** $t = 6, 18$

Exercise 10G

1 a $f'(x) = (x-1)(3x-2b-1)$

b $\left(\dfrac{2b+1}{3}, \dfrac{-4(b-1)^3}{27}\right)$ and $(1, 0)$

c $\dfrac{2b+1}{3} > 1$ since $b > 1$ **d** $b = \dfrac{11}{2}$

2 a $(0, 0), (\sqrt{2}, -4), (-\sqrt{2}, -4)$

b $(a, b), (\sqrt{2}+a, -4+b), (\sqrt{2}+a, -4+b)$

3 a $a = c - 20,\ b = 30 - 2c$ **b** $\dfrac{45}{2}$

4 a Increasing $\left[0, \dfrac{2}{3a}\right]$; Decreasing $\left[\dfrac{2}{3a}, \infty\right)$

b $y = \dfrac{-1}{a}x + \dfrac{1}{a^2}$ **c** $y = ax - 1$

d $\left(-\infty, \dfrac{4}{27a^2}\right]$

5 a i $2(a-3)$ **ii** $m = 2(a-3)$

b $(a, (a-3)^2)$

c $y = 2(a-3)x - a^2 + 9$ **d** $\dfrac{3+a}{2}$

6 a $h = 1$ or $h = -3$ **b** $a = 2$

c $a = -48,\ b = -64$

7 a $(a, 0),\ \left(\dfrac{a+2}{3}, \dfrac{4(a-1)^3}{27}\right)$

b $(a, 0)$ local minimum;
$\left(\dfrac{a+2}{3}, \dfrac{4(a-1)^3}{27}\right)$ local maximum

c i $y = (a-1)^2x - (a-1)^2$ **ii** $y = 0$

iii $y = -\dfrac{(a-1)^2}{4}(x-a)$

8 a $2(x-1)(x-b)(2x-b-1)$

b $(1, 0), (b, 0), \left(\dfrac{b+1}{2}, \dfrac{(b-1)^4}{16}\right)$ **c** $b = 3$

9 $a = \dfrac{1}{486},\ b = 0,\ c = \dfrac{-1}{161},\ d = \dfrac{1459}{243}$

10 a $a = \dfrac{4d-1}{4},\ b = \dfrac{-(2d+3)}{2},\ c = \dfrac{11-4d}{4}$

b $d = \dfrac{38}{67}$

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Chapter 10 review

Technology-free questions

1 a $y = -x$ **b** $(0, 0)$

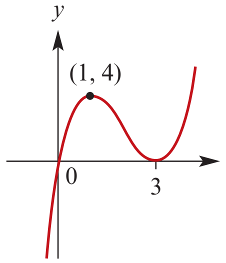
2 $y = 6ax - 3a^2$, $P(0, -3a^2)$

3 a $y = 3x - 3$ **b** $x = \frac{11}{3}$

4 a 5π square units/unit **b** 6π square units/unit

5 a $(1, 1)$ max; $(0, 0)$ inflection
b $(-1, 0)$ max; $(1, -4)$ min
c $(-\sqrt{3}, 6\sqrt{3} + 1)$ max; $(\sqrt{3}, -6\sqrt{3} + 1)$ min

6



7 $x = 1$ (inflection); $x = 2$ (minimum)

8 $y = -9x + 7$

9 $(0, 2)$

10 $x = \frac{-1}{2}$

11 a $\mathbb{R} \setminus \{1\}$; $f'(x) = \frac{4}{5}(x - 1)^{-\frac{1}{5}}$
b $y = \frac{4}{5}x - \frac{3}{5}$ and $y = -\frac{4}{5}x + 1$ **c** $\left(1, \frac{1}{5}\right)$

12 a 64π cm^3/cm **b** 64π cm^3/s

13 0.25 m/s, $0.25e$ m/s, $0.25e^2$ m/s, $0.25e^4$ m/s

14 a $(25e^{100t})$°C/s **b** $(25e^5)$°C/s

15 $y = ex$

16 b 20 cm/year

17 2

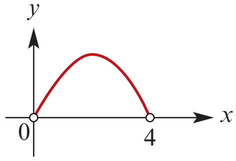
18 a $y = \frac{1}{e}x$ **b** $y = \frac{x}{\sqrt{2}} - \frac{\pi}{2\sqrt{2}} + \sqrt{2}$
c $y = x - \frac{3\pi}{2}$ **d** $y = \frac{-2}{\sqrt{e}}x - 1$

Multiple-choice questions

1 A **2** E **3** E **4** E **5** B
6 C **7** B **8** B **9** D **10** E
11 A **12** E **13** B **14** C

Extended-response questions

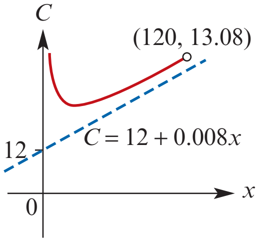
1 a $y = 4x - x^2$ **b** $0 < x < 4$ **c** $y = 4, x = 2$
d Gradient is positive to the left of $x = 2$, and negative to the right
e



f $0 < y < 4$

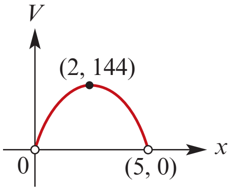
2 a $A = 4xy$ **b** $y = -\frac{2}{3}x + 8$
c $A = 32x - \frac{8}{3}x^2$ **d** $x = 6, y = 4$
e 96 m^2

3 a i \$12.68 **ii** \$12.74
b $C = 12 + 0.008x + \frac{14.40}{x}$
c



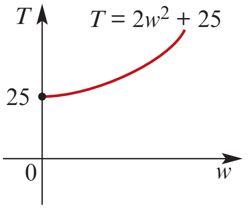
d 42.43 km/h

4 a $V = 4(x^3 - 13x^2 + 40x)$
b $0 < x < 5$ **c** $x = 2$
d 2 cm, 12 cm, 6 cm **e** 144 cm^3
f



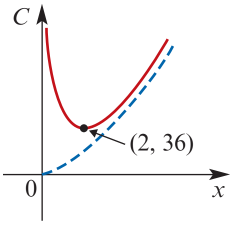
5 32

6 a $T = 2w^2 + 25$
b



c $A = \frac{25}{w} + 2w$
d i $\frac{5\sqrt{2}}{2} \approx 3.54$ kg **ii** $10\sqrt{2} \approx 14.14$ s

7 b $C = 3x^2 + \frac{48}{x}$
c



d i $x = 2, h = 3$; i.e. 2 m, 2 m, 3 m
ii 36 m^2

8 10 m, 10 m, 5 m; Area 400 m^2

9 a $A = \frac{1}{2}a^2\theta$ **b** $A = \frac{1}{2}\left(\frac{100}{\theta + 2}\right)^2\theta$
c $\theta = 2$ **d** 625 cm^2

10 b i $r = \frac{L}{4}$ **ii** $\theta = 2$ **iii** Maximum

11 b $\frac{dT}{dx} = \frac{x}{\sqrt{x^2 + 900}} - \frac{3}{5}$

c i $x = 22.5$ **ii** 71 seconds

d 63 seconds

12 a $y = ex$

b $y = 2ex$

c $y = kex$

e i $k = \frac{1}{e}$ or $k \le 0$ **ii** $k > \frac{1}{e}$

13 b $T = \frac{20 + 16\sqrt{2}}{15} \approx 2.84$ hours

14 $t = 1.16$, 1.2 km apart

15 b $0 < x < 1$ **c** $x = \frac{1}{\sqrt{2}}$, $y = \pm 1$

d $A = 2\sqrt{2}$

16 c ii $\frac{dA}{dx} = -3x^2 - 2ax + a^2$

17 $t = 5$, $N(5) = \frac{120}{e}$

18 a $b = 5$, $c = 6$

b i 6 weeks **ii** 3.852 weeks

c $190\frac{2}{3}$ cm^2

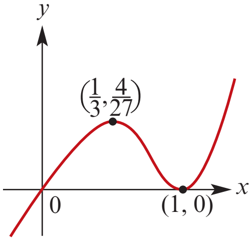
19 a $(1, -6)$ **b** $3(x - 1)^2 + 3$

c $3(x - 1)^2 + 3 > 3$ for all $x \in \mathbb{R} \setminus \{1\}$

20 a $a = 1$, $c = 1$, $b = -2$, $d = 0$

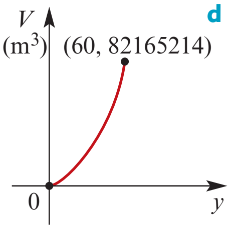
b $\{x : \frac{1}{3} < x < 1\}$

c



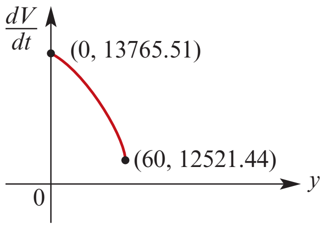
21 a 53 109 671 m^3 **b** $\frac{dV}{dy} = \pi(y + 630)^2$

c



d 82 165 214 m^3

e



22 a i $r = \frac{2\pi - \theta}{2\pi}$

ii $h = \sqrt{1 - \left(\frac{2\pi - \theta}{2\pi}\right)^2}$

b $\frac{49\sqrt{15}\pi}{1536}$

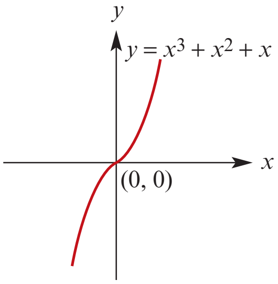
c 0.3281, 2.5271

d i $\theta = 1.153$

ii $V_{\max} = 0.403$ m^3

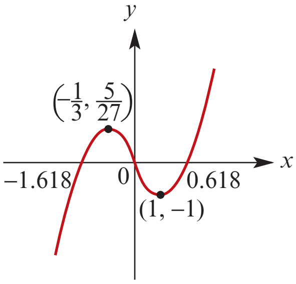
e 0.403 m^3

23 a i

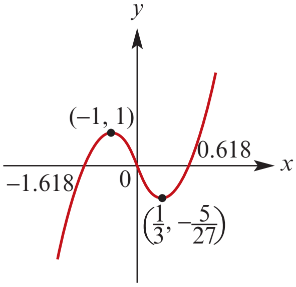


No stationary points

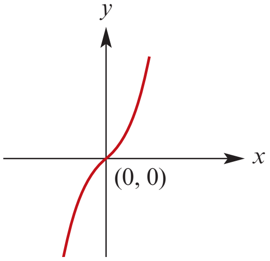
ii



iii



iv



No stationary points

b i $f'(x) = 3x^2 + 2ax + b$

ii $x = \frac{-a \pm \sqrt{a^2 - 3b}}{3}$

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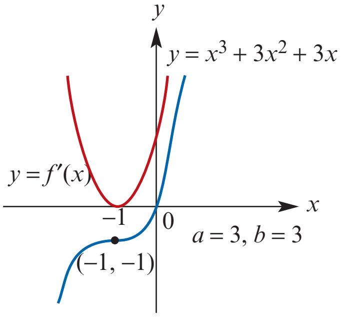
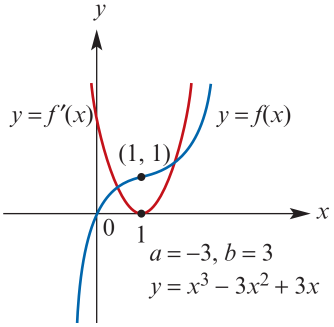
c **ii** $a = -3$ or $a = 3$; $(-1, -1), (1, 1)$ stationary points of inflection

iii **iv**



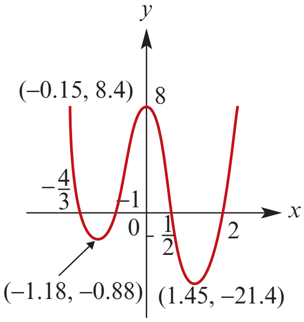


d $a^2 < 3b$

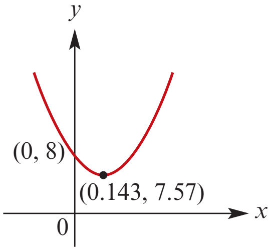
24 $x = e$

25 **a** **i** $a = -21$

ii



b **i**



ii Min at $(0.143, 7.57)$

iii $g'(x) = 24x^3 - 3x^2 + 42x - 6$

iv 0.1427

v $g'(0) = -6$, $g'(10) = 24\,114$

vi $g''(x) = 72x^2 - 6x + 42$

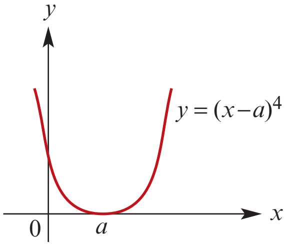
vii $g''(x) > 0$ for all x; thus $y = g'(x)$ has no turning points and crosses the x-axis only once

26 **b** **i** $x = a$ or $x = b$ or $x = \dfrac{b + a}{2}$

ii $x = a$ or $x = b$

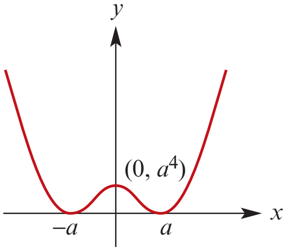
c $(a, 0)$, $(b, 0)$, $\left(\dfrac{a + b}{2}, \dfrac{(a - b)^4}{16}\right)$

e **i**



ii $(a, 0)$, $(-a, 0)$, $(0, a^4)$

iii



27 **b** **i** $x = a$ or $x = \dfrac{3b + a}{4}$

ii $x = a$ or $x = b$

c Local min at $\left(\dfrac{3b + a}{4}, \dfrac{-27}{256}(b - a)^4\right)$; Stationary point of inflection at $(a, 0)$

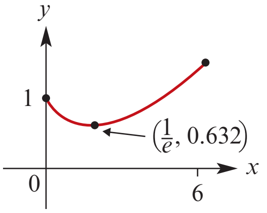
e $\left(-\dfrac{a}{2}, \dfrac{-27a^4}{16}\right)$ and $(a, 0)$

f **i** $b = -\dfrac{a}{3}$

28 **a** $f'(x) = \log_e x + 1$

b $x = \dfrac{1}{e} \approx 0.37$, i.e. during the first month

c

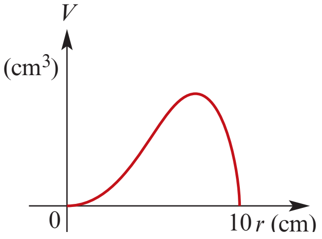


d When $x = 6$

29 **a** **i** $y = \sqrt{100 - r^2}$, $h = 2\sqrt{100 - r^2}$

ii $V = 2\pi r^2\sqrt{100 - r^2}$

b **i**



ii $V = 2418.4$, $r = 8.165$, $h = 11.55$

iii $r = 6.456$ or $r = 9.297$

c **i** $\dfrac{dV}{dr} = \dfrac{400\pi r - 6\pi r^3}{\sqrt{100 - r^2}}$

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ii $V_{\max} = \frac{4000\pi\sqrt{3}}{9}$ when $r = \frac{10\sqrt{6}}{3}$

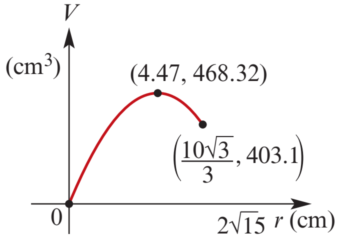
d **ii** $\frac{dV}{dr} > 0$ for $r \in \left(0, \frac{20\sqrt{6}}{6}\right)$

iii $\frac{dV}{dr}$ is increasing for $r \in (0, 5.21)$

30 **a** $h = \frac{100 - 3r^2}{2r}$ **b** $V = \frac{\pi r}{6}(300 - 5r^2)$

c $0 < r < \frac{10\sqrt{3}}{3}$ **d** $\frac{dV}{dr} = \frac{\pi}{6}(300 - 15r^2)$

e



31 **a** **i** $y = \frac{100}{x^2}$ **ii** $S = \frac{3000}{x} + 60x^2$

b **i** $\frac{dS}{dx} = -\frac{3000}{x^2} + 120x$ **ii** 1538.99 cm^2

c 585 cm^2/s

32 **a** $f'(x) = \frac{3\,000\,000e^{-0.3x}}{(1 + 100e^{-0.3x})^2}$

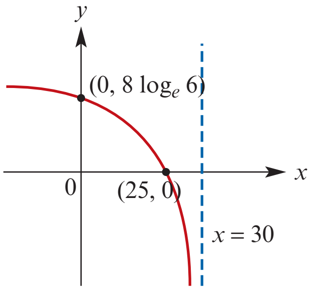
b **i** 294 kangaroos per year

ii 933 kangaroos per year

33 **a** $a = 30$ **b** $(0, 8\log_e 6)$, $(25, 0)$

c $f'(20) = -0.8$ **d** $f^{-1}(x) = 5\left(6 - e^{\frac{x}{8}}\right)$

e $\mathbb{R}$ **f**



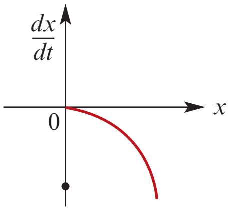
34 **b** $\left(\frac{\pi}{2}, e\right), \left(\frac{3\pi}{2}, \frac{1}{e}\right)$ **c** $\left[\frac{1}{e}, e\right]$

d Period 2π, since $g(x + 2\pi) = g(x)$

36 **a** **i** 30 g **ii** 12.28 g

b $\frac{dx}{dt} = \frac{-300\lambda e^{\lambda t}}{(5e^{\lambda t} - 3)^2}$

c **ii**



37 **b** $MP = \frac{2}{\tan\theta}$ **c** $NQ = 8\tan\theta$

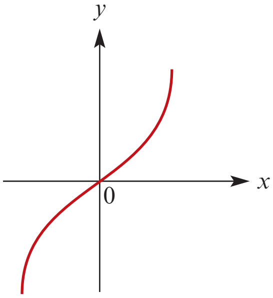
d $x = \frac{2}{\tan\theta} + 8\tan\theta + 10$

e $\frac{dx}{d\theta} = -2\,\text{cosec}^2\,\theta + 8\sec^2\theta$

f $x = 18$, $\theta = 26.6°$

38 **a** $f'(x) = e^x + e^{-x}$ **b** $\{0\}$

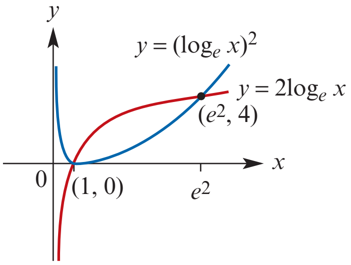
d



39 **a** $x = 1$ or $x = e^2$

b When $x = 1$, gradient of $y = 2\log_e x$ is 2 and gradient of $y = (\log_e x)^2$ is 0

c



d $\{x : 2\log_e x > (\log_e x)^2\} = (1, e^2)$

40 **a** $h = a(1 + \cos\theta)$

b $r = a\sin\theta$

d $\frac{dV}{d\theta} = \frac{\pi a^3}{3}[2\sin\theta\cos\theta\,(1 + \cos\theta) - \sin^3\theta]$

$\theta = \cos^{-1}\left(\frac{1}{3}\right) \approx 70.53°$

e $V = \frac{32\pi a^3}{81}$ cm^3

41 **b** $\frac{dy}{dt} = \frac{bAe^{bt}}{(1 + Ae^{bt})^2}$

e After 7 hours (to the nearest hour)

42 **a** $f'(x) = \frac{xe^x - e^x}{x^2}$

b $x = 1$

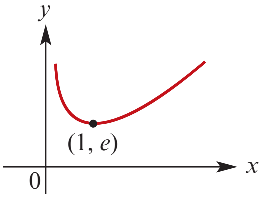
c $(1, e)$ minimum

d **i** $\frac{f'(x)}{f(x)} = \frac{x - 1}{x}$

ii $\lim\limits_{x \to \infty} \frac{f'(x)}{f(x)} = 1$

i.e. $f'(x) \to f(x)$ as $x \to \infty$

e



f $t = \frac{1}{k} \approx 45.27$ years, i.e. during 1945

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43 a $A = 1000,\ k = \frac{1}{5}\log_e 10 \approx 0.46$

b $\frac{dN}{dt} = kAe^{kt}$ **c** $\frac{dN}{dt} = kN$

d i $\frac{dN}{dt} \approx 2905.7$ **ii** $\frac{dN}{dt} \approx 4.61 \times 10^{12}$

44 a $t \approx 34.66$ years **b** $t \approx 9.12$ years

45 a Max height 0.7 m first occurs at $t = \frac{1}{6}$ s

b $\frac{2}{3}$ **c** 0.6π m/s, 0.6π m/s, 0 m/s

46 a i $r = \frac{1}{6}$ **ii** $p = 12,\ q = 8$

b $T'(3) = -\frac{4\pi}{3}$, i.e. length of night decreasing by $\frac{4\pi}{3}$ hours/month; $T'(9) = \frac{4\pi}{3}$, i.e. length of night increasing by $\frac{4\pi}{3}$ hours/month

c $-\frac{8}{3}$ hours/month

d $t = 9$, i.e. after 9 months

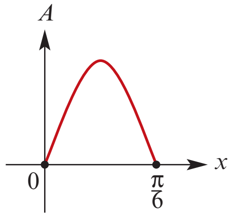
47 a $A = 2x\cos(3x)$

b i $\frac{dA}{dx} = 2\cos(3x) - 6x\sin(3x)$

ii When $x = 0$, $\frac{dA}{dx} = 2$;

When $x = \frac{\pi}{6}$, $\frac{dA}{dx} = \frac{dA}{dx} = -\pi$

c i



ii $x = 0.105$ or $x = 0.449$

iii Max area 0.374, occurs when $x = 0.287$

d ii (0.287, 1.162)

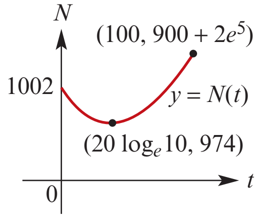
48 a i $N'(t) = -1 + \frac{1}{10}e^{\frac{t}{20}}$

ii Minimum population is 974, occurs when $t = 20\log_e 10$

iii $N(0) = 1002$

iv $N(100) = 900 + 2e^5$

v



b i $N_2(0) = 1002$

ii $N_2(100) = 990 + 2e^{\frac{1}{2}}$

iv Minimum population is 974, occurs when $t = (20\log_e 10)^2$

c ii Minimum population is 297, occurs when $t = 100.24$

d i $N_3'(t) = -\frac{3}{2}t^{\frac{1}{2}} + \frac{1}{10}e^{\frac{t}{20}}$

49 a $a = \frac{1}{3}\log_e\left(\frac{10}{3}\right)$

b i $x = 0$ and $x = \frac{5}{2}$

ii $x = \frac{-4 + 5a \pm \sqrt{25a^2 + 16}}{4a}$

Chapter 11

Exercise 11A

1 a 3.81 square units **b** 1.34 square units

c $\frac{35}{2}$ square units

2 a 13.2 square units **b** 10.2 square units

3 a 10 square units **b** 10.64 square units

4 a 0.72 square units

b 2.88, decrease strip width

5 a 36.8 square units **b** 36.7 square units

6 11.9 square units

7 a ≈ 48 square units **b** Distance travelled

8 a $\frac{9}{2}$ **b** 9 **c** 4

Exercise 11B

1 a $\frac{x^4}{8} + c$ **b** $\frac{5}{4}x^4 - x^2 + c$

c $\frac{x^4}{5} - x^3 + c$ **d** $2z + \frac{5}{2}z^2 - z^3 + c$

2 a $y = -\frac{1}{2x^2} + c$ **b** $y = 3x^{\frac{4}{3}} + c$

c $y = \frac{4}{5}x^{\frac{5}{4}} + \frac{5}{2}x^{\frac{2}{5}} + c$

3 a $-\frac{3}{x} + c$ **b** $-\frac{2}{3x^3} + 3x^2 + c$

c $-\frac{2}{x} - \frac{3}{x^2} + c$ **d** $\frac{9}{4}x^{\frac{4}{3}} - \frac{20}{9}x^{\frac{9}{4}} + c$

e $\frac{12}{7}x^{\frac{7}{4}} - \frac{14}{3}x^{\frac{3}{2}} + c$ **f** $\frac{5}{2}x^{\frac{8}{5}} + \frac{9}{2}x^{\frac{8}{3}} + c$

4 a $y = x^2 - 3x + 3$ **b** $y = \frac{x^4}{4} + 6$

c $y = \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{2}x^2 - \frac{22}{3}$

5 a $\frac{4}{3}x^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}} + c$ **b** $\frac{3z^3 - 4}{2z} + c$

c $\frac{5}{3}x^3 + x^2 + c$ **d** $\frac{4}{5}x^{\frac{5}{2}} + \frac{2}{7}x^{\frac{7}{2}} + c$

e $\frac{2x^3}{3} + \frac{3x^5}{5} + c$ **f** $\frac{3}{7}x^{\frac{7}{3}} + \frac{3}{16}x^{\frac{16}{3}} + c$

6 $f(x) = x^3 + \frac{1}{x} - \frac{17}{2}$

7 $s = \frac{3}{2}t^2 + \frac{8}{t} - 8$

8 a $k = -32$ **b** $f(7) = 201$

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Exercise 11C

1 a $\frac{1}{6}(2x-1)^3+c$ **b** $-\frac{1}{4}(t-2)^4+c$

c $\frac{1}{20}(5x-2)^4+c$ **d** $\frac{1}{24-16x}+c$

e $\frac{1}{8(6-4x)^2}+c$ **f** $\frac{-1}{8(3+4x)^2}+c$

g $\frac{2}{9}(3x+6)^{\frac{3}{2}}+c$ **h** $\frac{2}{3}(3x+6)^{\frac{1}{2}}+c$

i $\frac{1}{9}(2x-4)^{\frac{9}{2}}+c$ **j** $\frac{1}{7}(3x+11)^{\frac{7}{3}}+c$

k $-\frac{2}{9}(2-3x)^{\frac{3}{2}}+c$ **l** $-\frac{1}{10}(5-2x)^5+c$

2 a $\frac{1}{2}\log_e(x)+c$ **b** $\frac{1}{3}\log_e(3x+2)+c$

c $\log_e(1+4x)+c$ **d** $\frac{5}{3}\log_e(3x-2)+c$

e $-\frac{3}{4}\log_e(1-4x)+c$ **f** $-6\log_e(x-4)+c$

3 a $5\log_e|x|+c$ **b** $3\log_e|x-4|+c$

c $5\log_e|2x+1|+c$ **d** $-3\log_e|2x-5|+c$

e $-3\log_e|1-2x|+c$ **f** $-\frac{1}{3}\log_e|3x-4|+c$

4 a $3x+\log_e|x|+c$ **b** $x+\log_e|x|+c$

c $-\frac{1}{x+1}+c$ **d** $2x+\frac{x^2}{2}+\log_e|x|+c$

e $-\frac{3}{2(x-1)^2}+c$ **f** $-2x+\log_e|x|+c$

5 a $y=\frac{1}{2}\log_e(x)+1,\ x>0$

b $y=10-\log_e(5-2x),\ x<\frac{5}{2}$

6 $y=10\log_e(x-5)$

7 a $x-\log_e|x+1|+c$

b $-2(x+1)+3\log_e|x+1|+c$

c $2(x+1)-\log_e|x+1|+c$

8 $y=3\log_e\left(\frac{2-x}{2}\right)+10$

9 $y=\frac{5}{4}\log_e\left(\frac{5}{1-2x}\right)+10$

10 $y=\frac{5}{4}\log_e\left(\frac{1}{2x-1}\right)+10$

Exercise 11D

1 a $\frac{1}{6}e^{6x}+c$ **b** $\frac{1}{2}e^{2x}+\frac{3}{2}x^2+c$

c $-\frac{1}{3}e^{-3x}+x^2+c$ **d** $-\frac{1}{2}e^{-2x}+\frac{1}{2}e^{2x}+c$

2 a $\frac{1}{2}e^{2x}-2e^{\frac{x}{2}}+c$ **b** $e^x-e^{-x}+c$

c $\frac{2}{3}e^{3x}+e^{-x}+c$ **d** $15e^{\frac{x}{3}}-10e^{\frac{x}{5}}+c$

e $\frac{9}{2}e^{\frac{2x}{3}}-\frac{15}{7}e^{\frac{7x}{5}}+c$ **f** $\frac{15}{4}e^{\frac{4x}{3}}-\frac{9}{2}e^{\frac{2x}{3}}+c$

3 a $y=\frac{1}{2}(e^{2x}-x^2+9)$ **b** $y=-\frac{3}{e^x}-e^x+8$

4 $y=9-2e^{-2}$

5 a $k=2$ **b** $y=\frac{1}{2}e^{2x}+\frac{1}{2}e^2$

6 a $k=3$ **b** $y=-\frac{1}{3}e^{3x}-\frac{2}{3}e^3$

Exercise 11E

1 a $\frac{7}{3}$ **b** 20 **c** $-\frac{1}{4}$ **d** 9

e $\frac{1}{2}$ **f** $\frac{140}{3}$ **g** $15\frac{1}{3}$ **h** $343\frac{11}{20}$

2 a 10 **b** 1 **c** $\frac{13}{3}$ **d** $\frac{1}{3}$

e $\frac{10}{441}$ **f** 34 **g** $\frac{2}{3}\left(2^{\frac{3}{2}}-1\right)$

h $2-2^{\frac{1}{2}}$ **i** $\frac{1}{15}$

3 a $\frac{1}{2}(e^2-1)$ **b** $\frac{1}{2}(3-e^{-2})$

c $6e^{\frac{1}{3}}-4$ **d** e^2-e^{-2}

4 a 10 **b** 17 **c** -5 **d** 9 **e** -3

5 a $\log_e\left(\frac{1}{3}\right)$ **b** $\frac{1}{2}\log_e 5$ **c** $\frac{3}{2}\log_e\left(\frac{19}{17}\right)$

Exercise 11F

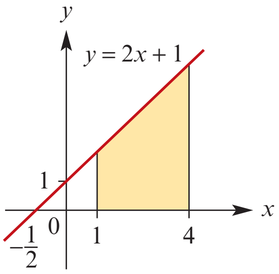
1 a 3

b 44

c i 8 **ii** 10

2 a $\frac{4}{3}$ **b** $\frac{1}{6}$ **c** $121\frac{1}{2}$ **d** $\frac{1}{6}$ **e** $4\sqrt{3}$ **f** 108

3 a



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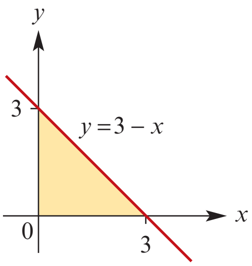

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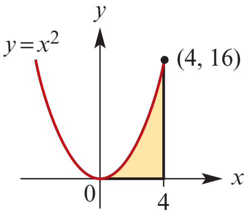

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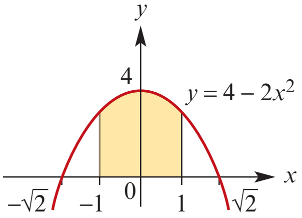
d



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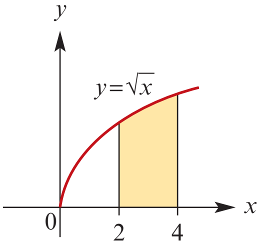

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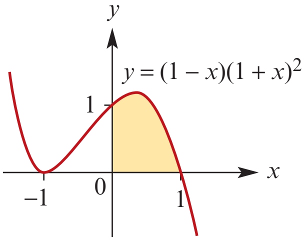
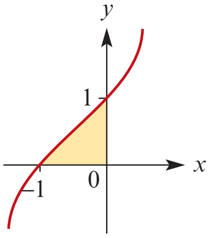


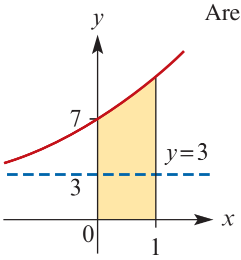
4 $\frac{321}{10}$ square units

5 Area = $\frac{3}{4}$ square units

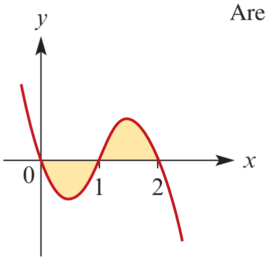


6 Area = $2e^2 + 1$

≈ 15.78 square units



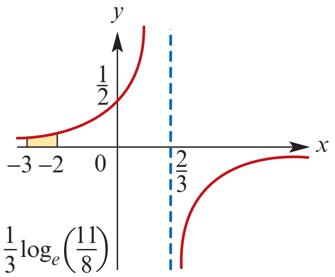
7 Area = 0.5 square units



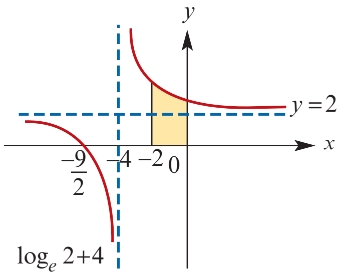
8 **a** $\frac{5}{6}$ square units **b** $8\frac{1}{6}$ square units

9 **a** $A(0, 3)$, $B(1, 0)$ **b** 2 square units

10



11



12 **b** Derivative: $(\log_e a)e^{x \log_e a}$

Antiderivative: $\dfrac{e^{x \log_e a}}{\log_e a}$

Exercise 11G

1 **a** $\frac{1}{3}\sin(3x)$ **b** $-2\cos\left(\frac{1}{2}x\right)$ **c** $\sin(3x)$

d $-4\cos\left(\frac{1}{2}x\right)$ **e** $-\frac{1}{2}\cos\left(2x - \frac{\pi}{3}\right)$

f $\frac{1}{3}\sin(3x) - \frac{1}{2}\cos(2x)$

g $\frac{1}{4}\sin(4x) + \frac{1}{4}\cos(4x)$

h $\frac{1}{4}\cos(2x) + \frac{1}{3}\sin(3x)$

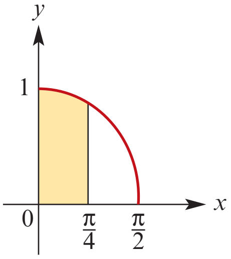
i $-\frac{1}{4}\sin\left(2x + \frac{\pi}{3}\right)$ **j** $-\frac{1}{\pi}\cos(\pi x)$

2 **a** $1 - \frac{1}{\sqrt{2}}$ **b** $\frac{1}{2}$ **c** $1 + \frac{1}{\sqrt{2}}$ **d** 2 **e** 1

f $\frac{2}{3}$ **g** $-\frac{1}{2}$ **h** 4 **i** $\frac{1-\sqrt{3}}{4}$ **j** −2

3 $-\sqrt{2} + 2$ square units

4 **a** $\int_0^{\frac{\pi}{4}} \cos x \, dx = \frac{1}{\sqrt{2}}$



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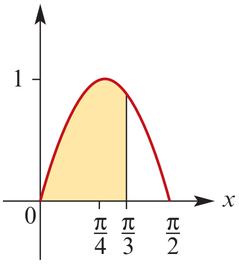
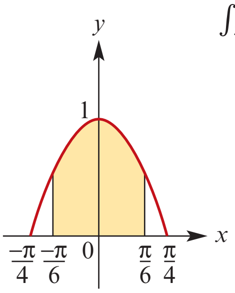
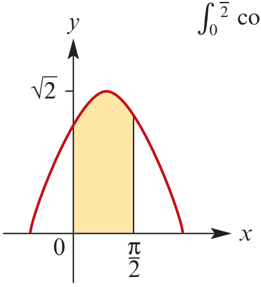
b $\int_0^{\frac{\pi}{3}} \sin(2x)\, dx = \frac{3}{4}$



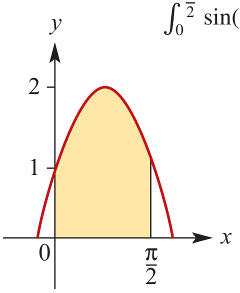
c $\int_{\frac{-\pi}{6}}^{\frac{\pi}{6}} \cos(2x)\, dx = \frac{\sqrt{3}}{2}$



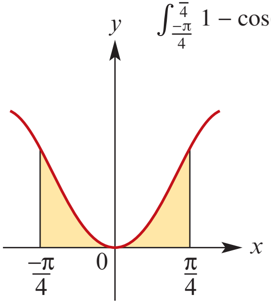
d $\int_0^{\frac{\pi}{2}} \cos\theta + \sin\theta\, d\theta = 2$



e $\int_0^{\frac{\pi}{2}} \sin(2\theta) + 1\, d\theta = 1 + \frac{\pi}{2}$



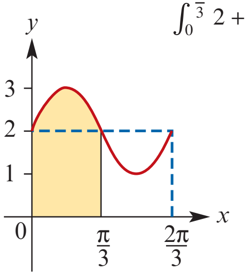
f $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} 1 - \cos(2\theta)\, d\theta = \frac{\pi}{2} - 1$



5 a $\frac{\sqrt{2}}{2}$ **b** $-\frac{1}{3}$

c $-\frac{\sqrt{3}}{3}$ **d** $-\frac{1}{\sqrt{2}}$

6 $\int_0^{\frac{\pi}{3}} 2 + \sin(3x)\, dx = \frac{2\pi}{3} + \frac{2}{3}$



Exercise 11H

1 a $4\frac{2}{3}$ **b** $2\frac{2}{3}$ **c** 12 **d** $\frac{5\sqrt{3}}{4} - 2$

e $\frac{e^4}{2} + 4\log_e 2 - \frac{e^2}{2}$ **f** $\frac{2}{3}$ **g** 4

h $\frac{5\pi^2}{8} + 1$ **i** $8\log_e 2 + \frac{51}{4}$ **j** $\frac{1}{12}$

2 0.5 square units

3 a $\frac{1}{\cos^2 x},\ \tan x$ **b** $-\frac{2}{\sin^2(2x)},\ -\frac{\cos(2x)}{2\sin(2x)}$

c $\frac{6x}{3x^2+7},\ \frac{1}{6}\log_e\left(\frac{19}{7}\right)$

d $\sin(x) + x\cos(x),\ -1 + \frac{1}{\sqrt{2}} + \frac{\pi}{4\sqrt{2}}$

4 a $1 + \log_e(2x),\ -x + x\log_e(2x)$

b $x + 2x\log_e(2x),\ \frac{1}{2}x^2\log_e(2x) - \frac{x^2}{4}$

c $1 + \frac{x}{\sqrt{1+x^2}},\ \log_e(1+\sqrt{2})$

5 $\frac{e^{\sqrt{x}}}{2\sqrt{x}},\ 2e^{\sqrt{2}} - 2e$

6 $6\sin^2(2x)\cos(2x),\ \frac{1}{6}$

7 a 139.68 **b** 18.50 **c** −0.66

d −23.76 **e** 2.06 **f** 0.43

8 b $5\log_e 3 + 4$

9 b $5 + 6\log_e 2$

10 a $\frac{dy}{dx} = -4\left(1 - \frac{1}{2}x\right)^7$

Hence $\int \left(1 - \frac{1}{2}x\right)^7 dx = -\frac{1}{4}\left(1 - \frac{1}{2}x\right)^8 + c$

b $\frac{dy}{dx} = -\tan x$; Hence $\int_0^{\frac{\pi}{3}} \tan x\, dx = \log_e 2$

11 $f(x) = 1 - 2\cos\left(\frac{1}{2}x\right)$

12 a $f(x) = \frac{1}{2}\sin 2x + 1$ **b** $f(x) = 3\log_e x + 6$

c $f(x) = 2e^{\frac{x}{2}} - 1$

13 $\sin(3x) + 3x\cos(3x)$

Hence $\int_0^{\frac{\pi}{6}} x\cos(3x)\, dx = \frac{\pi}{18} - \frac{1}{9}$

14 $a = 1,\ b = -2$; Area $= \frac{12\sqrt{3} - \pi - 12}{3\pi}$

15 a 1.450 square units **b** 1.716 square units

16 0.1345

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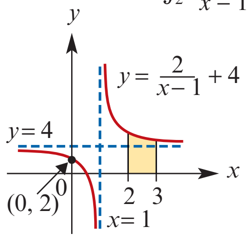
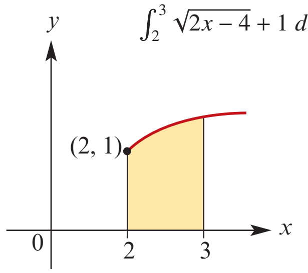
17 $f(x) = \frac{1}{2}\left(x^2 - \cos(2x) + 3\right)$

18 **a** $(x^2+1)^3 + c$ **b** $\sin(x^2) + c$
c $(x^2+1)^3 + \sin(x^2) + c$
d $-(x^2+1)^3 + c$ **e** $(x^2+1)^3 - 4x + c$
f $3\sin(x^2) + c$

19 $\int_2^3 \frac{2}{x-1} + 4\,dx = 2\log_e 2 + 4$



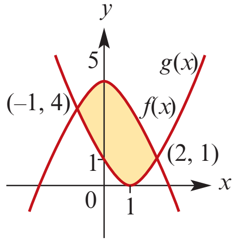
20 $\int_2^3 \sqrt{2x-4} + 1\,dx = \frac{1}{3} \times 2^{\frac{3}{2}} + 1$



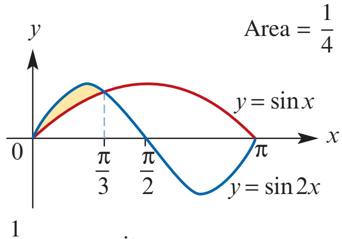
21 **a** $\frac{4\sqrt{2}}{3} - \frac{2}{3}$ **b** $\frac{2^{\frac{5}{2}}}{3}$ **c** $\frac{1}{3}\log_e 4$
d $\frac{1}{2}\log_e 3 + 3$ **e** $-2\frac{2}{3}$ **f** $2\sqrt{2} - 2$

Exercise 11I

1 36 square units
2 Area = 9 square units



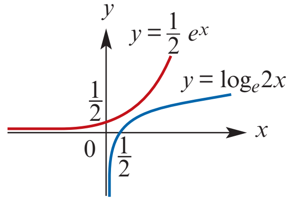
3 **a** 36 square units **b** $20\frac{5}{6}$ square units
c 4 square units **d** $4\frac{1}{2}$ square units
e $4\frac{1}{2}$ square units
4 **a** 2 square units
b $e + e^{-1} - 2 \approx 1.086$ square units
5 3.699 square units
6 Area = $\frac{1}{4}$ square units



7 $\frac{1}{2}$ square units

8 $P(\log_e 3, 3)$; Area ≈ 2.197 square units

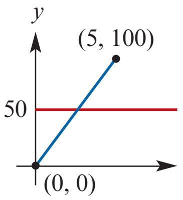
9 **a** $f^{-1}(x) = \frac{1}{2}e^x$ **b** $\frac{3}{2}$



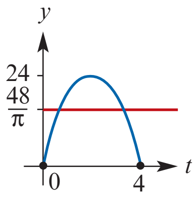
c $4\log_e(2) - \frac{3}{2}$

Exercise 11J

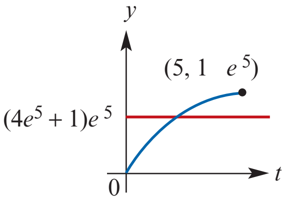
1 **a** $\frac{2}{3}$ **b** $\frac{2}{\pi}$ **c** $\frac{2}{\pi}$ **d** 0 **e** $\frac{1}{2}(e^2 - e^{-2})$
2 $10(e^5 - 1)e^{-5} \approx 9.93$°C
3 **a**



b



c



4 $\frac{147}{10}$ m/s
5 $\frac{a^2}{6}$
6 **a** $3000(2 - 2^{0.9})$ N/m^2
b $1000(4^{0.1} - 1)$ N/m^2
7 **a** $x = t^2 - 3t$ **b** $x = 0$ **c** 0
d $\frac{9}{2}$ m **e** $\frac{3}{2}$ m/s
8 **a** $x = \frac{2t^3}{3} - 4t^2 + 6t + 4,\ a = 4t - 8$
b When $t = 1$, $x = \frac{20}{3}$ m;
When $t = 3$, $x = 4$ m
c When $t = 1$, $a = -4$ m/s^2;
When $t = 3$, $a = 4$ m/s^2
9 Initial position is 3 m to the left of O
10 Velocity = 73 m/s; Position = $\frac{646}{3}$ m
11 **a** Velocity = $-10t + 25$
b Height = $-5t^2 + 25t$ **c** $\frac{5}{2}$ s
d $\frac{125}{4}$ m **e** 5 s

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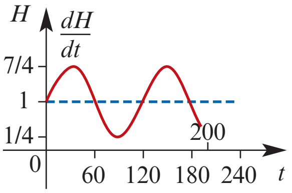
12 **a**



b $t \in (10, 50) \cup (130, 170)$

c $t = 30$ or $t = 150$

d **i** 120 kilojoules

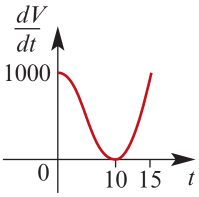
ii 221.48 kilojoules

13 **a** When $t = 0$, 1000 million litres per hour; When $t = 2$, 896 million litres per hour

b **i** $t = 0$ and $t = 15$

ii 1000 million litres per hour

c

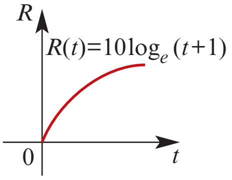


d **i** 5000

ii 5000 million litres flowed out in the first 10 hours

14 **a** When $t = 5$, ≈ 17.9 penguins per year; When $t = 10$, ≈ 23.97 penguins per year; When $t = 100$, ≈ 46.15 penguins per year

b



c $t = e^{\frac{R}{10}} - 1, R \geq 0$; i.e. $R^{-1}(t) = e^{\frac{t}{10}} - 1$

d **i** 3661

ii The growth in the size of the penguin population over 100 years (assuming zero death rate)

15 $71\,466\frac{2}{3}$ m^3

16 **a** 465 m^2 **b** 46 500 m^3

17 1.26 m

18 **a** 6 metres

b 18π m^2

c **i** $y - 3 + 3\cos\left(\frac{a}{3}\right) = \dfrac{-1}{\sin\left(\frac{a}{3}\right)}(x - a)$

ii 5.409

19 **a** **i** 9 **ii** $\dfrac{3(\sqrt{2} + 2)}{2}$ **iii** 12

b Max value is 12; Min value is 0.834

c $\dfrac{48(\pi + 1)}{\pi}$ litres

Chapter 11 review

Technology-free questions

1 **a** $\dfrac{65}{4}$ **b** 0 **c** $\dfrac{-5a^{\frac{3}{2}}}{3}$ **d** $-\dfrac{55}{3}$

e $\dfrac{1}{2}$ **f** 1 **g** 0 **h** 0

2 $\dfrac{23}{2}$ **3** 3 **4** 4 **5** 820

6 $\dfrac{85}{4}$ **7** $\dfrac{5}{3}$ **8** $\dfrac{5}{3}$

9 $\int_a^b f(x) - g(x)\,dx + \int_b^c g(x) - f(x)\,dx + \int_c^d f(x) - g(x)\,dx$

10 **a** $P(3, 9)$, $Q(7.5, 0)$ **b** 29.25 square units

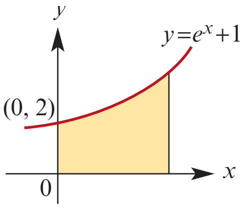
11 **a** 5 **b** $p = \dfrac{20}{7}$

12 3.45 square units

13 **a** $A(0, 6)$, $B(5, 5)$ **b** $15\frac{1}{6}$ square units

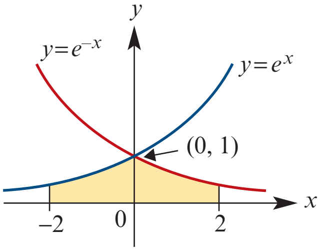
c $\dfrac{125}{6}$ square units

14 **a**



b $e^2 + 1 \approx 8.39$

15 **a**



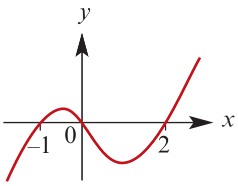
b $2 - 2e^{-2}$

16 **a** $e - 1 \approx 1.72$

b $2(e - 1) \approx 3.44$ square units

17 $2 + e^2 \approx 9.39$ square units

18 $3\frac{1}{12}$ square units



19 **a** $3 - e^{-2}$ **b** $\log_e\left(\dfrac{2}{3}\right) - \dfrac{3}{2}$ **c** $\dfrac{\pi^2}{8} + 1$

d $\dfrac{1}{2}\log_e\left(\dfrac{5}{6}\right) - e^{-4} + e^{-5}$

Multiple-choice questions

1 C	**2** C	**3** B	**4** B	**5** A
6 D	**7** E	**8** C	**9** C	**10** D

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Extended-response questions

1 a $4y - 5x = -3$ **b** $\left(\frac{3}{5}, 0\right)$

c $(1, 0)$ **d** $\frac{9}{40}$ **e** $9 : 49$

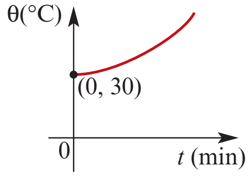
3 a $\frac{1}{3}$ square units

d $1 - \frac{n-1}{n+1} = \frac{2}{n+1}$ square units

e $\frac{9}{11}, \frac{99}{101}, \frac{999}{1001}$

f Area between the curves approaches 1

4 a 968.3° **b**



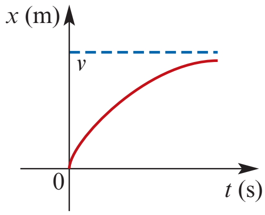
c 2.7 minutes **d** 64.5°C/min

5 a 5×10^4 m/s^2

b Magnitude of velocity becomes very small

c $5 \times 10^4(1 - e^{-20})$ m **d** $x = v(1 - e^{-t})$

e



6 a $\frac{d}{dx}\left(e^{-3x}\sin(2x)\right) = -3e^{-3x}\sin(2x) + 2e^{-3x}\cos(2x)$

c $\int e^{-3x}\sin(2x)\,dx = \frac{-1}{13}\left(3e^{-3x}\sin(2x) + 2e^{-3x}\cos(2x)\right)$

7 a i $\tan a = \frac{4}{3}$ **ii** $\sin a = \frac{4}{5}$, $\cos a = \frac{3}{5}$

b 2 square units

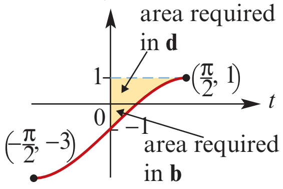
8 a $\frac{dy}{dx} = \log_e x + 1$, $\int_1^e \log_e x\,dx = 1$

b $\frac{dy}{dx} = (\log_e x)^n + n(\log_e x)^{n-1}$

d $\int_1^e (\log_e x)^3\,dx = 6 - 2e$

9 $s = \sqrt[3]{a^2b}$, $r = \sqrt[3]{ab^2}$

10 a



b $\int_0^{\frac{\pi}{6}} f(x)\,dx = 2 - \sqrt{3} - \frac{\pi}{6}$

c $f^{-1}: [-3, 1] \to \mathbb{R}$, $f^{-1}(x) = \sin^{-1}\left(\frac{x+1}{2}\right)$

d $\int_0^1 f^{-1}(x)\,dx = \frac{\pi}{2} - \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} f(x)\,dx$

$= \frac{5\pi}{6} - \sqrt{3}$

11 a $\frac{dy}{dx} = -\frac{x}{10}e^{\frac{x}{10}}$, $\frac{dy}{dx} = -x(100 - x^2)^{-\frac{1}{2}}$

b When $x = 0$, $\frac{dy}{dx} = 0$ for both functions

c $-e$

d 6.71 square units

e 8.55%

f $(25\pi - 50)$ square units

g i $10(10e - 20)$

ii $(25\pi - 100e + 200)$ square units

12 a i $R(0) = 0$ **ii** $R(3) = 0$

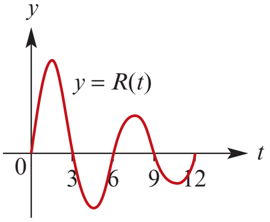
b $R'(t) = e^{-\frac{t}{10}}\left(\frac{10\pi}{3}\cos\left(\frac{\pi t}{3}\right) - \sin\left(\frac{\pi t}{3}\right)\right)$

c i 1.41, 4.41, 7.41, 10.41

ii Local max: (1.41, 8.65), (7.41, 4.75);
Local min: (4.41, −6.41) (10.41, −3.52)

d $t = 0, 3, 6, 9$ or 12

e



f i 16.47 litres **ii** 12.20 litres **iii** 8.27 litres

g 12.99 litres

13 b $1 - \frac{\pi}{4}$

Chapter 12

Technology-free questions

1 a $\frac{-2x^5 + 4x^3 - 2x}{(x^4 - 1)^2}$ **b** $\{0\}$

2 $4(6x - 4)(3x^2 - 4x)^3$ **3** $2x\log_e(2x) + x$

4 a $b = \frac{1}{2}$ **b** $k = (2b - 1)e^{2b+1}$

5 $m = \frac{1}{12}$, $a = -\frac{22}{3}$, $c = -\frac{28}{3}$

6 $\frac{1}{6}\log_e 7$

7 a $\frac{3}{5}\log_e(5x - 2)$ **b** $\frac{3}{10 - 25x}$

8 a −7 **b** −14 **c** −20

9 a $5\frac{1}{2}$ **b** $\frac{1}{8}$

10 $2x\sqrt{3x^2 + 1}$

11 a $4x - 3$ **b** −3 **c** $\{1\}$

12 $\frac{f'(x)}{f(x)}$ **13** $a = \frac{145}{144}$

14 $m = \frac{1}{4}(-3 + \sqrt{105})$

15 a (0, −4) and (−2, 0) **b** 0 **c** 4 **d** $9\frac{1}{2}$

16 $-\frac{3}{25}$ **18** 24π **19** $(0.28, 0.14)$

20 $\frac{2}{3}$ **21** $-\frac{8x}{(x^2-2)^2}$ **22** 6

23 $-63(5-7x)^8$ **24** $\frac{1}{9}$

25 $\frac{2}{3}$ **26** -70 **27** 0

28 -1

29 a $-\frac{2}{(2x+1)^2}$ **b** -2

30 a $x = 0$ or $x = -2$ **b** $x > 0$ or $x < -2$
c $-2 < x < 0$

31 a $\frac{1}{(1-x)^2}$ **b** $(y+1)^2$

32 $-3x(x^2+1)^{-\frac{5}{2}}$

34 $f'(x) = 10x^4 > 0$ for all $x \in \mathbb{R} \setminus \{0\}$. If $b > 0$, then $b^5 > 0$, and if $b < 0$, then $b^5 < 0$.

35 a $4 - 2\sqrt{2}$ **b** $2(e^{\frac{3}{4}} - 1)$ **c** $\frac{1}{2}\log_e 2$
d $-\frac{1}{2}\log_e 2$ **e** $\frac{1}{4}$ **f** $\frac{1}{20}$

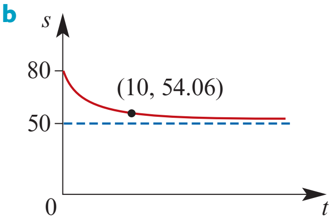
37 a $\frac{1}{m}\left(2 - \log_e\left(\frac{4}{m}\right)\right)$ **b** $0 < m < 4e^{-2}$

Multiple-choice questions

1 E **2** D **3** E **4** D **5** B **6** C
7 E **8** A **9** D **10** A **11** E **12** B
13 B **14** E **15** D **16** E **17** B **18** C
19 E **20** D **21** B **22** E **23** B **24** D
25 B **26** A **27** D **28** D **29** A **30** C
31 C **32** D **33** B **34** A **35** E **36** D
37 A **38** A **39** A **40** B **41** D **42** B
43 B **44** C **45** D **46** B **47** D **48** C
49 A **50** C **51** B **52** E **53** C **54** C
55 D **56** B **57** D **58** C **59** B **60** E
61 B **62** D

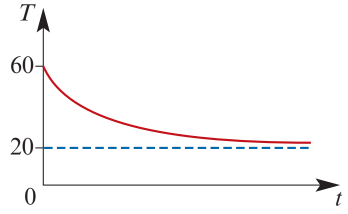
Extended-response questions

1 a 54.06 g **b**



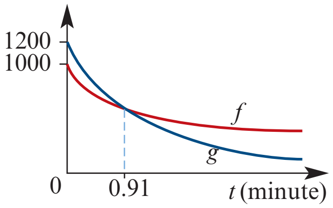
c $\frac{ds}{dt} = -6e^{-\frac{1}{5}t}$ **d** $\frac{ds}{dt} = -\frac{1}{5}(s - 50)$
e 0.8 g/L **f** 17 seconds

2 a 60°C
b

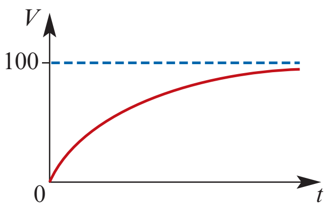


c $\frac{dT}{dt} = -14.4e^{-0.36t}$
d $\frac{dT}{dt} = -0.36(T - 20)$

3 a 1.386 minutes **b** 2200, 5.38%
c 66.4 spores/minute **d** 0.9116 minutes
e



4 a



b i $20e^{-0.2t}$ m/s^2 **ii** $20 - \frac{V}{5}$ m/s^2
c 8.05 seconds

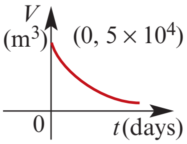
5 100

6 b $k = 0.028$ **c** 0.846°C/min

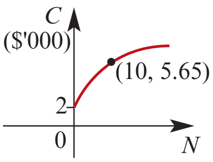
7 a i 0.1155 **ii** 0.2
b 13.86 days

8 \$600

9 a 5×10^4 m^3 **b** $-12\,500$ m^3/day
c -3500 m^3/day **d** After 222.61 days
e



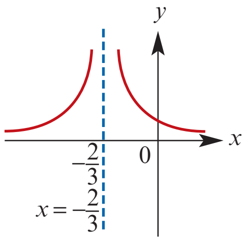
10 a



b $\frac{3N^2}{4(N^3+16)^{\frac{3}{4}}}$
c Rate of change of cost in \$1000s with respect to the increase in the number of bottle tops produced

11 $p = 4$; Number of items sold = 50

12 $a = \pm 3$, $b = \pm 2$



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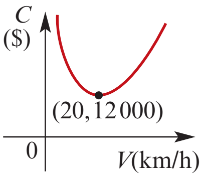
13 **a** \$17 000 **b** $C = \dfrac{160\,000}{V} + 10V^2$

c



d $V = 20,\ C = 12\,000$ **e** \$12 560

14 **a** $\dfrac{60}{\sqrt{39}} \approx 9.61$ km **b** $\dfrac{60}{\sqrt{39}} \approx 9.61$ km

15 $4 - \sqrt{\dfrac{12}{7}} \approx 2.7$ m

16 **a** $\{x : x > 1\}$ **b** $\{x : 0 < x < 2\}$

c $\{x : x > 1\},\ \left\{x : 0 < x < \dfrac{3}{2}\right\}$

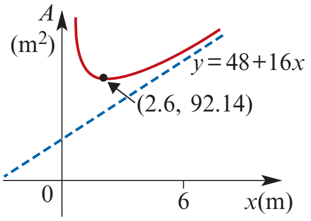
d $\{x : x > 1\},\ \left\{x : 0 < x < \dfrac{n+1}{n}\right\}$

17 **a** $(1, 1), (-1, -1)$ **b** $\left(\pm 2^{\frac{1}{6}}, 2^{-\frac{1}{3}}\right)$

c $\left(\pm n^{\frac{1}{2n+2}}, n^{-\frac{n}{2n+2}}\right)$

18 **a** $A = 48 + 16x + \dfrac{108}{x}$

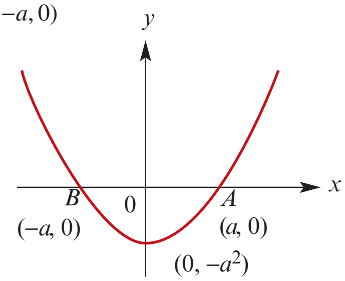
b



c Height $= \dfrac{3\sqrt{3}}{2}$ m, width $= \dfrac{4\sqrt{3}}{3}$ m

d 172 m^2

19 **a** $(a, 0), (-a, 0)$



b $\dfrac{4a^3}{3}$ **c** $\dfrac{2a^2}{3}$ **d** 2 : 3

20 **a** $-5\frac{1}{3}$ **c** $a = 1$ or $a = -2$

21 **a** **i** $50e^{-1}$ litres/minute

ii $t = 5$

iii 2 minutes 18 seconds

iv 3 minutes 48 seconds

b 14.74 litres

c 53 seconds

Chapter 13

Exercise 13A

1 1H, 1T, 2H, 2T, 3H, 3T, 4H, 4T, 5H, 5T, 6H, 6T

2 HH1, HH2, HH3, HH4, HH5, HH6,
HT1, HT2, HT3, HT4, HT5, HT6,
TH1, TH2, TH3, TH4, TH5, TH6,
TT1, TT2, TT3, TT4, TT5, TT6

3 **a** $\dfrac{1}{13}$ **b** $\dfrac{3}{4}$ **c** $\dfrac{4}{13}$ **d** $\dfrac{2}{13}$

4 **a** $\dfrac{1}{2}$ **b** $\dfrac{2}{3}$

5 0.8

6 0.65

7 **a** 0.067 **b** 0.047

8 5%

9 $\dfrac{6}{7}$

10 **a** $\dfrac{17}{500}$ **b** $\dfrac{9}{250}$ **c** $\dfrac{41}{125}$ **d** $\dfrac{41}{500}$

11 **a** $\dfrac{13}{20}$ **b** $\dfrac{7}{20}$

12 **a** $\dfrac{57}{100}$ **b** $\dfrac{2}{19}$ **c** $\dfrac{27}{100}$ **d** $\dfrac{1}{250}$

13 $\dfrac{9}{25}$

14 **a** $\dfrac{1}{2}$ **b** $\dfrac{1}{6}$ **c** $\dfrac{5}{6}$

15 **a** 0.13 **b** 0.32

16 **a** 0.40 **b** 0.67 **c** 0.18

17 **a** 0.35 **b** 0.18 **c** 0.12 **d** 0.17

Exercise 13B

1 **a** 0.2 **b** 0.675 **c** 0.275

2 **a** $\dfrac{1}{6}$ **b** $\dfrac{1}{3}$

3 **a** 0.06 **b** $\dfrac{1}{5}$

4 $\dfrac{3}{5}$ **5** $\dfrac{24}{59}$

6 **a** $\dfrac{1}{2}$ **b** $\dfrac{77}{200}$ **c** $\dfrac{40}{77}$ **d** 0.4

7 **a** $\dfrac{65}{224}$ **b** $\dfrac{115}{448}$ **c** $\dfrac{21}{65}$ **d** $\dfrac{61}{246}$

8 **a** 0.24 **b** 0.86

9 **a** Yes **b** No **c** No

10 **a** 0.5 **b** 0.2 **c** 0.7

11 0.39

12 $\dfrac{1}{7}$

13 0.0479

14 **a** 0.486 **b** 0.012 **c** 0.138

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15 **a** $\frac{2}{5}$ **b** $\frac{1}{15}$ **c** $\frac{8}{15}$

16 **a** $\frac{2}{5}$ **b** $\frac{7}{40}$ **c** $\frac{7}{16}$ **d** $\frac{7}{15}$

17 **a** $\frac{5}{14}$ **b** $\frac{3}{5}$

18 **a** 0.735 **b** 0.453

19 $\frac{3}{44}$

Exercise 13C

1 **a** Discrete **b** Not discrete
c Discrete **d** Discrete

2 **a** Not discrete **b** Discrete
c Not discrete **d** Discrete

3 **a** $\{HHH, THH, HTH, HHT, HTT, THT, TTH, TTT\}$

b

x	Outcomes
0	TTT
1	HTT, THT, TTH
2	THH, HTH, HHT
3	HHH

c $\frac{1}{2}$

4 **a** Yes, as the sum of the probabilities is 1 and $p(x) \geq 0$ for all x
b 0.8

5 **a**

x	0	1	2	3
$p(x)$	$\frac{125}{729}$	$\frac{300}{729}$	$\frac{240}{729}$	$\frac{64}{729}$

b $\frac{604}{729}$ **c** $\frac{304}{729}$

6 **a** $\{(1,1),(1,2),(1,3),\ldots,(6,4),(6,5),(6,6)\}$
b $Y = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$

		2nd die					
		1	2	3	4	5	6
1st die	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

c **i** $\frac{1}{6}$ **ii** $\frac{1}{3}$ **iii** $\frac{1}{5}$ **iv** $\frac{7}{10}$ **v** $\frac{1}{5}$ **vi** $\frac{2}{7}$

7 **a**

		2nd die					
		1	2	3	4	5	6
1st die	1	1	1	1	1	1	1
	2	1	2	2	2	2	2
	3	1	2	3	3	3	3
	4	1	2	3	4	4	4
	5	1	2	3	4	5	5
	6	1	2	3	4	5	6

b 1, 2, 3, 4, 5, 6 **c** 0.19

8 **a** 0.288 **b** 0.064 **c** 0.352 **d** 0.182

9 **a** $\{(1,1),(1,2),(1,3),\ldots,(6,4),(6,5),(6,6)\}$
b $\Pr(A) = \frac{1}{6},\ \Pr(B) = \frac{1}{6},\ \Pr(C) = \frac{5}{12},$
$\Pr(D) = \frac{1}{6}$
c $\Pr(A \mid B) = \frac{1}{6},\ \Pr(A \mid C) = \frac{1}{5},\ \Pr(A \mid D) = \frac{1}{6}$
d **i** Independent **ii** Not independent
iii Independent

10 **a** Yes **b** 0.5

11 **a** and **c**

12 **a**

x	0	1	2	3
$p(x)$	$\frac{27}{125}$	$\frac{54}{125}$	$\frac{36}{125}$	$\frac{8}{125}$

b

x	0	1	2	3
$p(x)$	$\frac{5}{30}$	$\frac{15}{30}$	$\frac{9}{30}$	$\frac{1}{30}$

13

x	0	1	2
$p(x)$	0.36	0.48	0.16

14 **a**

x	1	2	3	4	5
$p(x)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

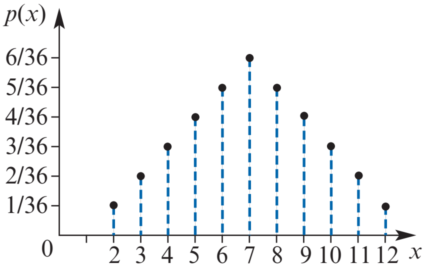
b $\frac{3}{5}$ **c** $\frac{1}{3}$

15 **a** $\{(1,1),(1,2),(1,3),\ldots,(6,4),(6,5),(6,6)\}$

b

x	2	3	4	5	6	7	8	9	10	11	12
$p(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

c



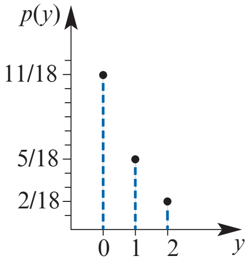
d $\frac{5}{18}$ **e** $\frac{7}{10}$

16 **a** $\{(1,1),(1,2),(1,3),\ldots,(6,4),(6,5),(6,6)\}$

b

y	0	1	2
$p(y)$	$\frac{11}{18}$	$\frac{5}{18}$	$\frac{2}{18}$

c



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17 a

x	0	1	2
$p(x)$	$\frac{1}{3}$	$\frac{8}{15}$	$\frac{2}{15}$

b $\frac{7}{15}$

18 a

x	10	20	100
$p(x)$	$\frac{3}{4}$	$\frac{6}{25}$	$\frac{1}{100}$

b

y	20	30	40	110	120	200
$p(y)$	$\frac{9}{16}$	$\frac{9}{25}$	$\frac{36}{625}$	$\frac{3}{200}$	$\frac{3}{625}$	$\frac{1}{10000}$

19 a $\frac{1}{4}$

b $\{EENE, ENEE, ENNN, NEEE, NENN, NNEN\}$, $\Pr(X = 4) = \frac{3}{8}$

c $\Pr(X = 5) = \frac{3}{8}$

Exercise 13D

1 \$60

2 a $\mathrm{E}(X) = 4.6$ **b** $\mathrm{E}(X) = 0.5$
c $\mathrm{E}(X) = 2.38$ **d** $\mathrm{E}(X) = 0.569$
e $\mathrm{E}(X) = 7$ **f** $\mathrm{E}(X) = 0$

3 Expected profit = \$3000

4 A loss of 17c

5 1.54

6

x	1	2	3	4	5	6	7	8	9	10	11	12
$p(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	0	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{36}$

$\mathrm{E}(X) = \frac{49}{12}$

7 a $\mathrm{E}(X) = 4.11$ **b** $\mathrm{E}(X^3) = 78.57$
c $\mathrm{E}(5X - 4) = 16.55$ **d** $\mathrm{E}\left(\frac{1}{X}\right) = 0.255$

8 \$5940

9 a $p = \frac{1}{16}$ **b** $\mathrm{E}(X) = 2$ **c** $\mathrm{Var}(X) = 3.5$

10 a $k = \frac{1}{21}$ **b** $\mathrm{E}(X) = \frac{91}{21}$ **c** $\mathrm{Var}(X) = \frac{20}{9}$

11 a

x	1	2	3	4	6	8	9	12	16
$p(x)$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{16}$

b i $\frac{1}{4}$ **ii** $\frac{25}{4}$ **iii** $\frac{275}{16}$

12 a $\frac{21}{4}$ **b** $\frac{7}{12}$ **c** $\frac{497}{48}$

13 a $\mathrm{Var}(2X) = 64$ **b** $\mathrm{Var}(X + 2) = 16$
c $\mathrm{Var}(1 - X) = 16$ **d** $\mathrm{sd}(3X) = 12$

14 a $c = 0.35$ **b** $\mathrm{E}(X) = 2.3$
c $\mathrm{Var}(X) = 1.61$, $\mathrm{sd}(X) = 1.27$ **d** 0.95

15 a $k = \frac{1}{15}$ **b** $\mathrm{E}(X) = 3.667$
c $\mathrm{Var}(X) = 1.556$ **d** 0.9333

16 a 7 **b** 5.83 **c** 0.944

17 a 3 **b** 1.5 **c** 0.9688

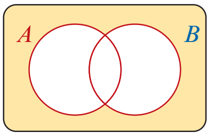
18 $c_1 = 40$, $c_2 = 60$

Chapter 13 review

Technology-free questions

1 Yes, as $\Pr(A \cap B) = 0$

2



$\Pr(A' \cap B')$
$= \Pr(A \cup B)'$

3 a $\frac{40}{81}$ **b** $\frac{5}{9}$

4 0.4

6 a 0.1 **b** 1.3 **c** 2.01

7 a 21.5 **b** $\frac{53}{256}$ **c** 630.75 **d** $\frac{29\sqrt{3}}{2}$

8 a

p	$x - 2$	-2
$\Pr(P = p)$	$\frac{4}{5}$	$\frac{1}{5}$

b $\frac{4}{5}x - 2$

c $x >$ \$2.50

9 a 0.47 **b** $\frac{47}{70}$

10 21.5%

11 a $\frac{1}{24}$ **b** $\frac{17}{24}$ **c** $\frac{5}{6}$ **d** $\frac{11}{18}$

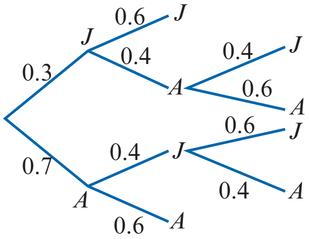
Multiple-choice questions

1 A **2** D **3** D **4** E **5** C
6 C **7** B

Extended-response questions

1 a 0.1 **b** 0.2 **c** 4

2 a



b i 0.396 **ii** 0.604

c i

x	2	3
$\Pr(X = x)$	0.6	0.4

ii 2.4

d 0.46

3 \$14

4 a 0.5 **b** 0.05 **c** 0.033 **d** $\frac{25}{33}$

5 a i 1.21
ii $\mathrm{Var}(P) = 1.6659$, $\mathrm{sd}(P) = 1.2907$
iii 0.94

b i

t	1	0.40	0
$p(t)$	0.39	0.27	0.34

ii $\mathrm{E}(T) = 0.498 \approx 0.50$ **iii** 1

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6 a E(Y) = 2.002

b Var(Y) = 2.014, sd(Y) = 1.419

c i

b	0	100	200
$p(b)$	0.677	0.270	0.053

ii E(B) = \$37.60

7 a $\mu = 13.5\%$, $\sigma = 16.2\%$ **b** 0.95

c E(G) = 6.9%, sd(G) = 9.726%

8 Yes

9 \$1.00

10 a i 0.65 **ii** 0.2275 **iii** 0.079625

iv 0.042875

b Expected cost: \$8 439 375

c Expected profit: \$10 703 125

11 a $\frac{1}{3}(304 - 2x)$ **b** $x = 2$ **c** $27\frac{2}{9}$

12 b $x = \frac{1}{2}, \frac{49}{288}$

13 a i $\frac{1}{81}$ **ii** $\frac{8}{81}$ **iii** $\frac{4}{81}$ **iv** $\frac{4}{27}$ **v** $\frac{56}{81}$

b 4.4197 cents (as the lowest value coin is 5c, he can settle for that)

Chapter 14

Exercise 14A

1 a and **b**

2 0.2734

3 a 0.0256 **b** 0.0016

4 a 0.0778 **b** 0.2304 **c** 0.01024

5 a $\Pr(X = x) = \binom{3}{x}(0.5)^x(0.5)^{3-x}$, $x = 0, 1, 2, 3$

b 0.375

6 a $\Pr(X = x) = \binom{6}{x}(0.48)^x(0.52)^{6-x}$, $x = 0, 1, 2, 3, \dots, 6$

b 0.2527

7 a 0.0536 **b** 0.0087 **c** 0.0623

8 a $\Pr(X = x) = \binom{10}{x}(0.1)^x(0.9)^{10-x}$, $x = 0, 1, 2, 3, \dots, 10$

b i 0.3487 **ii** 0.6513

9 a $\Pr(X = x) = \binom{11}{x}(0.2)^x(0.8)^{11-x}$, $x = 0, 1, 2, 3, \dots, 11$

b i 0.2953 **ii** 0.0859 **iii** 0.9141

10 a $\Pr(X = x) = \binom{7}{x}(0.2)^x(0.8)^{7-x}$, $x = 0, 1, 2, 3, \dots, 7$

b i 0.000013 **ii** 0.2097 **iii** 0.3899

11 0.624

12 a $\left(\frac{x}{100}\right)^6$ **b** $\frac{6x^5(100 - x)}{100^6}$

c $\frac{x^6}{100^6} + \frac{6x^5(100 - x)}{100^6} + \frac{15x^4(100 - x)^2}{100^6}$

13 0.6836

14 a 0.1156 **b** 0.7986 **c** 0.3170

15 0.6791

16 a 0.1123 **b** 0.5561 **c** 0.000 01

d 0.000 01

17 0.544

18 a $\left(\frac{1}{4}\right)^6 \approx 0.000\ 24$ **b** 0.1694

19 a 0.0138 **b** 0.2765 **c** 0.8208 **d** 0.3368

20 a $(0.8)^8 \approx 0.168$ **b** 0.001 23

c 0.0021

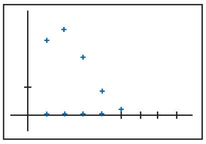
21 a $(0.15)^{10} \approx 0.000\ 000\ 006$

b $1 - (0.85)^{10} \approx 0.8031$ **c** 0.5674

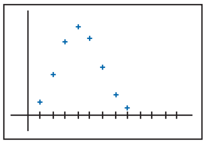
22 a 0.011 529 **b** 0.002 59 **c** 0.0392

Exercise 14B

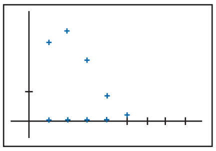
1



2



3 a b



c The distribution in part b is the reflection of the distribution in part a in the line $X = 5$.

4 a Mean = 5; Variance = 4

b Mean = 6; Variance = 2.4

c Mean = $\frac{500}{3}$; Variance = $\frac{1000}{9}$

d Mean = 8; Variance = 6.4

5 a 1 **b** 0.2632

6 37.5

7 $n = 48$, $p = \frac{1}{4}$, $\Pr(X = 7) = 0.0339$

8 $n = 100$, $p = \frac{3}{10}$, $\Pr(X = 20) = 0.0076$

9 Mean = 10, sd = $\sqrt{5}$; The probability of obtaining between 6 and 14 heads is 0.95

10 Mean = 120, sd = $4\sqrt{3}$; The probability that between 107 and 133 students attend a state school is 0.95

Exercise 14C

1 a i $(0.8)^5 \approx 0.3277$ **ii** 0.6723

b 14

c 22

2 a i 0.1937 **ii** $1 - (0.9)^{10} \approx 0.6513$

b 12

3 7 **4** 7 **5** 10 **6** 42 **7** 86

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Chapter 14 review

Technology-free questions

1 a $\frac{16}{81}$ **b** $\frac{32}{81}$ **c** $\frac{16}{27}$ **d** $\frac{65}{81}$

2 $\frac{54}{125}$

3 0.40951

4 a 2 **b** $\frac{3\sqrt{5}}{5}$

5 a $(1-p)^4$ **b** $4p(1-p)^3$ **c** $1-(1-p)^4$
d p^4 **e** $1-(1-p)^4-4p(1-p)^3$

6 120 **7** $\frac{5p(1-p)^4}{1-(1-p)^5}$ **8** $\frac{5}{16}$ **9** $\frac{32}{625}$

Multiple-choice questions

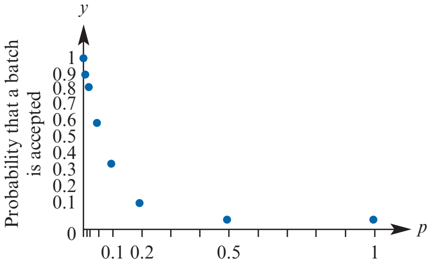
1 D **2** A **3** E **4** B **5** A
6 B **7** C **8** C **9** E **10** B

Extended-response questions

1 a 0.0173 **b** 0.2131

2

p	Probability that a batch is accepted
0	1
0.01	0.9044
0.02	0.8171
0.05	0.5987
0.1	0.3487
0.2	0.1074
0.5	0.00098
1	0



3 a $0.0582 = 5.82\%$
b $\mu = 0.4,\ \sigma = 0.6197,\ \mu \pm 2\sigma = 0.4 \pm 1.2394$
c Yes

4 0.0327

5 a i 0.0819 **ii** 0.9011
b i $P = 15p^2(1-p)^4$
ii $\frac{dP}{dp} = 30p(1-p)^3(1-3p)$

6 a 2 **b** $n = 6,\ p = \frac{1}{3}$

c

x	0	1	2	3	4	5	6
Freq	17.56	52.68	65.84	43.90	16.46	3.29	0.27

7 a 0.9139 **b** 0.041 45 **c** 10.702

8 a 0.0735 **b** 0.5015 **c** 27

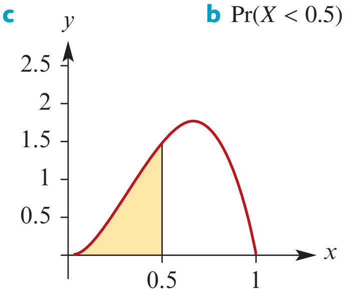
9 $\frac{1}{3} \le q \le 1$

Chapter 15

Exercise 15A

2 $k = -\frac{11}{6}$

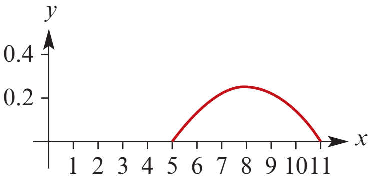
3 a c



b $\Pr(X < 0.5) = \frac{5}{16}$

4 a $k = 1$ **b** 0.865

5 a

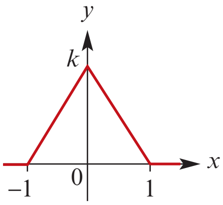


b 0.259 **c** 0.244 **d** 0.28

6 b i $0.024 = 2.4\%$ **ii** $0.155 = 15.5\%$

7 a $k = 0.005$ **b** 0.007

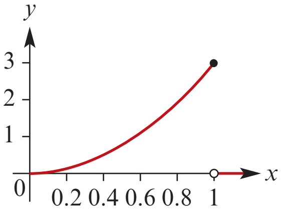
8 a



b $k = 1$

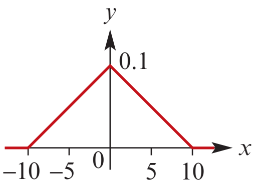
c $\frac{3}{4}$

9 a



b 0.406

10 a



b 0.190

11 a $k = 1000$ **b** 0.5

12 a $\frac{2}{3}$ **b** $\frac{17}{30}$

13 a 0.202 **b** 0.449

14 a 0.45 **b** 0.711

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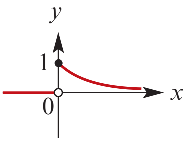
15 a



b i $1 - e^{-\frac{1}{2}}$ **ii** e^{-1} **iii** $e^{-\frac{1}{2}}$

Exercise 15B

1 a $\frac{2}{3}$ **b** $\frac{1}{3}$ **c** $\frac{1}{2}$ **d** Does not exist

2 a 1 **b** 2.097 **c** 1.132 **d** 0.4444

3 a 0.567 **b** 0.458

4 0

5 $A = \frac{2}{9}$, $B = 3$

6 a 2 **b** 1.858

7 a 0.632 **b** 0.233 **c** 0.693

8 a 1 **b** 0.5

9 0.1294

10 2.773 minutes

11 a 1 **b** 1

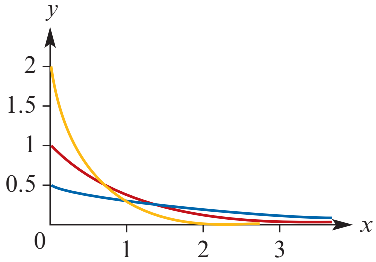
12 a 0.714 **b** 0.736

13 12

14 a 0.4 **b** $\frac{\sqrt{19} - 1}{6}$

15 a $ke^{-kx} - k^2xe^{-kx}$, $\frac{-(kx+1)}{k}e^{-kx}$

c



d $y = e^{-x}$ is dilated by factor $\frac{1}{\lambda}$ from the x-axis and by factor λ from the y-axis

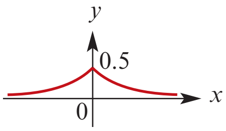
Exercise 15C

1 $\text{Var}(X) = \frac{1}{18}$, $\text{sd}(X) = \frac{\sqrt{2}}{6}$

2 (384, 416)

3 a 0.630 **b** 0.909 **c** 0.279

4 a



b 1.386

5 a $\frac{1}{\log_e 9}$ **b** $\text{E}(X) = 3.641$, $\text{Var}(X) = 4.948$

6 a 0.366 **b** $\text{E}(X) = 0.333$, $\text{Var}(X) = 0.056$

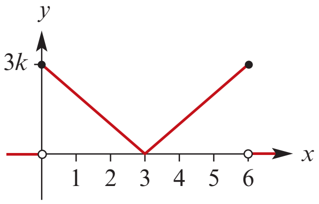
7 0.641

8 a 0.732 **b** $\text{E}(X) = \frac{4}{3}$, $\text{Var}(X) = \frac{2}{9}$

9 a 0.0004 **b** $\frac{16}{3}$ **c** 2.21

10 a $\frac{3}{4a^3}$ **b** $2\sqrt{5}$

11 a



b $k = \frac{1}{9}$ **d** 4.5

Exercise 15D

1 1300

2 a 0.708, 0.048 **b** \$98.94, \$0.33

3 a 0, 5.4 **b** 3, 0.6 **c** 1, 5.4

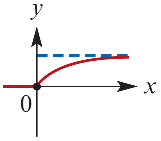
d $g(x) = \begin{cases} \frac{x^2}{18} & \text{if } -3 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$

e $h(x) = \begin{cases} \frac{(x-1)^2}{18} & \text{if } -2 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$

Exercise 15E

1 a $F(x) = \begin{cases} 0 & \text{if } x \le 0 \\ \frac{1}{5}x & \text{if } 0 < x < 5 \\ 1 & \text{if } x \ge 5 \end{cases}$ **b** $\frac{3}{5}$

2 a



b e^{-4} **c** 0.0182

3 a $k = \frac{1}{36}$ **b** $\frac{1}{48}$

Chapter 15 review

Technology-free questions

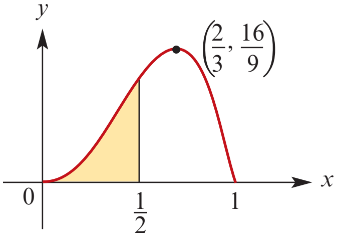
1 a 2 **b** 0.21 **c** 0.44

2 $a = \frac{1}{3}$, $b = 2$

3 $\frac{\pi}{2}$

4 a $\frac{1}{2}$ **b** $\frac{1}{2}$ **c** $\frac{1}{3}$

5 a



b $\Pr(X < 0.5) = \frac{5}{16}$

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6 a $k = 12$ **b** $\Pr(X < \frac{2}{3}) = \dfrac{16}{27}$

c $\Pr(X < \frac{1}{3} \mid X < \frac{2}{3}) = \dfrac{3}{16}$

7 a 0.008 **b** $\dfrac{8}{27}$

8 $\dfrac{2}{3}$

9 a $\dfrac{7}{3}$ **b** $a = 1$

10 a $c = \dfrac{3}{4}$ **b** 0

12 a 1.649 **b** 0.833

13 (320, 340) **14** (246, 254)

Multiple-choice questions

1 B **2** D **3** D **4** A **5** E
6 B **7** C **8** D **9** C **10** A

Extended-response questions

1 a $a = \dfrac{-2}{81}$ **b** 700 hours **c** 736.4 hours

2 a $\dfrac{1}{4}$ **b** $\dfrac{5 - \sqrt{5}}{5}$ **c** $\dfrac{1}{3}, \dfrac{8}{15}$

3 a Median = 6, IQR = $\dfrac{10}{3}$

b $E(X) = 6$, $\text{Var}(X) = 4.736$

4 a $\dfrac{7}{25}$ **b** \$22.13

5 $c = \dfrac{8}{3}$ or $c = 4$

6 b $E(X) = 2$, $\text{Var}(X) = 0.2$ **c** $\dfrac{5}{32}$

7 a 25 **b** $\dfrac{2}{3}$ **d** $\dfrac{2}{3}$

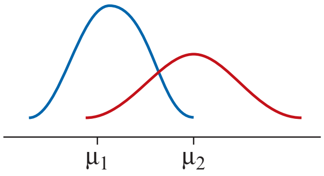
8 a $k = \dfrac{1}{4}$ **b** $\mu = 2$, $\text{Var}(X) = \dfrac{2}{3}$ **c** $\dfrac{3}{4}$

d $\dfrac{4\sqrt{5}}{5} \approx 1.8$

Chapter 16

Exercise 16A

1



2 c

3 a 1

b i $E(X) = \dfrac{1}{3\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} x e^{-\frac{1}{2}\left(\frac{x-2}{3}\right)^2} \, dx$

ii 2

c i $E(X^2) = \dfrac{1}{3\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} x^2 e^{-\frac{1}{2}\left(\frac{x-2}{3}\right)^2} \, dx$

ii 13 **iii** 3

4 a 1

b i $E(X) = \dfrac{1}{5\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} x e^{-\frac{1}{2}\left(\frac{x+4}{5}\right)^2} \, dx$

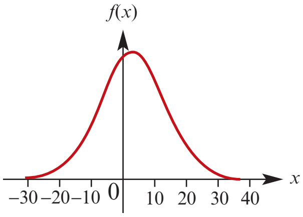
ii −4

c i $E(X^2) = \dfrac{1}{5\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} x^2 e^{-\frac{1}{2}\left(\frac{x+4}{5}\right)^2} \, dx$

ii 41 **iii** 5

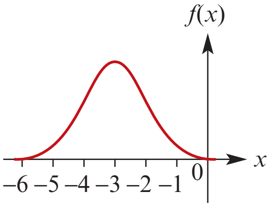
5 a $\mu = 3$, $\sigma = 10$

b



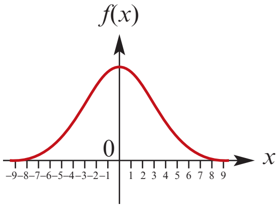
6 a $\mu = -3$, $\sigma = 1$

b



7 a $\mu = 0$, $\sigma = 3$

b



8 a Dilation of factor 2 from the y-axis and dilation of factor $\frac{1}{2}$ from the x-axis, then translation 3 units to the right

b Dilation of factor $\frac{1}{2}$ from the y-axis and dilation of factor 2 from the x-axis, then translation 3 units to the right

c Dilation of factor 2 from the y-axis and dilation of factor $\frac{1}{2}$ from the x-axis, then translation 3 units to the left

9 a Translation 3 units to the left, then dilation of factor $\frac{1}{2}$ from the y-axis and dilation of factor 2 from the x-axis

b Translation 3 units to the left, then dilation of factor 2 from the y-axis and dilation of factor $\frac{1}{2}$ from the x-axis

c Translation 3 units to the right, then dilation of factor $\frac{1}{2}$ from the y-axis and dilation of factor 2 from the x-axis

Exercise 16B

1 a 16% **b** 16% **c** 2.5% **d** 2.5%

2 a $\mu = 135$, $\sigma = 5$ **b** $\mu = 10$, $\sigma = \dfrac{4}{3}$

3 a 68% **b** 16% **c** 0.15%

4 21.1, 33.5

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5 one, 95, 99.7, three

6 2.5%

7 a 16% **b** 16%

8 a 68% **b** 16% **c** 2.5%

9 a 95% **b** 16% **c** 50% **d** 99.7%

10 a 0 **b** $-\frac{5}{4}$ **c** 1.5

11 a −1.4 **b** 1.1 **c** 3.5

12 Michael 1.4, Cheryl 1.5; Cheryl

13 Biology 1.73, History 0.90; Biology

14 a

Student	French	English	Maths
Mary	1	0.875	0
Steve	−0.5	−1	1.25
Sue	0	0.7	−0.2

b i Mary **ii** Mary **iii** Steve

c Mary

Exercise 16C

1 a 0.9772 **b** 0.9938 **c** 0.9938 **d** 0.9943
e 0.0228 **f** 0.0668 **g** 0.3669 **h** 0.1562

2 a 0.9772 **b** 0.6915 **c** 0.9938 **d** 0.9003
e 0.0228 **f** 0.0099 **g** 0.0359 **h** 0.1711

3 a 0.6827 **b** 0.9545 **c** 0.9973

4 a 0.0214 **b** 0.9270 **c** 0.0441 **d** 0.1311

5 $c = 1.2816$ **6** $c = 0.6745$

7 $c = 1.96$ **8** $c = -1.6449$

9 $c = -0.8416$ **10** $c = -1.2816$

11 $c = -1.9600$

12 a 0.9522 **b** 0.7977 **c** 0.0478 **d** 0.1547

13 a 0.9452 **b** 0.2119 **c** 0.9452 **d** 0.1571

14 a $c = 9.2897$ **b** $k = 8.5631$

15 a $c = 10$ **b** $k = 15.88$

16 a $a = 0.994$ **b** $b = 1.96$ **c** $c = 2.968$

17 a 0.7161 **b** 0.0966 **c** 0.5204
d $c = 33.5143$ **e** $k = 13.02913$
f $c_1 = 8.28,\ c_2 = 35.72$

18 a 0.9772 **b** 0.9772 **c** $c = 10.822$
d $k = 9.5792$ **e** $c_1 = 9.02,\ c_2 = 10.98$

Exercise 16D

1 a i 0.2525 **ii** 0.0478 **iii** 0.0901
b 124.7

2 a i 0.7340 **ii** 0.8944 **iii** 0.5530
b 170.25 cm
c 153.267 cm

3 a i 0.0766 **ii** 0.9998 **iii** 0.153
b 57.3

4 a 10.56% **b** 78.51%

5 Mean = 1.55 kg; sd = 0.194 kg

6 a 36.9% **b** $c = 69$

7 a 0.0228 **b** 0.0005 **c** 0.0206

8 a 3.04 **b** 350.27

9 1004 mL

10 a Small 0.1587; Med 0.7745; Large 0.0668
b $348.92

11 a i 0.1169 **ii** 17.7
b 0.0284

12 a A: 0.0228; B: 0.1587 **b** $c = \frac{34}{3}$

Exercise 16E

1 0.9632

2 0.2442

3 a 0.0478 **b** 0.2525

4 a 0.7834 **b** 0.0108

5 0.2819

6 a 0.0416 **b** 0.0038

Chapter 16 review

Technology-free questions

1 a $1 - p$ **b** $1 - p$ **c** $2p - 1$

2 a $a = -1$ **b** $b = 1$ **c** 0.5

3 $(x, y) \to \left(\frac{x-8}{3}, 3y\right)$

4 a $\frac{q}{p}$ **b** $1 - q$ **c** $\frac{1-p}{1-q}$

5 a $\Pr(Z < \frac{1}{2})$ **b** $\Pr(Z < -\frac{1}{2})$ **c** $\Pr(Z > \frac{1}{2})$
d $\Pr(-\frac{1}{2} < Z < \frac{1}{2})$ **e** $\Pr(-\frac{1}{2} < Z < 1)$

6 a 0.84 **b** 0.5 **c** 0.16 **d** 0.68

7 a 0.16 **b** 0.34 **c** 0.32 **d** 0.02

8 a 0.69 **b** 0.19 **c** 0.15 **d** 0.68

9 Best C, worst B

10 a 0.5 **b** $b = -1.5$

Multiple-choice questions

1 A **2** C **3** B **4** B **5** E **6** E
7 C **8** D **9** A **10** D **11** D **12** C

Extended-response questions

1

Category	Range
High	> 63
Moderate	$[56, 62]$
Average	$[45, 55]$
Little	$[37, 44]$
Low	< 37

2 $k = 3.92$

3 a i 0.1587 **ii** 0.9747 **iii** 0.0164
b $c = 53\,592$
c 3.7×10^{-11}

4 a 3.17×10^{-5} **b** False
c $c_1 = 13.53,\ c_2 = 16.47$

5 0.0802

6 0.92%

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7 **a** $\mu = 60.058, \ \sigma = 0.2$ **b** 10%

8 **a** 0.9044 **b** 5.88 **c** 9.044
d 0.2651 **e** \$17.61, 54.0281

9 **a** $\mu = 0, \ \sigma = 2.658$ **b** 0.882

10 **a** 0.1056 **b** 0.0803 **c** 0.5944

Chapter 17

Exercise 17A

1 No; sample will be biased towards the type of movie being shown.

2 **a** No; biased towards shoppers.
b Randomly select a sample from telephone lists or an electoral roll.

3 No; only interested people will call, and they may call more than once.

4 **a** No; biased towards older, friendly or sick guinea pigs which may be easier to catch.
b Number guinea pigs and then generate random numbers to select a sample.

5 No; a student from a large school has less chance of being selected than a student from a small school.

7 **a** Unemployed will be under represented.
b Unemployed or employed may be under represented, depending on time of day.
c Unemployed will be over represented.
Use random sampling based on the whole population (e.g. electoral roll).

8 **a** Divide platform into a grid of 1 m^2 squares. Select squares using a random number generator to give two digits, one a vertical reference and one a horizontal reference.
b Yes, if crabs are fairly evenly distributed; otherwise, five squares may not be enough.

9 No; a parent's chance of selection depends on how many children they have at the school.

10 Not a random sample; only interested people will call, and they may call more than once.

11 People who go out in the evenings will not be included in the sample.

12 **a** All students at this school
b $p = 0.35$ **c** $\hat{p} = 0.42$

13 **a** 0.22 **b** $\hat{p}$

Exercise 17B

1 **a** 0.5 **b** $0, \frac{1}{3}, \frac{2}{3}, 1$

c

$\hat{p}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
$\Pr(\hat{P} = \hat{p})$	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{5}{12}$	$\frac{1}{12}$

d $\frac{1}{2}$

2 **a** $\frac{3}{5}$ **b** $0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1$

c

$\hat{p}$	0	$\frac{1}{5}$	$\frac{2}{5}$
$\Pr(\hat{P} = \hat{p})$	0.0036	0.0542	0.2384
$\hat{p}$	$\frac{3}{5}$	$\frac{4}{5}$	1
$\Pr(\hat{P} = \hat{p})$	0.3973	0.2554	0.0511

d 0.3065 **e** 0.6924

3 **a** 0.5 **b** $0, \frac{1}{3}, \frac{2}{3}, 1$

c

$\hat{p}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
$\Pr(\hat{P} = \hat{p})$	0.1	0.4	0.4	0.1

d 0.9

4 **a** 0.4 **b** $0, \frac{1}{3}, \frac{2}{3}, 1$

c

$\hat{p}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
$\Pr(\hat{P} = \hat{p})$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{30}$

d $\frac{1}{3}$

e $\frac{3}{5}$

5 **a** 0.5 **b** $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$

c

$\hat{p}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$\Pr(\hat{P} = \hat{p})$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

d $\frac{5}{16}$

6 **a** $0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1$

b

$\hat{p}$	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	1
$\Pr(\hat{P} = \hat{p})$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{5}{16}$	$\frac{5}{16}$	$\frac{5}{32}$	$\frac{1}{32}$

c $\frac{3}{16}$ **d** $\frac{25}{26}$

7 **a** $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$

b

$\hat{p}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$\Pr(\hat{P} = \hat{p})$	$\frac{256}{625}$	$\frac{256}{625}$	$\frac{96}{625}$	$\frac{16}{625}$	$\frac{1}{625}$

c $\frac{17}{369}$

8 $\mu = 0.5, \ \sigma = 0.25$ **9** $\mu = 0.5, \ \sigma = 0.224$

10 $\mu = 0.2, \ \sigma = 0.2$ **11** $\mu = 0.3, \ \sigma = 0.084$

12 $\mu = 0.4, \ \sigma = 0.049$ **13** $\mu = 0.2, \ \sigma = 0.04$

14 **a** 0.1844 **b** 0.7600 **c** 0.9683

Exercise 17C

1 0.2858 **2** 0.8568 **3** 0.1587

4 0.0092 **5** 0.0614

6 **a** 1 **b** 0.5000 **c** 0.0412

7 0.9545

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8 **a** 0.9650 **b** 0.9650
9 **a** 0.575 **b** 0.0139
10 **a** 0.848 **b** 0.0031 **c** Yes

Exercise 17D

1 **a** 0.08 **b** $(0.0268, 0.1332)$
2 **a** 0.192 **b** $(0.1432, 0.2408)$
3 **a** 0.2 **b** $(0.1216, 0.2784)$
4 $(0.2839, 0.3761)$
5 **a** $(0.4761, 0.5739)$ **b** $(0.5095, 0.5405)$
c The second interval is narrower because the sample size is larger
6 **a** $(0.8035, 0.8925)$ **b** $(0.8339, 0.8621)$
c The second interval is narrower because the sample size is larger
7 1537
8 246
9 **a** 897 **b** 2017
c Reducing margin of error by 1% requires the sample size to be more than doubled
10 **a** 2017 **b** 2401
c **i** $M = 1.8\%$ **ii** $M = 2.2\%$
d 2401, as this ensures that M is 2% or less, whoever is correct
11 90%: $(0.5194, 0.6801)$, 95%: $(0.5034, 0.6940)$, 99%: $(0.4738, 0.7262)$; Interval width increases as confidence level increases
12 90%: $(0.5111, 0.5629)$, 95%: $(0.5061, 0.5679)$, 99%: $(0.4964, 0.5776)$; Interval width increases as confidence level increases

Chapter 17 review

Technology-free questions

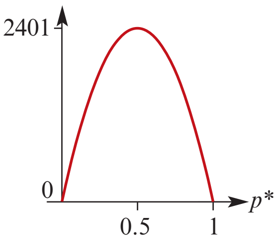
1 **a** All employees of the company
b $p = 0.35$ **c** $\hat{p} = 0.40$
2 **a** No; only people already interested in yoga
b Use electoral roll
3 **a** $\dfrac{k}{100}$ **b** $\dfrac{k}{100} \pm \dfrac{1.96\sqrt{k(100-k)}}{1000}$
4 **a** $\hat{p} = 0.9$ **b** $M = \dfrac{0.588}{\sqrt{n}}$
c Margin of error would decrease by a factor of $\sqrt{2}$
5 **a** 38 **b** $(0.95)^{40}$
6 **a** 45 **b** $5.9(0.9)^{49}$
7 **a** 0.60 **b** 0.10 **c** Increase sample size

Multiple-choice questions

1 B **2** C **3** D **4** E **5** C
6 E **7** B **8** E **9** C **10** E
11 A **12** B **13** C **14** D

Extended-response questions

1 **a**



b 0.5 **c** 2401
2 **a** 0.1537 **b** 0.8257
3 **a** $(0.4730, 0.6670)$
b **i** 0.7738 **ii** 0.0000003
iii 0.2262 **iv** 4.75
c $(0.5795, 0.6645)$
4 **a** $p = \dfrac{500}{N}$ **b** $\hat{p} = 0.15$
c $N = 3333.33 \approx 3333$ **e** $(2703, 4348)$

Chapter 18

Technology-free questions

1 **a** $\dfrac{\pi}{2}$ **b** 2 **c** $\dfrac{2-\sqrt{2}}{2}$ **d** $3 - 2\sqrt{2}$
2 **a** $\dfrac{1}{5}$ **b** $\dfrac{4}{9}$ **c** 1.7 **d** 2.01
3 **a** $\dfrac{1}{36}$ **b** $\dfrac{20}{27}$ **c** 3 **d** 3 **e** $\dfrac{14}{27}$ **f** $\dfrac{13}{20}$
4 **a** $\dfrac{3}{28}$ **b** $\dfrac{3}{14}$ **c** $\dfrac{5}{7}$ **d** $\dfrac{5}{14}$ **e** $\dfrac{5}{28}$
5 **a** $\dfrac{1}{7}$ **b** $\dfrac{1}{3}$
6 **a** $\dfrac{3}{20}$ **b** $\dfrac{9}{35}$ **c** $\dfrac{7}{12}$
7 **a** 0.3369 **b** 0.1348
8 **a** $a = 0.34,\ b = 0.06$ **b** 1.6644
9 **a** 0.75 **b** 0.28
10 **b** $\dfrac{6^{\frac{4}{3}}}{4}$
11 $\dfrac{1-q}{2}$
12 $a = \dfrac{1}{2}$
13 **a** $(1-p)^3$ **b** $p = \dfrac{1}{3}$
14 **a** $\dfrac{a+b}{2}$ **b** $\dfrac{b-a}{2}$

Multiple-choice questions

1 E **2** D **3** D **4** C **5** A **6** B
7 E **8** D **9** E **10** C **11** B **12** D
13 C **14** B **15** E **16** E **17** A **18** A
19 D **20** D **21** C **22** E **23** B **24** C
25 D **26** C **27** E **28** A **29** B **30** E
31 C **32** A **33** B **34** B **35** B **36** B
37 B **38** C **39** B

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Extended-response questions

1 b $\frac{9}{16}$

2 \$0.76

3 a $P = \begin{cases} 0.76x - 0.5s, & x \le s \\ 0.5s - 0.25x, & x > s \end{cases}$

b \$5.95

c $\mathrm{E}(P) = \sum_{x=24}^{s}(0.75x - 0.5s)p(x)$

$+ \sum_{x=s+1}^{30}(0.5s - 0.25x)p(x)$

d 27

5 a i $\frac{1}{6}$ **ii** $\frac{1}{36}$ **iii** $\frac{1}{6}$

b i $\frac{4}{25}$ **ii** $\frac{41}{100}$

c $\frac{121}{600}$

6 a 0.6915 **b** 0.1365

7 a 0.0436 **b** 26.67% **c** 183 **d** 59 271

8 a i 0.1587 **ii** 511.63

b 0.1809

9 a i $\frac{1}{8}$ **ii** A: 0.6915, B: 0.5625

iii $\mathrm{E}(X) = 10$, $\mathrm{E}(Y) = 10.67$; Machine A

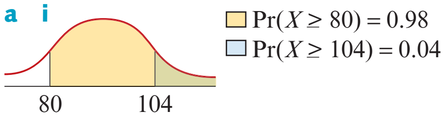
b $\frac{3}{4}$

10 a $c = \frac{20}{49}$ **b** $\mathrm{E}(X) = \frac{120}{49}$ **c** $\mathrm{Var}(X) = \frac{6180}{2401}$

11 a i $\frac{1}{2500}$ **ii** $\frac{16}{3}$ **iii** 0.8281 **iv** 0.7677

b 0.9971

12 a i



🟨 $\Pr(X \ge 80) = 0.98$

▭ $\Pr(X \ge 104) = 0.04$

ii $\mu = 92.956$, $\sigma = 6.3084$

b i 16.73% of sensors **ii** 81°C

13 a i 0.2 **ii** 0.7 **iii** 0.125 **iv** $\frac{3}{160}$

b i 0.360 15 **ii** $\frac{128}{625}$

c 0.163 08

14 a 0.1056 **b** 1027.92 g

15 a i 0.0105 **ii** 0.0455

b 0.4396

c $\frac{1149}{1909}$

16 a i $\mu = 4.25$ **ii** $\sigma = 0.9421$ **iii** 0.94 **iv** 0.9

b i Binomial **ii** 18 **iii** 1.342 **iv** 0.3917

17 a i $\Pr(\text{Black}) = \frac{n-3}{n}$ **ii** $\Pr(\text{White}) = \frac{3}{n}$

b $\frac{(n-3)^2}{n^2 - 3n + 3}$

18 a $(0.0814, 0.1186)$ **b** $(0.0792, 0.1208)$

c Larger sample of females

d 900 of each sex **e** 0.078 or 0.922

Chapter 19

Technology-free questions

1 $f(g(x)) = (3x+1)^2 + 6 = 9x^2 + 6x + 7$

2 $k = -1 - \sqrt{13}$ or $k = -1 + \sqrt{13}$

3 $y = -\frac{6}{x}$

Reflection in x-axis, dilation of factor 2 from the y-axis, dilation of factor 3 from the x-axis; Alternatively: reflection in the x-axis, dilation of factor 6 from the x-axis

4 a $f'(x) = 21x^6(5x^2 - 3)^6(5x^2 - 1)$

b $f'(0) = 2$

5 a $x(1 + 2\log_e(2x))$ **b** $f'\left(\frac{\pi}{2}\right) = \frac{-2}{(\pi+1)^2}$

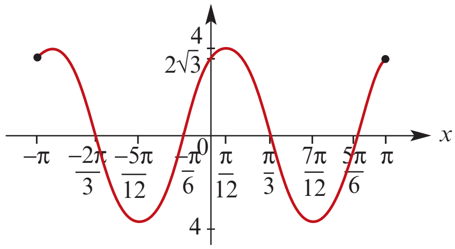
6 a $f'(x) = 2\cos(2x)e^{\sin(2x)}$

b $f'\left(\frac{\pi}{3}\right) = 8\pi - 3\sqrt{3}$

7 $x = \frac{(4n+1)\pi}{8}, n \in \mathbb{Z}$

8 a Amplitude = 4; Period = π

b



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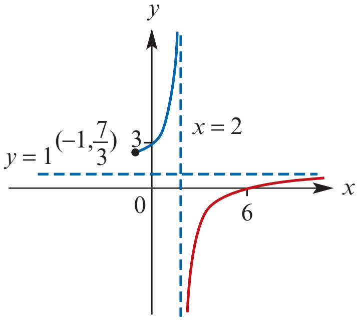

10 a $f^{-1}(x) = \log_e\left(\frac{x+3}{5}\right) + 1$

b $\operatorname{dom} f^{-1} = (-3, \infty)$

11 $x = -\frac{2\pi}{15}$ or $x = \frac{2\pi}{15}$

13 $\frac{1}{4}(e^4 - 1)$

14 a $c = 6$

b $4a + b - 3 = 0$, $3a + b = 0$

c $a = 3$, $b = -9$

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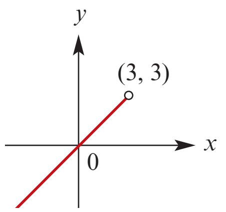
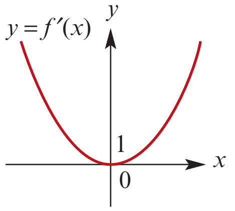
15 a $g^{-1}(x) = \frac{1}{2}\log_e(3-x)$, $\operatorname{dom} g^{-1} = (-\infty, 3)$

b



16 a



b $f'(x) = \begin{cases} -8x^3 & \text{if } x \le 0 \\ 8x^3 & \text{otherwise} \end{cases}$

17 $\frac{-1}{3}\log_e(1-3x) + c$ **18** $f^{-1}(x) = \frac{x}{2(x-3)}$

19 $x = \frac{-\pi}{6}$ or $x = \frac{\pi}{3}$

20 a 0.5 **b** 0.68 **c** 0.32

21 a $\frac{1}{6}$ **b** $\sqrt{31}$

22 $\frac{-2}{3}, \frac{2}{3}$

23 a $A = 32a - 8a^3$

b Max value $A = \frac{128\sqrt{3}}{9}$ when $a = \frac{2\sqrt{3}}{3}$

24 $b = 3$

25 0.36

26 a \$0.65 **b** 0.425

27 0.37

28 a $h = \frac{4000}{x^2}$ **c** $2000(2+\sqrt{2})$

29 a No **b** Electoral roll

30 a 0.53 **b** $(0.4322, 0.6278)$

31 a 0.37 **b** $\frac{0.9463}{\sqrt{n}}$

c Increased by a factor of $\sqrt{2}$

Multiple-choice questions

1 B	**2** A	**3** D	**4** B	**5** C
6 D	**7** A	**8** D	**9** C	**10** D
11 C	**12** A	**13** A	**14** B	**15** D
16 B	**17** C	**18** A	**19** A	**20** E
21 A	**22** E	**23** E	**24** D	**25** C

Extended-response questions

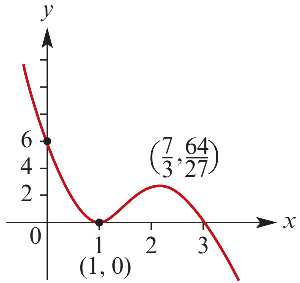
1 a i $\left(\frac{1}{2}, 8\right)$ **ii** Minimum

b ii $A = \frac{x}{12}(60-5x)$ **iii** Max area 15 cm^2

2 a $p = 1$, $q = 3$, $k = 2$

b i $m = -2$ **ii** $y = -2x^3 + 10x^2 - 14x + 6$

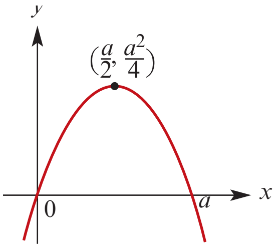
iii



Local minimum at (1, 0)

Local maximum at $\left(\frac{7}{3}, \frac{64}{27}\right)$

3 a



b $\frac{a^3}{6}$ square units

c i $y = \frac{2a^2}{9}$, $y = \frac{2a^2}{9}$ **ii** $\frac{a^3}{162}$ square units

4 a $y = -\frac{1}{2}x + \frac{3}{2}$

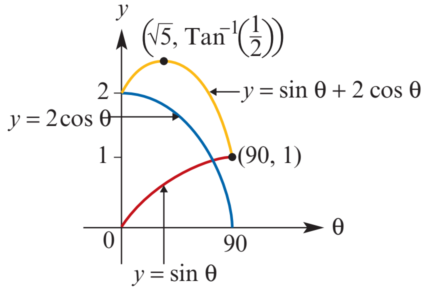
b i $\frac{dy}{d\theta} = \cos\theta - 2\sin\theta$

ii $\theta = \tan^{-1}\left(\frac{1}{2}\right) = 26.57°$

iii $(26.57, 2.2361)$; exact: $(\tan^{-1}(\frac{1}{2}), \sqrt{5})$

iv $r = \sqrt{5}$, $\alpha = 63.435°$

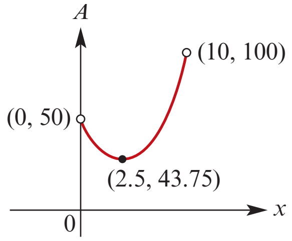
v



c i $Q(2\sin\theta, 2\cos\theta)$ **iii** $\theta = 74.4346°$

5 a i $A = x^2 - 5x + 50$ **ii** $(0, 10)$

iii



iv Minimum area = 43.75 cm^2

b i $f(x) = \frac{1}{2}(10-x)x$

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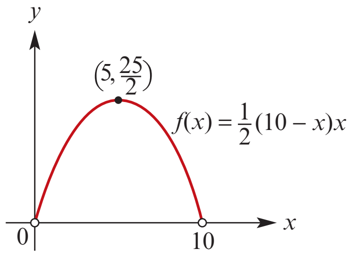
ii



c $AYX : OXYZ : ABY : CBYZ = 1 : 2 : 2 : 3$

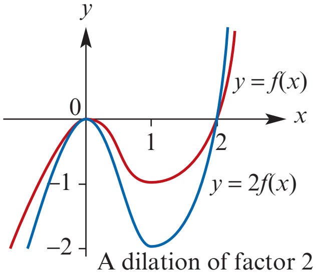
6 a i $f'(t) = -100e^{-\frac{t}{10}}(t^2 - 30t + 144)$

ii $f''(t) = 10e^{-\frac{t}{10}}(t^2 - 50t + 444)$

b i $t \in (6, 24)$ **ii** $t \in (11.546, 35)$;
exact: $t \in (25 - \sqrt{181}, 25 + \sqrt{181})$

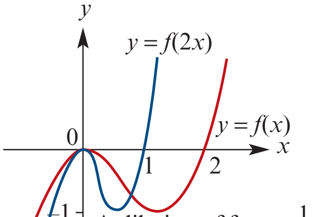
iii $t \in (11.546, 24)$

7 a i



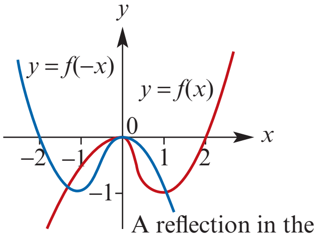
A dilation of factor 2 from the x-axis

ii



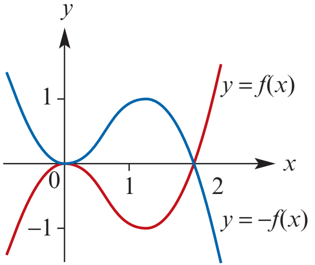
A dilation of factor $\frac{1}{2}$ from the y-axis

iii



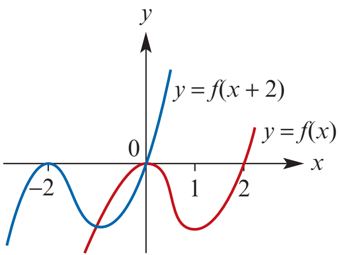
A reflection in the y-axis

iv



A reflection in the x-axis

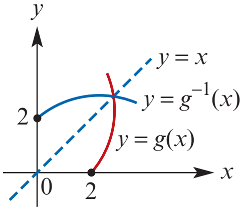
v



A translation of 2 to the left

b f is not one-to-one

c



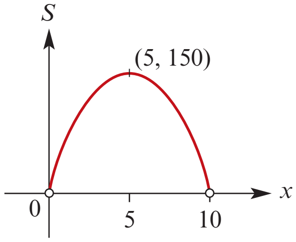
d i Gradient = 15 **ii** Gradient = $\frac{1}{15}$

8 a i 0.995 **ii** $x = \pm 0.2$

b i $h(x) = \frac{1}{2}(x - \pi)^2 - 1$ **ii** −0.989 99

9 a $S = (60x - 6x^2)$ **b** $0 < x < 10$

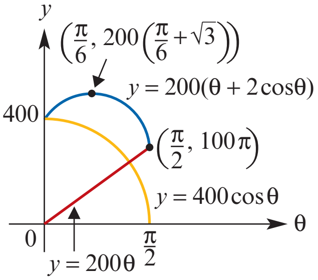
c $x = 5$ **d**



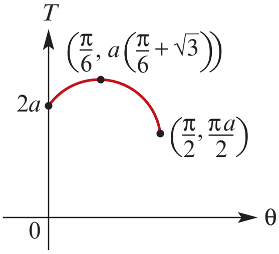
10 a i $OP = \frac{1}{\sin\theta}$ **ii** $BQ = \frac{1 - \cos\theta}{\sin\theta}$

d Min value $S = \frac{\sqrt{3}}{2}$ when $AP = \frac{2\sqrt{3} - 3}{3}$

11 b



c



Minimum value for T is $\frac{\pi a}{2}$

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12 a iii $x = 1$ or $x = k - 2$

b i $b = 3 - 2a,\ c = a - 2$

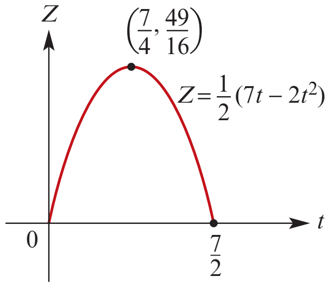
ii $h = a - 2$

iii $a = 0,\ b = 3,\ c = -2$

iv $a = -1,\ b = 5,\ c = -3$

13 a $Z = \frac{1}{2}(7t - 2t^2)$

b



c Max value $Z = \frac{49}{16}$ when $t = \frac{7}{4}$

14 a i $\frac{3}{8}$ **ii** $\frac{4}{15}$

b i $\frac{27}{125}$ **ii** $\frac{8}{125}$ **iii** $\frac{38}{125}$

15 a $k = \frac{b}{a^2}$

b i $y = \frac{b}{2a}x + \frac{b}{2}$ **ii** $\left(\frac{-a}{2}, \frac{b}{4}\right)$

d $S_1 : S_2 = 27 : 37$

16 a i 0.9332 **ii** 0.0668 **iii** 0.1151 **iv** 0.1151

b i 33.3% **ii** 866.4 **iii** 199.4

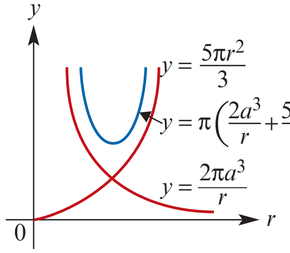
17 $90 - 8\sqrt{3}$ metres from A towards E

18 a i $y = -e^{-n}x + e^{-n}(n + 1)$ **ii** $x = n + 1$

b i $\frac{1}{e^n}\left(1 - \frac{1}{e}\right)$ **ii** $e : e - 2$

19 b i $h = \frac{3a^3 - 2r^3}{3r^2}$ **ii** $S = \pi\left(\frac{2a^3}{r} + \frac{5r^2}{3}\right)$

c i



ii Local minimum at

$$\left((\sqrt[3]{0.6})a,\ \pi a^2\left(\frac{2}{\sqrt[3]{0.6}} + \frac{5}{3}(\sqrt[3]{0.6})^2\right)\right)$$

20 a 0.0023

b

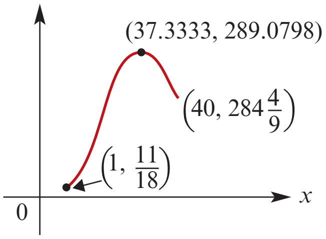
Q	$s - 1$	-1
$Pr(Q = q)$	$\frac{3}{4}$	$\frac{1}{4}$

c $\mathrm{E}(Q) = \frac{3}{4}s - 1,\ \mathrm{sd}(Q) = \frac{\sqrt{3}}{4}s$

21 a 0.091 21 **b** 0.2611 **c** 0.275

22 a $\frac{dP}{dx} = \frac{1}{90}(112x - 3x^2)$

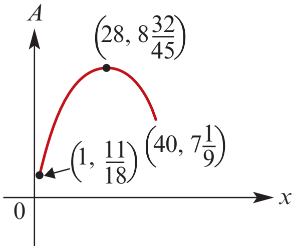
b i



ii Max value of P is 289.0798 tonnes

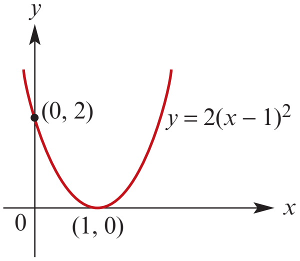
c $A = \frac{x}{90}(56 - x)$

i

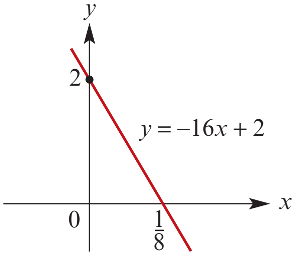


ii Max value of A is $8\frac{32}{45}$ tonnes per miner, occurs when $x = 28$

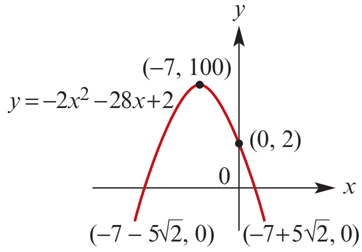
23 a i



ii



iii



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b $\left(\dfrac{2-3k}{k+2}, \dfrac{-2(2-3k)^2+2(k+2)}{k+2}\right)$

i $\left(-2, \dfrac{2}{3}\right)$ **ii** $\left\{\dfrac{2}{3}\right\}$

iii $(-\infty,-2)\cup\left(0,\dfrac{14}{9}\right)$ **iv** $(-2,0)\cup\left(\dfrac{14}{9},\infty\right)$

c $k < -2$

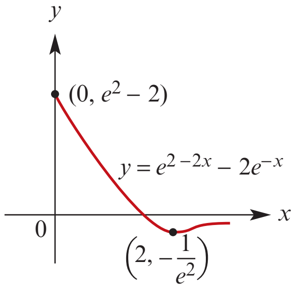
d **i** $k = 0$ or $k = \dfrac{14}{9}$ **ii** $0 < k < \dfrac{14}{9}$

24 **a** $x = 2 - \log_e 2$

b **i** $\dfrac{dy}{dx} = -2e^{2-2x} + 2e^{-x}$ **ii** $x = 2$

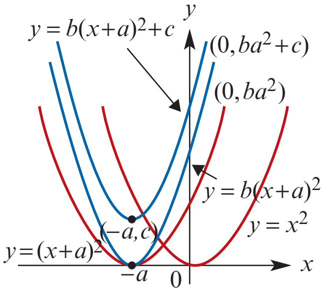
iii $\left(2, -\dfrac{1}{e^2}\right)$

iv



c $\left(-\dfrac{1}{e^2}, 0\right)$

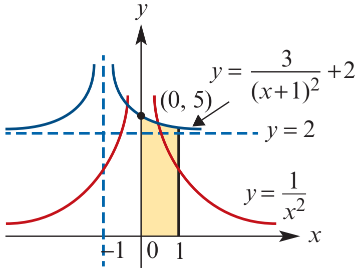
25 **a**



c Dilation of factor 3 from the x-axis, then translation 1 unit to the left and 2 units up

d $\dfrac{7}{2}$

e



26 **a** **i** $y = 50$ **ii** $y = x - 25$

b $a = -\dfrac{1}{15}$, $c = -\dfrac{25}{3}$

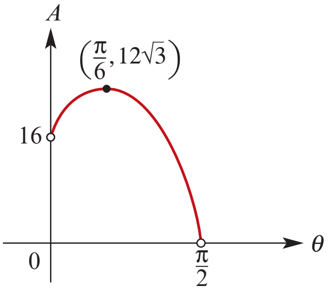
c **i** 1250 square units

ii $\dfrac{14\,375}{18}$ square units

iii $\dfrac{36\,875}{18}$ square units

27 **c** $\theta = \dfrac{\pi}{6}$

d



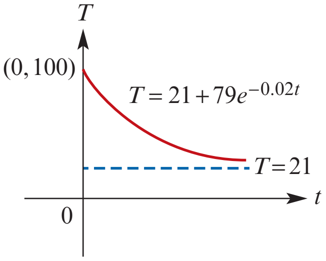
Maximum value of A is $12\sqrt{3}$

28 **a** $\mu = 5.0290$, $\sigma = 0.0909$ **b** \$409.28

29 **a** $k = \dfrac{1}{10}\log_e\left(\dfrac{79}{63}\right) \approx 0.02$, $A = 79$

b Approx 2:44 p.m.

c



d Average rate of change = −1.6°C/minute

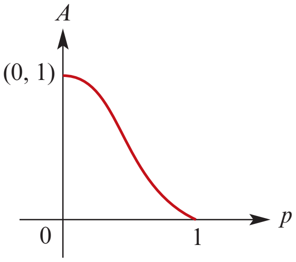
e **i** 2.0479°C/minute

ii −0.8826°C/minute

30 **a** $\dfrac{3}{16}$

b $b = 4$

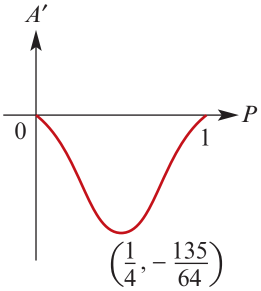
c



d **i** 0.076 **ii** 0.657

e **i** $A'(p) = -20p(1-p)^3$

ii



iii $p = \dfrac{1}{4}$

iv Most rapid rate of change of probability occurs when $p = \frac{1}{4}$

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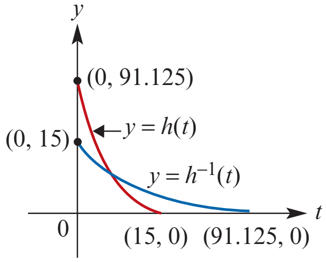
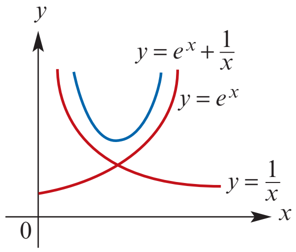
31 **a** 91.125 cm
b [0, 15]
c $V = 0.64(4.5 - 0.3t)^3$
d h is a one-to-one function;
$h^{-1}(t) = 15 - \frac{10t^{\frac{1}{3}}}{3}$, dom h^{-1} = [0, 91.125]
e



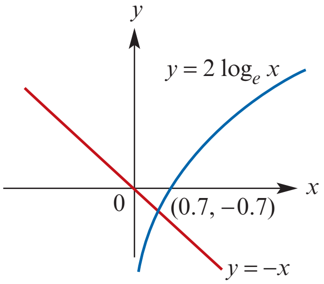
32 **a** 0.065 36
b **i** 0.6595 **ii** 0.198 14
c **i** 23.3% **ii** $c = 0.1075$

33 **a** **i** 0.32 **ii** 0.18 **iii** 0.5
b 0.64
c **i** 0.043 95 **ii** 0.999 **iii** $\frac{7}{128}$

34 **a** **b**

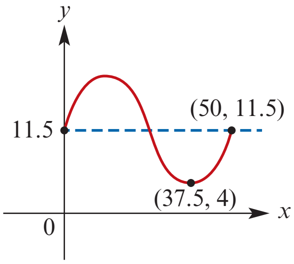


c $y = \frac{1}{x} + e^x$, $\frac{dy}{dx} = -\frac{1}{x^2} + e^x$
d **ii** $2\log_e x < 0, x \in (0, 1)$
iii

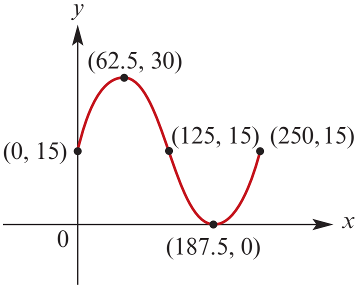


iv (0.7, 3.4)

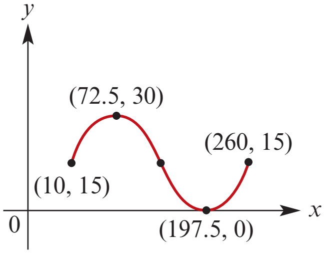
35 **a** **i** $m = 12.5, n = 15, d = 37.5, a = 7.5,$ $b = 7.5$
ii



b (2.704, 10), (22.296, 10)
c



d **i** $h(x) = 15 + 15\sin\left(\frac{\pi(x - 10)}{125}\right)$
ii

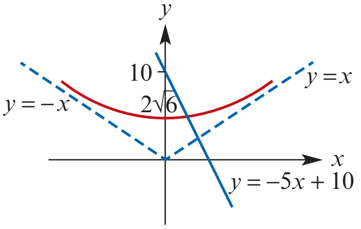


36 **a** $k = 4$
b **i** $\mathrm{E}(X) = \frac{13}{6}$ **ii** $\frac{10 - \sqrt{2}}{4}$ **iii** $\frac{\sqrt{2}}{12}$
c 0.1857

37 **a** $k = \frac{2}{a^2}$ **b** $\mathrm{E}(X) = \frac{a}{3}$, $\mathrm{Var}(X) = \frac{a^2}{18}$
c $\frac{6 - 4\sqrt{2}}{9}$ **d** $a = 1000(\sqrt{2} + 2)$

38 **a** 7 **c** $y = \frac{1}{10}x - \log_e 10$ **d** **ii** 36.852

39 **a** $\frac{dy}{dx} = \frac{x}{\sqrt{x^2 + 24}}$ **b** $(0, 2\sqrt{6})$ **c** Even
d



e $y = -5x + 10$ **f** 14 units/second
h $12\log_e(\sqrt{7} - 1) - 2\sqrt{7} + \frac{35}{2}$

40 **a** **i** 0 **ii** −0.6745 **iii** 0.6745
iv 1.3490 **v** 99.3% **vi** 0.7%
b **i** μ **ii** μ − 0.645σ
iii μ + 0.645σ **iv** 1.3490σ
v 0.9930% **vi** 0.7%

41 **a** $k = n + 1$ **b** $\mathrm{E}(X) = \frac{n + 1}{n + 2}$
c $\frac{n + 1}{(n + 2)^2(n + 3)}$ **d** Median = $\sqrt[n+1]{\frac{1}{2}}$

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42 a i $\frac{1}{1-b}$ **ii** $1+\sqrt{b-1}$

b i 1 **ii** $1+e^{-7}$

c i $\frac{b(b-2)}{2(b-1)}$ **ii** $9+\sqrt{65}$

d e

43 a i $\frac{-(b+1)}{b^2}$ **ii** $\left(\frac{2b^2}{b+1}\right)^{\frac{1}{3}}$

b i $\frac{(b^2+1)(b-1)}{2b^2}$ **ii** $b=3$

44 a Local maximum $\left(\frac{m}{n}, \left(\frac{m}{n}\right)^m e^{n-m}\right)$

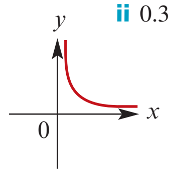
b $\left(\frac{m-1}{n}, \left(\frac{m-1}{n}\right)^m e^{n-m+1}\right)$

c i $\frac{4}{e^2}$ **ii** $1-\frac{5}{e^2}$

45 a i q **ii** $\frac{1}{q}$ **iii** $\frac{1}{q^2}$

b $\frac{1}{3}$

c i



ii 0.37 **iii** 69.31

46 a $\frac{527}{1000}$ **b** $(0.4961, 0.5580)$

c 67% **d** 99.9955%

Appendix A

Exercise A1

1 63 **2** 26 **3** 336

4 a 5040 **b** 210

5 a 120 **b** 120

6 18

7 a 5 852 925 **b** 1 744 200

8 100 386

9 a 792 **b** 336

10 a 200 **b** 75 **c** 6 **d** 462

e 81

Exercise A2

1 a $\sum_{i=1}^{4} i^3 = 1+8+27+64 = 100$

b $\sum_{k=1}^{5} k^3 = 1+8+27+64+125 = 225$

c $\sum_{i=1}^{5} (-1)^i\, i = -1+2-3+4-5 = -3$

d $\frac{1}{5}\sum_{i=1}^{5} i = \frac{1}{5}(1+2+3+4+5) = 3$

e $\sum_{i=1}^{6} i = 1+2+3+4+5+6 = 21$

f $\sum_{k=1}^{4} (k-1)^2 = 0+1+4+9 = 14$

g $\frac{1}{3}\sum_{i=1}^{4} (i-2)^2 = \frac{1}{3}(1+0+1+4) = 2$

h $\sum_{i=1}^{6} i^2 = 1+4+9+16+25+36 = 91$

2 a $\sum_{i=1}^{n} i$ **b** $\sum_{i=1}^{11} x_i$ **c** $\frac{1}{10}\sum_{i=1}^{10} x_i$

d $\sum_{i=1}^{n+1} i^4$ **e** $\sum_{i=1}^{5} \frac{1}{i}$

3 a $x + x^2 + x^3 + \cdots + x^n$

b $32 + 16x + 8x^2 + 4x^3 + 2x^4 + x^5$

c $3^6 + (2x)\cdot 3^5 + (2x)^2\cdot 3^4 + (2x)^3\cdot 3^3 + (2x)^4\cdot 3^2 + (2x)^5\cdot 3 + (2x)^6$

d $(x-x_1) + (x-x_2)^2 + (x-x_3)^3 + (x-x_4)^4$

4 a $\sum_{i=0}^{5} x^{5-i}\cdot 3^i$ **b** $\sum_{i=0}^{5} x^{5-i}\cdot(-3)^i$

c $\sum_{i=0}^{2} x^{2-i}\cdot 2^{2-i}$ **d** $\sum_{i=0}^{3} (2x)^{3-i}\cdot 3^i$

Exercise A3

1 a $x^6 + 36x^5 + 540x^4 + 4320x^3 + 19\,440x^2 + 46\,656x + 46\,656$

b $32x^5 + 80x^4 + 80x^3 + 40x^2 + 10x + 1$

c $32x^5 - 80x^4 + 80x^3 - 40x^2 + 10x - 1$

d $64x^6 + 576x^5 + 2160x^4 + 4320x^3 + 4860x^2 + 2916x + 729$

e $64x^6 - 1152x^5 + 8640x^4 - 34\,560x^3 + 77\,760x^2 - 93\,312x + 46\,656$

f $16x^4 - 96x^3 + 216x^2 - 216x + 81$

g $x^6 - 12x^5 + 60x^4 - 160x^3 + 240x^2 - 192x + 64$

h $x^{10} + 10x^9 + 45x^8 + 120x^7 + 210x^6 + 252x^5 + 210x^4 + 120x^3 + 45x^2 + 10x + 1$

2 a $-960x^3$ **b** $960x^3$

c $-960x^3$ **d** $192\,456x^5$

e $1732\,104x^5$ **f** $-25\,344b^7x^5$

3 $-\frac{16}{243}x^7$ **4** $-336\,798x^6$

5 $(-x+1)^{11} = -x^{11} + 11x^{10} - 55x^9 + 165x^8 - 330x^7 + 462x^6 - 462x^5 + 330x^4 - 165x^3 + 55x^2 - 11x + 1$

6 a 40 **b** −160 **c** −80

d 181 440 **e** 432 **f** 1080

7 83 026 944 **8** −768

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